# Problems 3 - Trigonometric Interpolation

**Approximation Theory (MATH3081/4221)** — Epiphany 2015 — anthony.yeates@dur.ac.uk

The problem marked $\star$ should be handed in for marking at the lecture on **Monday 9th March**. There will be a problem class on this chapter on Monday 2nd March.
I use † to indicate (what I consider to be) trickier problems.

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29. *Discrete Fourier transforms.* Compute the discrete Fourier transform of the following vectors, and interpret your results: (a) $\mathbf{x} = (1,1,1,1)^\top$; (b) $\mathbf{x} = (0,1,0,-1,0,1,0,-1)^\top$.

**Solution:** (a) Let $\omega = e^{i2\pi/4} = e^{i\pi/2} = i$. Then

$$F_4^{-1}\mathbf{x} = \frac{1}{4}\begin{pmatrix}\omega^0 & \omega^0 & \omega^0 & \omega^0\\ \omega^0 & \omega^{-1} & \omega^{-2} & \omega^{-3}\\ \omega^0 & \omega^{-2} & \omega^{-4} & \omega^{-6}\\ \omega^0 & \omega^{-3} & \omega^{-6} & \omega^{-9}\end{pmatrix}\begin{pmatrix}1\\1\\1\\1\end{pmatrix} = \frac{1}{4}\begin{pmatrix}1 & 1 & 1 & 1\\ 1 & -i & -1 & i\\ 1 & -1 & 1 & -1\\ 1 & i & -1 & -i\end{pmatrix}\begin{pmatrix}1\\1\\1\\1\end{pmatrix} = \begin{pmatrix}1\\0\\0\\0\end{pmatrix}.$$

Interpretation: the only non-zero coefficient is $c_0$, which is the constant term in the trigonometric polynomial (as expected).

(b) Let $\omega = e^{i2\pi/8} = e^{i\pi/4} = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i$. Then

$$F_8^{-1}\mathbf{x} = \frac{1}{8}\begin{pmatrix}\omega^0 & \omega^0 & \omega^0 & \omega^0 & \omega^0 & \omega^0 & \omega^0 & \omega^0\\ \omega^0 & \omega^{-1} & \omega^{-2} & \omega^{-3} & \omega^{-4} & \omega^{-5} & \omega^{-6} & \omega^{-7}\\ \omega^0 & \omega^{-2} & \omega^{-4} & \omega^{-6} & \omega^{-8} & \omega^{-10} & \omega^{-12} & \omega^{-14}\\ \omega^0 & \omega^{-3} & \omega^{-6} & \omega^{-9} & \omega^{-12} & \omega^{-15} & \omega^{-18} & \omega^{-21}\\ \omega^0 & \omega^{-4} & \omega^{-8} & \omega^{-12} & \omega^{-16} & \omega^{-20} & \omega^{-24} & \omega^{-28}\\ \omega^0 & \omega^{-5} & \omega^{-10} & \omega^{-15} & \omega^{-20} & \omega^{-25} & \omega^{-30} & \omega^{-35}\\ \omega^0 & \omega^{-6} & \omega^{-12} & \omega^{-18} & \omega^{-24} & \omega^{-30} & \omega^{-36} & \omega^{-42}\\ \omega^0 & \omega^{-7} & \omega^{-14} & \omega^{-21} & \omega^{-28} & \omega^{-35} & \omega^{-42} & \omega^{-49}\end{pmatrix}\begin{pmatrix}0\\1\\0\\-1\\0\\1\\0\\-1\end{pmatrix}$$

$$= \frac{1}{8}\begin{pmatrix}1 & 1 & 1 & 1 & 1 & 1 & 1 & 1\\ 1 & \omega^{-1} & \omega^{-2} & \omega^{-3} & \omega^{-4} & \omega^{-5} & \omega^{-6} & \omega^{-7}\\ 1 & \omega^{-2} & \omega^{-4} & \omega^{-6} & 1 & \omega^{-2} & \omega^{-4} & \omega^{-6}\\ 1 & \omega^{-3} & \omega^{-6} & \omega^{-1} & \omega^{-4} & \omega^{-7} & \omega^{-2} & \omega^{-5}\\ 1 & \omega^{-4} & 1 & \omega^{-4} & 1 & \omega^{-4} & 1 & \omega^{-4}\\ 1 & \omega^{-5} & \omega^{-2} & \omega^{-7} & \omega^{-4} & \omega^{-1} & \omega^{-6} & \omega^{-3}\\ 1 & \omega^{-6} & \omega^{-4} & \omega^{-2} & 1 & \omega^{-6} & 1 & \omega^{-2}\\ 1 & \omega^{-7} & \omega^{-6} & \omega^{-5} & \omega^{-4} & \omega^{-3} & \omega^{-2} & \omega^{-1}\end{pmatrix}\begin{pmatrix}0\\1\\0\\-1\\0\\1\\0\\-1\end{pmatrix}$$

