Conway's Army

Durham University

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Department of Mathematical Sciences


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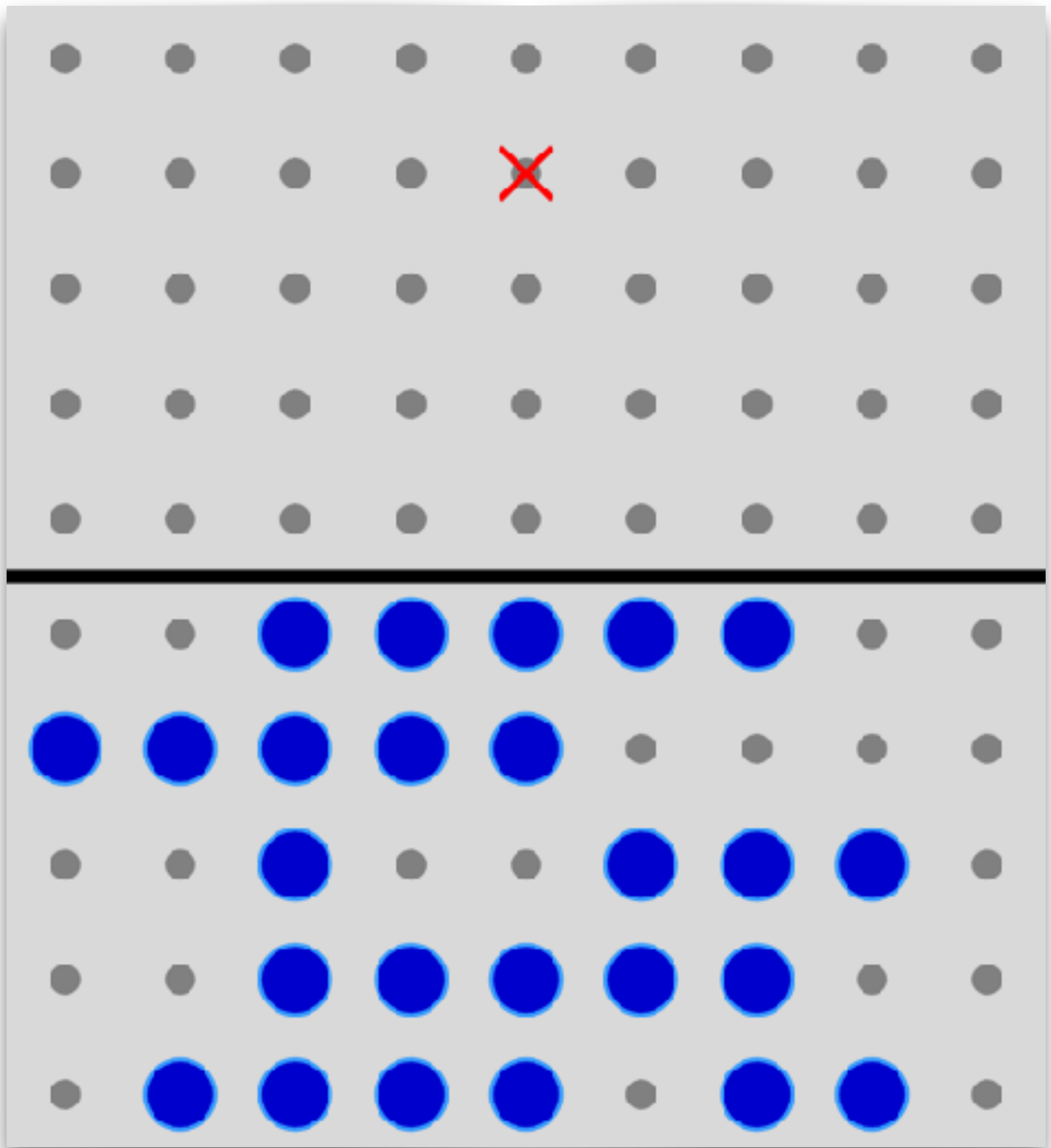
Conway's Army



Aim

Reach target x above the line.

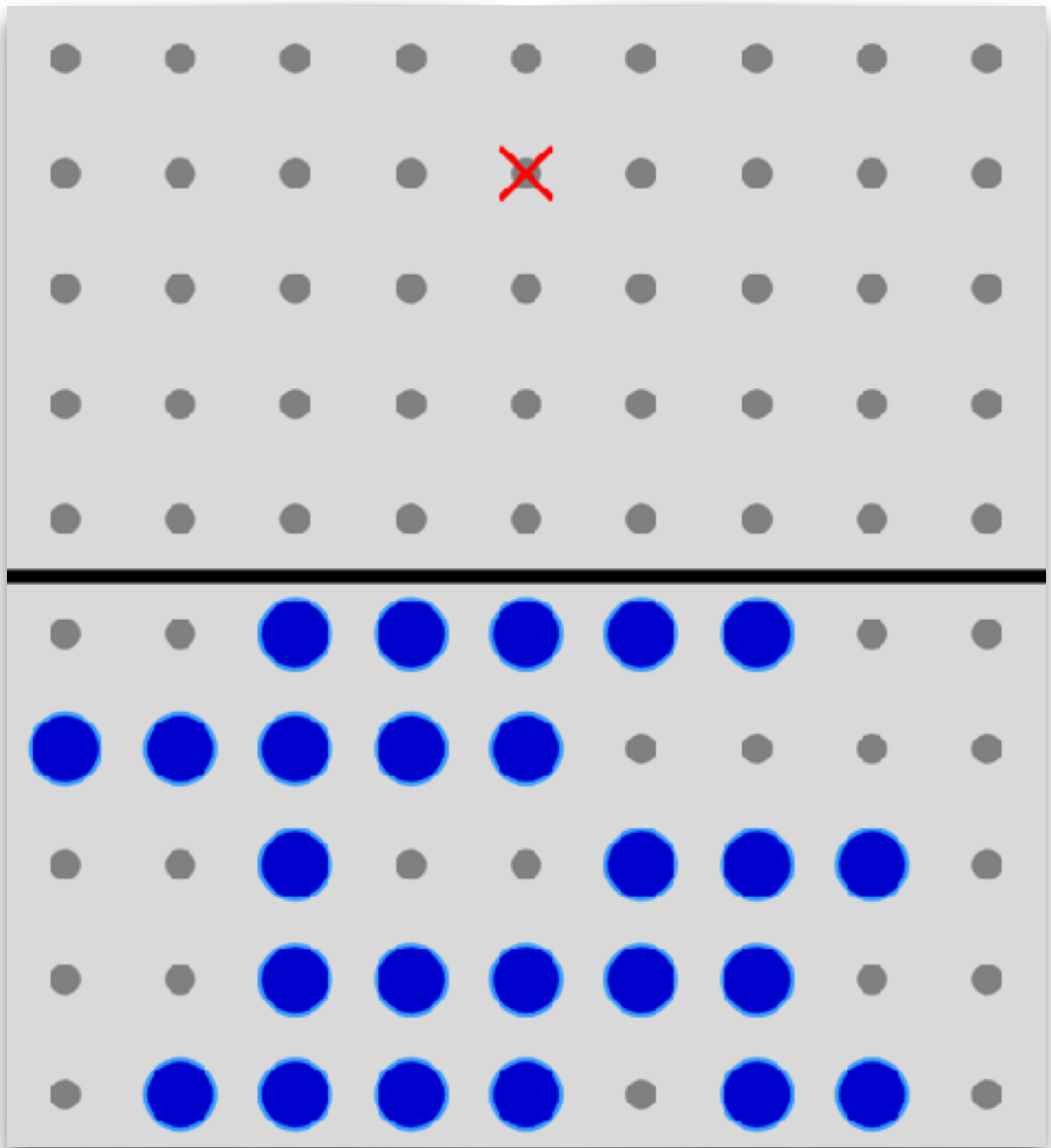
Start

Place as many pegs as you like, anywhere below the line.

Legal moves

Jump left/right or up/down into empty hole, with capture.