$$= \frac{1}{8}\begin{pmatrix}1-1+1-1\\ \omega^{-1}-\omega^{-3}+\omega^{-5}-\omega^{-7}\\ \omega^{-2}-\omega^{-6}+\omega^{-2}-\omega^{-6}\\ \omega^{-3}-\omega^{-1}+\omega^{-7}-\omega^{-5}\\ \omega^{-4}-\omega^{-4}+\omega^{-4}-\omega^{-4}\\ \omega^{-5}-\omega^{-7}+\omega^{-1}-\omega^{-3}\\ \omega^{-6}-\omega^{-2}+\omega^{-6}-\omega^{-2}\\ \omega^{-7}-\omega^{-5}+\omega^{-3}-\omega^{-1}\end{pmatrix} = \frac{1}{8}\begin{pmatrix}0\\0\\ \omega^{-2}-\omega^{-6}+\omega^{-2}-\omega^{-6}\\0\\0\\0\\ \omega^{-6}-\omega^{-2}+\omega^{-6}-\omega^{-2}\\0\end{pmatrix} = \frac{i}{2}\begin{pmatrix}0\\0\\-1\\0\\0\\0\\1\\0\end{pmatrix}.$$

Note that we used the fact that $\omega^{-3} = -\omega^{-5}$ and $\omega^{-1} = -\omega^{-7}$ (think of the unit circle), and that $\omega^{-2} = -i$, $\omega^{-6} = i$. Interpretation: the only non-zero terms are $c_2$ and $c_6$, and they are purely imaginary. Thus the real trigonometric polynomial has only a $\sin(2x)$ term (which matches the original function being interpolated) and a $\sin(6x)$ term (which is an alias of $\sin(2x)$).

30. *Real entries.* Suppose that the entries of $\mathbf{f}$ are all real. If $\mathbf{c} = F_n^{-1}\mathbf{f}$ is the discrete Fourier transform of $\mathbf{f}$, show that $\bar{c}_{n-k} = c_k$ for $k = 0,\ldots,n-1$.

**Solution:** We know that

$$c_k = (F_n^{-1})_{kj}f_j = \frac{1}{n}\sum_{j=0}^{n-1}\omega^{-jk}f_j,$$

so using the fact that $f_j$ are real,

$$\bar{c}_{n-k} = \overline{\frac{1}{n}\sum_{j=0}^{n-1}\omega^{-j(n-k)}f_j} = \frac{1}{n}\sum_{j=0}^{n-1}\omega^{-jk}\omega^{jn}f_j = \frac{1}{n}\sum_{j=0}^{n-1}\omega^{-jk}\omega^{i2\pi j}f_j = \frac{1}{n}\sum_{j=0}^{n-1}\omega^{-jk}(1)f_j = c_k.$$

⋆ **31.** *Trigonometric interpolation.* Consider the periodic function $f(x) = \sin(x) + 2\cos(2x)$.

(a) Write down the Fourier matrix $F_3$, and its inverse $F_3^{-1}$.

(b) Use this to find a real trigonometric polynomial of the form

$$p_3(x) = \sum_{k=0}^{2}\Big(a_k\cos(kx) - b_k\sin(kx)\Big)$$

that interpolates $f$ at three equally-spaced nodes on $[0, 2\pi)$.

(c) Explain how it can be that the interpolant $p_3$ you found in part (b) does not reproduce the original function $f$ exactly.

(d) Find a trigonometric polynomial of lower degree that interpolates the same data.

**Solution:** (a) Let $\omega = e^{i2\pi/3} = \cos(\frac{2\pi}{3}) + i\sin(\frac{2\pi}{3}) = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$. Then $\omega^2 = \omega^{-1} = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$. Thus

$$F_3 = \begin{pmatrix}\omega^0 & \omega^0 & \omega^0\\ \omega^0 & \omega^1 & \omega^2\\ \omega^0 & \omega^2 & \omega^4\end{pmatrix} = \begin{pmatrix}1 & 1 & 1\\ 1 & \omega & \omega^2\\ 1 & \omega^2 & \omega\end{pmatrix} = \begin{pmatrix}1 & 1 & 1\\ 1 & -\frac{1}{2}+\frac{\sqrt{3}}{2}i & -\frac{1}{2}-\frac{\sqrt{3}}{2}i\\ 1 & -\frac{1}{2}-\frac{\sqrt{3}}{2}i & -\frac{1}{2}+\frac{\sqrt{3}}{2}i\end{pmatrix},$$

and

$$F_3^{-1} = \tfrac{1}{3}\bar{F}_3 = \tfrac{1}{3}\begin{pmatrix}1 & 1 & 1\\ 1 & -\frac{1}{2}-\frac{\sqrt{3}}{2}i & -\frac{1}{2}+\frac{\sqrt{3}}{2}i\\ 1 & -\frac{1}{2}+\frac{\sqrt{3}}{2}i & -\frac{1}{2}-\frac{\sqrt{3}}{2}i\end{pmatrix},$$

(b) The coefficients $a_k$ and $b_k$ are the real and imaginary parts of $c_k$, where $\mathbf{c} = F_3^{-1}\mathbf{f}$ and the data are given by

$$f_0 = \sin(0) + 2\cos(0) = 2, \quad f_1 = \sin(\tfrac{2\pi}{3}) + 2\cos(\tfrac{2\pi}{3}) = \tfrac{\sqrt{3}}{2} - 1, \quad f_2 = -\tfrac{\sqrt{3}}{2} - 1.$$

Using the matrix from (a), we obtain

$$\mathbf{c} = \tfrac{1}{3}\begin{pmatrix}1 & 1 & 1\\ 1 & -\frac{1}{2}-\frac{\sqrt{3}}{2}i & -\frac{1}{2}+\frac{\sqrt{3}}{2}i\\ 1 & -\frac{1}{2}+\frac{\sqrt{3}}{2}i & -\frac{1}{2}-\frac{\sqrt{3}}{2}i\end{pmatrix}\begin{pmatrix}2\\ \frac{\sqrt{3}}{2}-1\\ -\frac{\sqrt{3}}{2}-1\end{pmatrix} = \begin{pmatrix}0\\ 1-\frac{1}{2}i\\ 1+\frac{1}{2}i\end{pmatrix},$$

so the real trigonometric interpolant is

$$p_3(x) = \cos(x) + \tfrac{1}{2}\sin(x) + \cos(2x) - \tfrac{1}{2}\sin(2x).$$

(c) We see that $p_3(x) \neq f(x)$ for most $x$. Although $f$ is itself a trigonometric polynomial with $n = 3$, this is not inconsistent, because the Nyquist frequency for $n = 3$ is $k_0 = \frac{3}{2}$. Since the function $f$ contains a component of frequency $k = 2 > k_0$, there is not a unique trigonometric polynomial with $n = 3$ that interpolates $f$.

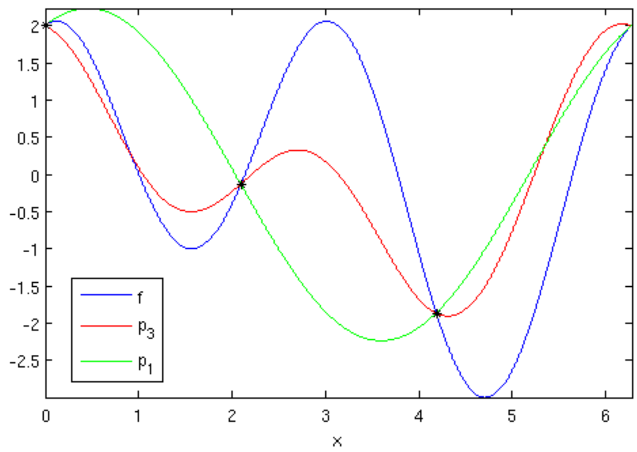
(d) We have that

$$\cos(2x_j) = \cos((3-1)x_j) = \cos(2\pi j - \tfrac{2\pi j}{3}) = \cos(2\pi j)\cos(\tfrac{2\pi j}{3}) + \sin(2\pi j)\sin(\tfrac{2\pi j}{3}) = \cos(x_j),$$
$$\sin(2x_j) = \sin((3-1)x_j) = \sin(2\pi j - \tfrac{2\pi j}{3}) = \sin(2\pi j)\cos(\tfrac{2\pi j}{3}) - \cos(2\pi j)\sin(\tfrac{2\pi j}{3}) = -\sin(x_j),$$

so replacing $\sin(2x)$ by $-\sin(x)$ and $\cos(2x)$ by $\cos(x)$ yields another interpolant

$$\tilde{p}_1(x) = 2\cos(x) + \sin(x).$$

Here is what the functions look like:




32. *Splitting.* Let $\mathbf{h} = \mathbf{f} + i\mathbf{g}$, where $\mathbf{f}$ and $\mathbf{g}$ are real vectors, and let $\mathbf{b}$ be the DFT of $\mathbf{h}$. Show that the DFTs of $\mathbf{f}$ and $\mathbf{g}$ are

$$c_k = \frac{1}{2}\big(b_k + \bar{b}_{n-k}\big), \qquad d_k = \frac{i}{2}\big(\bar{b}_{n-k} - b_k\big).$$

*Remark: One can speed up the DFT of a real vector* $\mathbf{f}$ *by splitting into* $\mathbf{f}_{\text{even}}$ *and* $\mathbf{f}_{\text{odd}}$ *and finding the size* $n/2$ *transform of* $\mathbf{h} = \mathbf{f}_{\text{even}} + i\mathbf{f}_{\text{odd}}$.