Conway's Army



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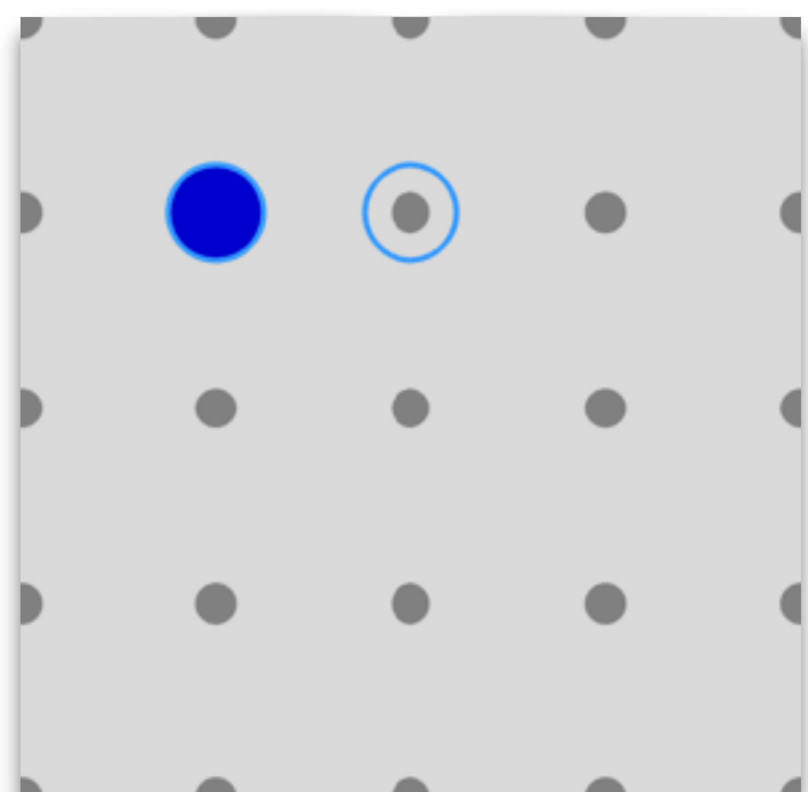