**Solution:** This is really an extension of Problem 30. We have

$$b_k = \frac{1}{n}\sum_{j=0}^{n-1}\omega^{-jk}h_j = \frac{1}{n}\sum_{j=0}^{n-1}\omega^{-jk}(f_j + ig_j), \tag{1}$$

and

$$\bar{b}_{n-k} = \frac{1}{n}\sum_{j=0}^{n-1}\overline{\omega^{-jn}\omega^{jk}(f_j + ig_j)} = \frac{1}{n}\sum_{j=0}^{n-1}\omega^{jn}\omega^{-jk}(f_j - ig_j) = \frac{1}{n}\sum_{j=0}^{n-1}\omega^{-jk}(f_j - ig_j). \tag{2}$$

Adding (1) and (2) gives

$$\frac{1}{2}\big(b_k + \bar{b}_{n-k}\big) = \frac{1}{n}\sum_{j=0}^{n-1}\omega^{-jk}f_j = c_k,$$

and subtracting gives

$$\frac{i}{2}\big(\bar{b}_{n-k} - b_k\big) = \frac{1}{n}\sum_{j=0}^{n-1}\omega^{-jk}g_j = d_k.$$

† 33. *Eigenvalues of* $F_4$.

(a) Find the $4\times 4$ matrix $P$ such that $F_4 = P\bar{F}_4$, and verify that $P^2 = I_4$.

(b) Show that $P = \frac{1}{4}F_4^2$.

(c) Hence show that $F_4^4 = 16I_4$, and deduce that the eigenvalues of $F_4$ must be either $\pm 2$ or $\pm 2i$.

*Remark: In fact, for any* $n$*, we have* $F_n^4 = n^2 I_n$.

**Solution:** (a) As in Problem 29(a), the matrices are

$$F_4 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{pmatrix}, \qquad \bar{F}_4 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{pmatrix}.$$

Hence the matrix $P$ needs to swap rows 2 and 4. This is achieved with

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},$$

which is easily shown to satisfy $P^2 = I_4$.

(b) Using the fact that $F_4^{-1} = \frac{1}{4}\bar{F}_4$, we get

$$P\bar{F}_4 = F_4 \implies P\bar{F}_4 F_4 = F_4^2 \implies P(4I_4) = F_4^2 \implies P = \tfrac{1}{4}F_4^2.$$

(c) Clearly it follows that $F_4^4 = (F_4^2)^2 = (4P)^2 = 16P^2 = 16I_4$. This shows that the eigenvalues of $F_4^4$ are 16 (with multiplicity 4). It follows that the eigenvalues of $F_4$ must satisfy $\lambda^2 = \pm 4$, so each $\lambda$ must be one of $\pm 2, \pm 2i$.

34. *The columns of $F_n$ as eigenvectors.* Show that the columns of the Fourier matrix $F_n$ are the eigenvectors of the cyclic permutation matrix

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & & \ddots & \vdots \\ 0 & & & & & 1 \\ 1 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

**Solution:** The columns of $F_n$ are vectors $\mathbf{v}^{(k)}$ with components $v_j^{(k)} = \omega^{jk}$, where $\omega = e^{i2\pi/n}$. For $j = 0, \ldots, n-2$ we have

$$(A\mathbf{v}^{(k)})_j = v_{j+1}^{(k)} = \omega^{k(j+1)} = \omega^k v_j^{(k)}.$$

The last element is

$$(A\mathbf{v}^{(k)})_{n-1} = v_0^{(k)} = \omega^0 = \omega^{nk} = \omega^{nk-k+k} = \omega^k \omega^{(n-1)k} = \omega^k v_{n-1}^{(k)}.$$

Hence $\mathbf{v}^{(k)}$ is an eigenvector of $A$ with eigenvalue $\omega^k$.

35. *Inverse Fast Fourier Transform.* Find $\frac{n}{2} \times \frac{n}{2}$ matrices $A$, $B$, $C$, $D$ such that

$$F_n^{-1} = \frac{1}{n}\begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} F_{n/2}^{-1} & 0 \\ 0 & F_{n/2}^{-1} \end{pmatrix} P_n,$$

where $F_n$ is the $n \times n$ Fourier matrix and $P_n$ is the odd-even permutation matrix (as defined in the lecture). This shows that the FFT algorithm works in both directions!