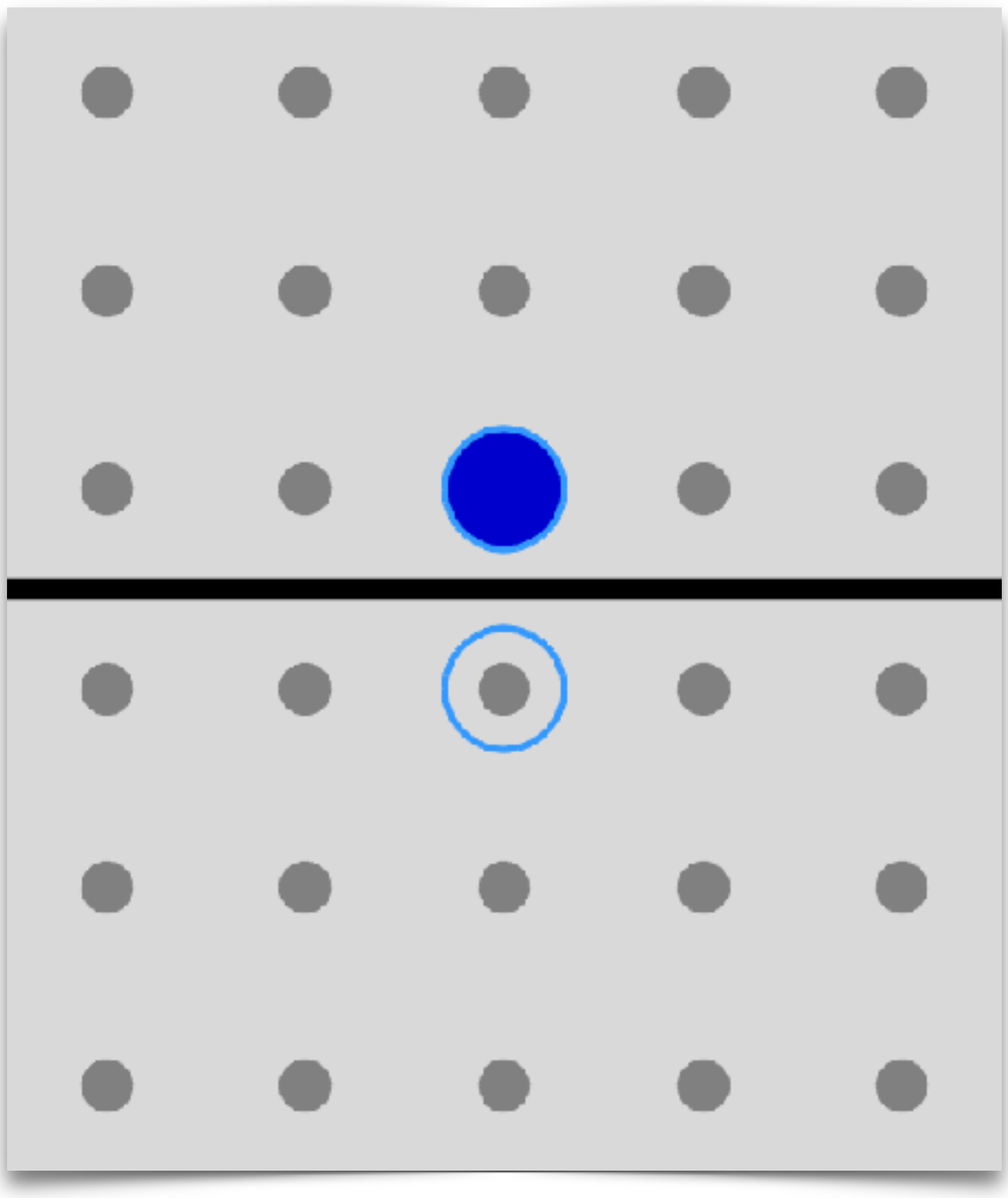
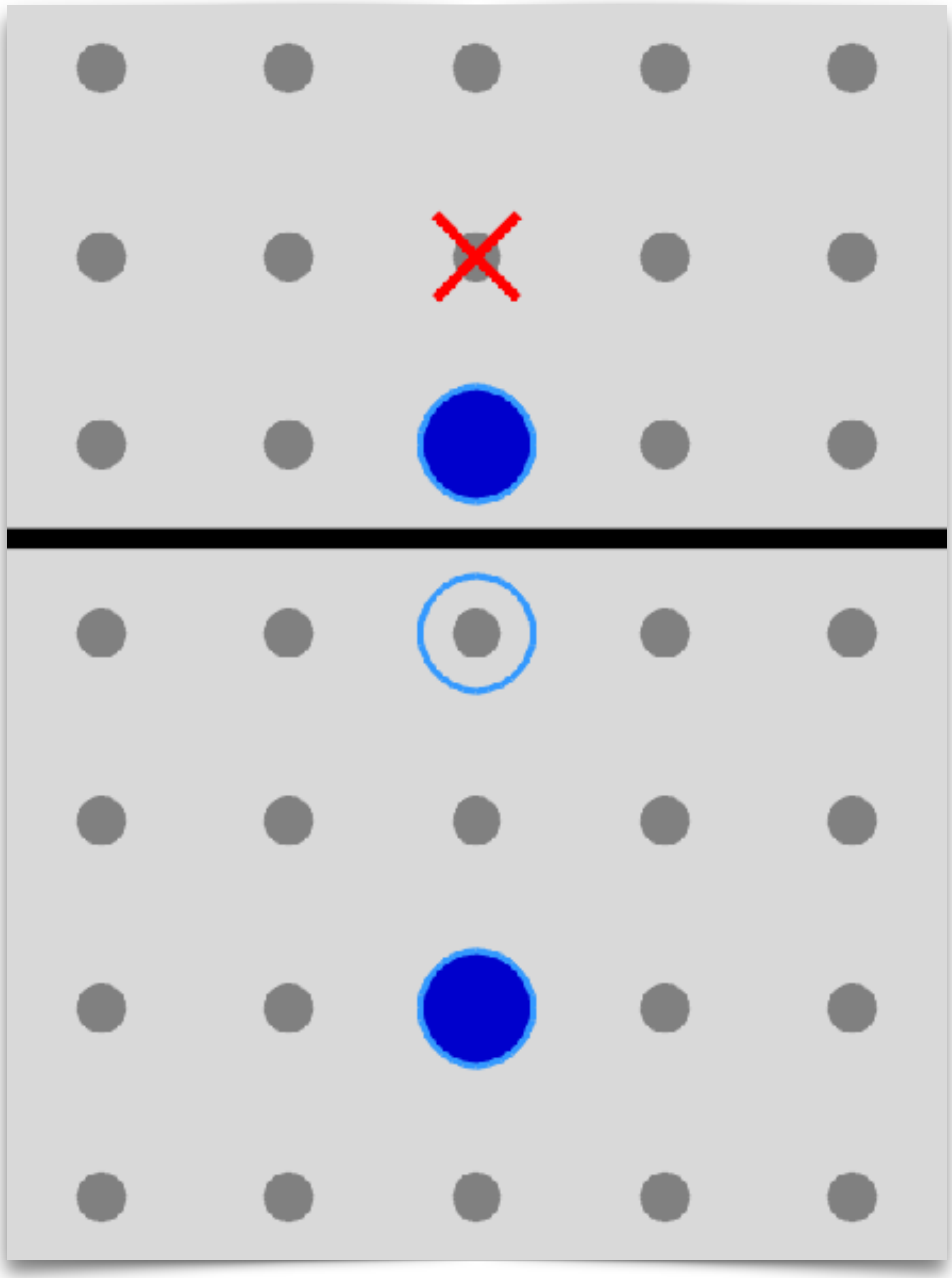