**Solution:** Since $F_n^{-1} = \frac{1}{n}\bar{F}_n$, the components of $\mathbf{f} = F_n^{-1}\mathbf{c}$ are

$$\begin{aligned} f_j = \frac{1}{n}\sum_{k=0}^{n-1}\omega_n^{-jk}c_k &= \frac{1}{n}\sum_{k=0}^{n/2-1}\omega_n^{-j2k}c_{2k} + \frac{1}{n}\sum_{k=0}^{n/2-1}\omega_n^{-j(2k+1)}c_{2k+1} \\ &= \frac{1}{n}\sum_{k=0}^{n/2-1}(\omega_n^2)^{-jk}c_{2k} + \frac{1}{n}\omega_n^{-j}\sum_{k=0}^{n/2-1}(\omega_n^2)^{-jk}c_{2k+1}. \end{aligned}$$

From this we see that the required matrices are

$$A = C = I_{n/2}, \quad B = \bar{D}_{n/2}, \quad D = -\bar{D}_{n/2},$$

where $\bar{D}_{n/2} = \mathrm{diag}\big(1, \omega^{-1}, \omega^{-2}, \ldots, \omega^{-(n/2-1)}\big)$.

† 36. *Applying the FFT.* Compute $F_8\mathbf{x}$ using the recursive FFT algorithm for $\mathbf{x} = (1,0,1,0,1,0,1,0)^\top$.

**Solution:** Applying the algorithm recursively gives (in block matrix form)

$$F_8 = \begin{pmatrix} I_4 & D_4 \\ I_4 & -D_4 \end{pmatrix} \begin{pmatrix} F_4 & 0 \\ 0 & F_4 \end{pmatrix} P$$

$$= \begin{pmatrix} I_4 & D_4 \\ I_4 & -D_4 \end{pmatrix} \begin{pmatrix} I_2 & D_2 & 0 & 0 \\ I_2 & -D_2 & 0 & 0 \\ 0 & 0 & I_2 & D_2 \\ 0 & 0 & I_2 & -D_2 \end{pmatrix} \begin{pmatrix} F_2 & 0 & 0 & 0 \\ 0 & F_2 & 0 & 0 \\ 0 & 0 & F_2 & 0 \\ 0 & 0 & 0 & F_2 \end{pmatrix} P'$$

$$= \begin{pmatrix} I_4 & D_4 \\ I_4 & -D_4 \end{pmatrix} \begin{pmatrix} I_2 & D_2 & 0 & 0 \\ I_2 & -D_2 & 0 & 0 \\ 0 & 0 & I_2 & D_2 \\ 0 & 0 & I_2 & -D_2 \end{pmatrix} \begin{pmatrix} I_1 & D_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ I_1 & -D_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & I_1 & D_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & I_1 & -D_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I_1 & D_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & I_1 & -D_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & I_1 & D_1 \\ 0 & 0 & 0 & 0 & 0 & 0 & I_1 & -D_1 \end{pmatrix} P''$$

where $P$, $P'$, $P''$ are the appropriate permutation matrices, and we have used that $F_1 = (1)$. We don't need to compute the matrix $P''$ explicitly, only to work out the re-ordering of the $\mathbf{x}$ components. This can be done easily by binary bit-reversal:

$$\begin{bmatrix} 000 \\ 001 \\ 010 \\ 011 \\ 100 \\ 101 \\ 110 \\ 111 \end{bmatrix} \to \begin{bmatrix} 000 \\ 100 \\ 010 \\ 110 \\ 001 \\ 101 \\ 011 \\ 111 \end{bmatrix} \quad \text{i.e. } P''\mathbf{x} = \begin{pmatrix} x_0 \\ x_4 \\ x_2 \\ x_6 \\ x_1 \\ x_5 \\ x_3 \\ x_7 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

We need

$$D_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \omega_8 & 0 & 0 \\ 0 & 0 & \omega_8^2 & 0 \\ 0 & 0 & 0 & \omega_8^3 \end{pmatrix}, \quad D_2 = \begin{pmatrix} 1 & 0 \\ 0 & \omega_4 \end{pmatrix}, \quad D_1 = (1),$$

where $\omega_8 = e^{i2\pi/8}$ and $\omega_4 = e^{i2\pi/4} = i$. Putting all of this together gives

$$F_8 = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & \omega_8 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & \omega_8^2 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & \omega_8^3 \\ 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -\omega_8 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -\omega_8^2 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -\omega_8^3 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & i & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & i \\ 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -i \end{pmatrix} \times$$

$$\times \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \ldots = \begin{pmatrix} 4 \\ 0 \\ 0 \\ 0 \\ 4 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

37. *Radix-3 FFT.* The FFT may be applied with more general splittings, instead of the radix-2 algorithm presented in the lecture. Suppose $\mathbf{f} = F_n\mathbf{c}$ but now $n$ is a power of 3.

   (a) Show that the entries in $\mathbf{f}$ may be written as

$$f_j = \sum_{k=0}^{n/3-1} (\omega_{n/3})^{jk} c_{3k} + (\omega_n)^j \sum_{k=0}^{n/3-1} (\omega_{n/3})^{jk} c_{3k+1} + (\omega_n)^{2j} \sum_{k=0}^{n/3-1} (\omega_{n/3})^{jk} c_{3k+2}.$$

   † (b) For $n = 6$, write out the explicit factorisation of $\mathbf{f} = F_n\mathbf{c}$ in matrix form, including the necessary permutation matrix.

*Remark: This can be generalised to any radix, which was known already to Gauss.*

**Solution:** (a) This is a simple generalisation of the argument given in the lecture. We split into three sums:

$$\begin{aligned} f_j = \sum_{k=0}^{n-1} \omega_n^{jk} c_k &= \sum_{k=0}^{n/3-1} \omega_n^{j3k} c_{3k} + \sum_{k=0}^{n/3-1} \omega_n^{j(3k+1)} c_{3k+1} + \sum_{k=0}^{n/3-1} \omega_n^{j(3k+2)} c_{3k+2} \\ &= \sum_{k=0}^{n/3-1} (\omega_n^3)^{jk} c_{3k} + (\omega_n)^j \sum_{k=0}^{n/3-1} (\omega_n^3)^{jk} c_{3k+1} + (\omega_n)^{2j} \sum_{k=0}^{n/3-1} (\omega_n^3)^{jk} c_{3k+2} \\ &= \sum_{k=0}^{n/3-1} (\omega_{n/3})^{jk} c_{3k} + (\omega_n)^j \sum_{k=0}^{n/3-1} (\omega_{n/3})^{jk} c_{3k+1} + (\omega_n)^{2j} \sum_{k=0}^{n/3-1} (\omega_{n/3})^{jk} c_{3k+2}. \end{aligned}$$

(b) For $n = 6$, each of the sums is an $n = 2$ transform, and the whole can be expressed in matrix form as

$$\mathbf{f} = \begin{pmatrix} I_2 & D_2 & E_2 \\ I_2 & \alpha D_2 & \alpha^2 D_2^2 \\ I_2 & \beta D_2 & \beta^2 D_2^2 \end{pmatrix} \begin{pmatrix} F_2 & 0 & 0 \\ 0 & F_2 & 0 \\ 0 & 0 & F_2 \end{pmatrix} \begin{pmatrix} c_0 \\ c_3 \\ c_1 \\ c_4 \\ c_2 \\ c_5 \end{pmatrix}$$

where the $2 \times 2$ blocks are

$$I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad D_2 = \begin{pmatrix} \omega_6^0 & 0 \\ 0 & \omega_6^1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/3} \end{pmatrix}, \quad F_2 = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

The factors $\alpha$, $\beta$ are required to renormalise the matrix $D_2$ for the three sections. We have that

$$\omega_n^{j-n/3} = e^{i2\pi j/n} e^{-(i2\pi/n)(n/3)} = \omega_n^j e^{-i2\pi/3} \qquad \Longrightarrow \qquad \omega_n^j = e^{i2\pi/3} \omega_n^{j-n/3},$$

so $\alpha = e^{i2\pi/3}$ and similarly

$$\omega_n^{j-2n/3} = e^{i2\pi j/n} e^{-(i2\pi/n)(2n/3)} = \omega_n^j e^{-i4\pi/3} \qquad \Longrightarrow \qquad \omega_n^j = e^{i4\pi/3} \omega_n^{j-2n/3},$$

so $\beta = e^{i4\pi/3} = \alpha^2$. Overall, including the permutation matrix, we get

$$\mathbf{f} = \begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & \omega_6 & 0 & \omega_6^2 \\ 1 & 0 & \alpha & 0 & \alpha^2 & 0 \\ 0 & 1 & 0 & \alpha\omega_6 & 0 & \alpha^2\omega_6^2 \\ 1 & 0 & \alpha^2 & 0 & \alpha^4 & 0 \\ 0 & 1 & 0 & \alpha^2\omega_6 & 0 & \alpha^4\omega_6^2 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \mathbf{c}.$$

38. *Discrete cosine transform.* Consider the data $(\frac{\pi}{8}, 2)$, $(\frac{3\pi}{8}, 0)$, $(\frac{5\pi}{8}, -2)$, $(\frac{7\pi}{8}, 0)$.

   (a) Use the DCT to find an interpolant $p_4(x)$ for these data.

(b) Hence find the least-squares approximations of the same form with $m = 1$, $m = 2$, and $m = 3$ terms, for the same data.