Level 1



Level 2



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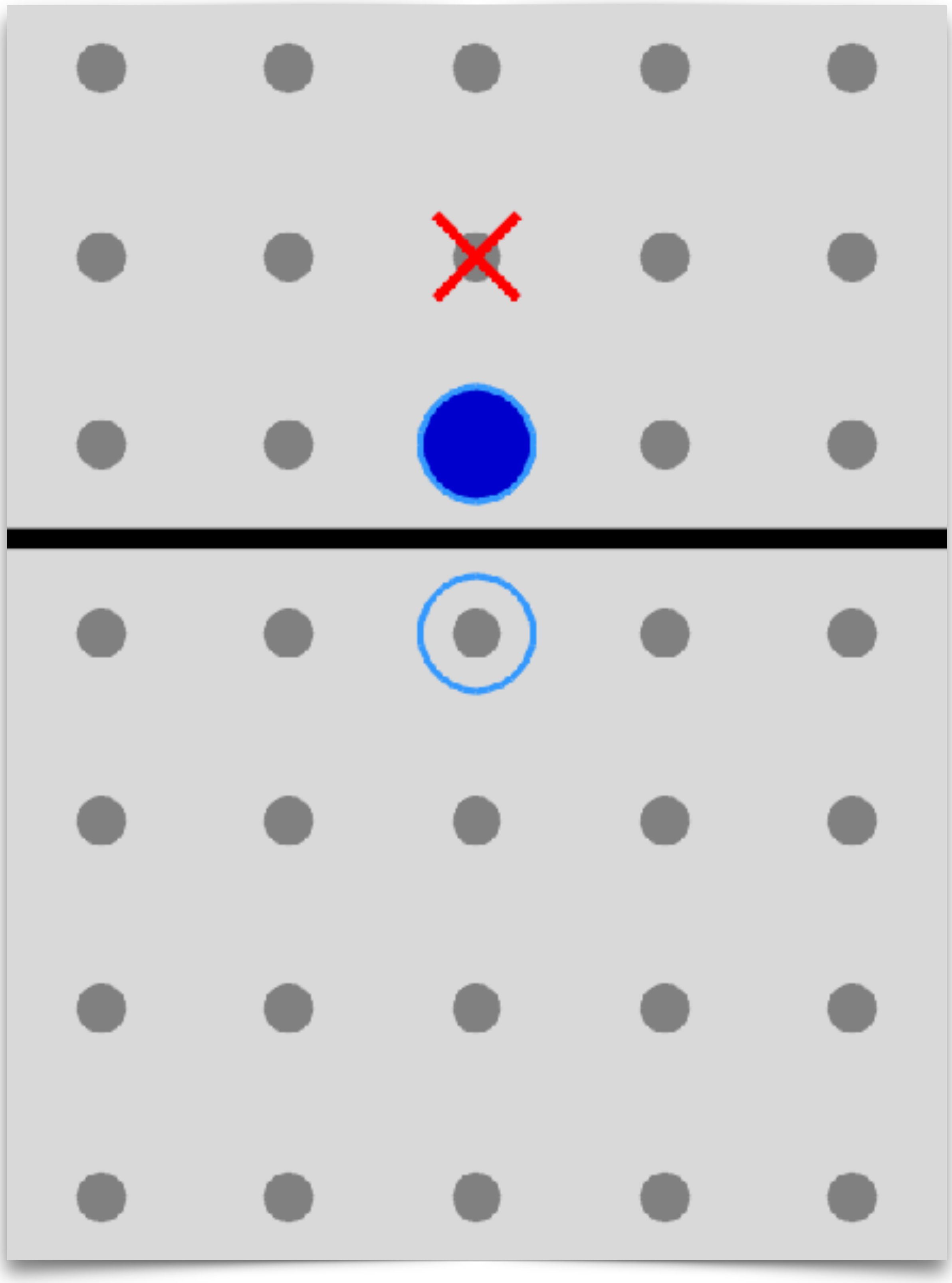