**Solution:** (a) The interpolation coefficients are given by the following DCT:

$$\mathbf{a} = C_4^{-1}\mathbf{f} = \sqrt{\frac{2}{4}}\begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 1/\sqrt{2} & 1/\sqrt{2} \\ \cos(\frac{\pi}{4}\frac{1}{2}) & \cos(\frac{\pi}{4}\frac{3}{2}) & \cos(\frac{\pi}{4}\frac{5}{2}) & \cos(\frac{\pi}{4}\frac{7}{2}) \\ \cos(\frac{2\pi}{4}\frac{1}{2}) & \cos(\frac{2\pi}{4}\frac{3}{2}) & \cos(\frac{2\pi}{4}\frac{5}{2}) & \cos(\frac{2\pi}{4}\frac{7}{2}) \\ \cos(\frac{3\pi}{4}\frac{1}{2}) & \cos(\frac{3\pi}{4}\frac{3}{2}) & \cos(\frac{3\pi}{4}\frac{5}{2}) & \cos(\frac{3\pi}{4}\frac{7}{2}) \end{pmatrix}\begin{pmatrix} 2 \\ 0 \\ -2 \\ 0 \end{pmatrix}$$

$$= \frac{1}{\sqrt{2}}\begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 1/\sqrt{2} & 1/\sqrt{2} \\ \cos(\frac{\pi}{8}) & \cos(\frac{3\pi}{8}) & \cos(\frac{5\pi}{8}) & \cos(\frac{7\pi}{8}) \\ \cos(\frac{\pi}{4}) & \cos(\frac{3\pi}{4}) & \cos(\frac{5\pi}{4}) & \cos(\frac{7\pi}{4}) \\ \cos(\frac{3\pi}{8}) & \cos(\frac{9\pi}{8}) & \cos(\frac{15\pi}{8}) & \cos(\frac{21\pi}{8}) \end{pmatrix}\begin{pmatrix} 2 \\ 0 \\ -2 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ \sqrt{2}\big(\cos(\frac{\pi}{8}) - \cos(\frac{5\pi}{8})\big) \\ \sqrt{2}\big(\cos(\frac{\pi}{4}) - \cos(\frac{5\pi}{4})\big) \\ \sqrt{2}\big(\cos(\frac{3\pi}{8}) - \cos(\frac{15\pi}{8})\big) \end{pmatrix} = \begin{pmatrix} 0 \\ 1.8478 \\ 2.0 \\ -0.7654 \end{pmatrix}.$$

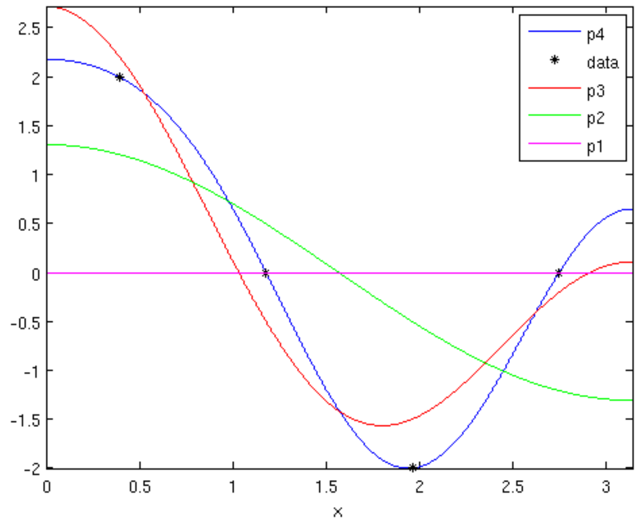
Thus the interpolating function is

$$p_4(x) = \tfrac{1}{2}a_0 + \tfrac{1}{\sqrt{2}}\sum_{k=1}^{3} a_k\cos(kx) = 1.3066\cos(x) + 1.4142\cos(2x) - 0.5412\cos(3x).$$

(b) To find the least-squares approximations we just leave off the subsequent terms of $p_4$. So

$$p_1(x) = 0, \qquad p_2(x) = 1.3066\cos(x), \qquad p_3(x) = 1.3066\cos(x) + 1.4142\cos(2x).$$

This is what the different functions look like:




39. *DCT-4.* An alternative version of the discrete cosine transform known as DCT-4 is used in sound compression. It is based on the $n \times n$ matrix $E_n$ with entries

$$(E_n)_{jk} = \sqrt{\frac{2}{n}}\cos\frac{\pi(j+\frac{1}{2})(k+\frac{1}{2})}{n}.$$

By considering the circulant matrix

$$\begin{pmatrix} 1 & -1 & & & & \\ -1 & 2 & -1 & & & \\ & -1 & 2 & -1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & -1 & 2 & -1 \\ & & & & -1 & 3 \end{pmatrix},$$

show that the matrix $E_n$ is orthogonal.

**Solution:** Let $\mathbf{v}^{(k)}$ denote column $k$ of $E_n$. We will show that these are the eigenvectors of the circulant matrix. Consider the first entry of $A\mathbf{v}^{(k)}$, where $A$ is the circulant matrix, and use the shorthand

$$\theta = \frac{\pi(k+\frac{1}{2})}{2n}.$$

We have

$$\begin{aligned}(A\mathbf{v}^{(k)})_0 = A_{0l}v_l^{(k)} = v_0^{(k)} - v_1^{(k)} &= \sqrt{\frac{2}{n}}\Big(\cos(\theta) - \cos(3\theta)\Big)\\ &= \sqrt{\frac{2}{n}}\Big(\cos(\theta) - \cos(\theta)\cos(2\theta) + \sin(\theta)\sin(2\theta)\Big)\\ &= \sqrt{\frac{2}{n}}\Big(\cos(\theta) - \cos(\theta)\cos(2\theta) + 2\sin^2(\theta)\cos(\theta)\Big)\\ &= \sqrt{\frac{2}{n}}\cos(\theta)\Big(2-2\cos(2\theta)\Big) = \Big(2-2\cos(2\theta)\Big)v_0^{(k)}.\end{aligned}$$

For $j = 1, \ldots, n-2$ we have

$$\begin{aligned}(A\mathbf{v}^{(k)})_j &= -v_{j-1}^{(k)} + 2v_j^{(k)} - v_{j+1}^{(k)}\\ &= \sqrt{\frac{2}{n}}\left(-\cos\frac{\pi(j+\frac{1}{2}-1)(k+\frac{1}{2})}{n} + 2\cos\frac{\pi(j+\frac{1}{2})(k+\frac{1}{2})}{n} - \cos\frac{\pi(j+\frac{1}{2}+1)(k+\frac{1}{2})}{n}\right)\\ &= \sqrt{\frac{2}{n}}\left(-2\cos\frac{\pi(j+\frac{1}{2})(k+\frac{1}{2})}{n}\cos\frac{\pi(k+\frac{1}{2})}{n} + 2\cos\frac{\pi(j+\frac{1}{2})(k+\frac{1}{2})}{n}\right)\\ &= \left(2-2\cos\frac{\pi(k+\frac{1}{2})}{n}\right)v_j^{(k)} = \Big(2-2\cos(2\theta)\Big)v_j^{(k)}.\end{aligned}$$

Finally, let

$$\phi = \frac{\pi(n-\frac{1}{2})(k+\frac{1}{2})}{n},$$

so

$$\begin{aligned}(A\mathbf{v}^{(k)})_{n-1} = -v_{n-2}^{(k)} + 3v_{n-1}^{(k)} &= \sqrt{\frac{2}{n}}\Big(-\cos(\phi-2\theta) + 3\cos(\phi)\Big)\\ &= \sqrt{\frac{2}{n}}\Big(-\cos(\phi)\cos(2\theta) - \sin(\phi)\sin(2\theta) + 3\cos(\phi)\Big)\\ &= \sqrt{\frac{2}{n}}\Big(-2\cos(\phi)\cos(2\theta) + 2\cos(\phi)\Big)\\ &= \Big(2-2\cos(2\theta)\Big)v_{n-1}^{(k)}.\end{aligned}$$

Therefore we see that each column $\mathbf{v}^{(k)}$ is an eigenvector of $A$ with eigenvalue

$$\lambda = 2 - 2\cos\frac{\pi(k+\frac{1}{2})}{n}.$$

Since $A$ is real and symmetric, it follows that the matrix $E_n$ is orthogonal.

40. *Two-dimensional DCT*. A very simple "image" is represented by the matrix

$$\begin{pmatrix}1 & 0\\ 1 & 0\end{pmatrix}.$$

Compute the two-dimensional DCT of this matrix, and hence the corresponding interpolation function $p_2(x,y)$ for the nodes $(\frac{\pi}{4},\frac{\pi}{4})$, $(\frac{3\pi}{4},\frac{\pi}{4})$, $(\frac{\pi}{4},\frac{3\pi}{4})$, $(\frac{3\pi}{4},\frac{3\pi}{4})$.

**Solution:** The one-dimensional DCT matrix we require is

$$C_2^{-1} = \sqrt{\frac{2}{2}} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \cos(\frac{\pi}{2}\frac{1}{2}) & \cos(\frac{\pi}{2}\frac{3}{2}) \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

To apply the two-dimensional DCT, we simply compute $C_2^{-1} X C_2$ where $X$ is the matrix of data values. Using the fact that $C_2^{-1} = C_2^{\top}$, this gives

$$Y = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}.$$

This means that the corresponding interpolation function is

$$p_2(x, y) = \frac{2}{2} \sum_{k=0}^{1} \sum_{l=0}^{1} y_{kl} \sigma_k \sigma_l \cos(lx) \cos(ky)$$

where

$$\sigma_j = \begin{cases} \frac{1}{\sqrt{2}} & j = 0, \\ 1 & j > 0. \end{cases}$$

Thus

$$p_2(x, y) = \tfrac{1}{2} y_{00} + \tfrac{1}{\sqrt{2}} y_{10} \cos(y) + \tfrac{1}{\sqrt{2}} y_{01} \cos(x) + \tfrac{1}{2} y_{11} \cos(x) \cos(y) = \tfrac{1}{2} + \tfrac{1}{\sqrt{2}} \cos(x).$$

You can see that this satisfies the required interpolation conditions.