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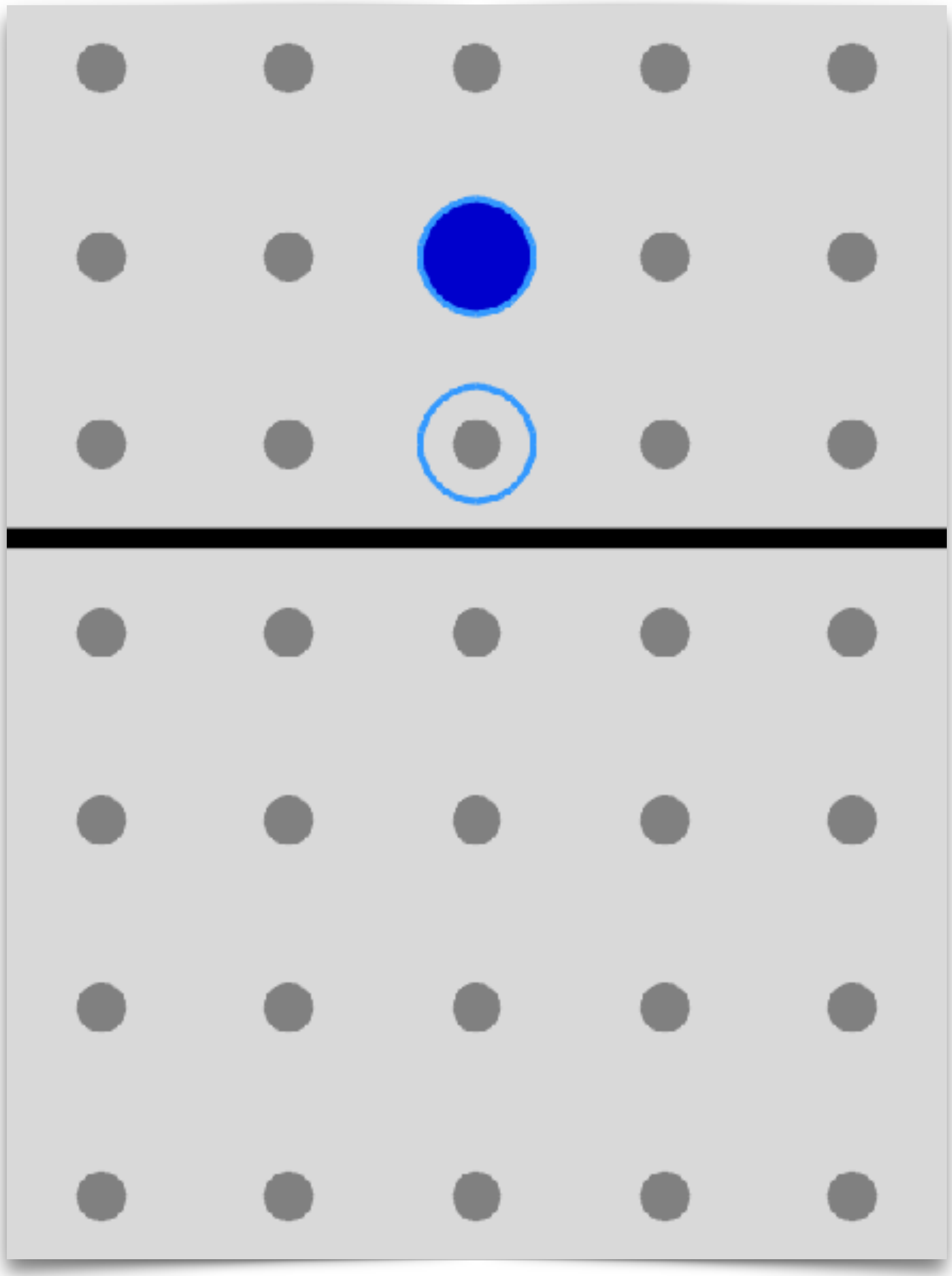
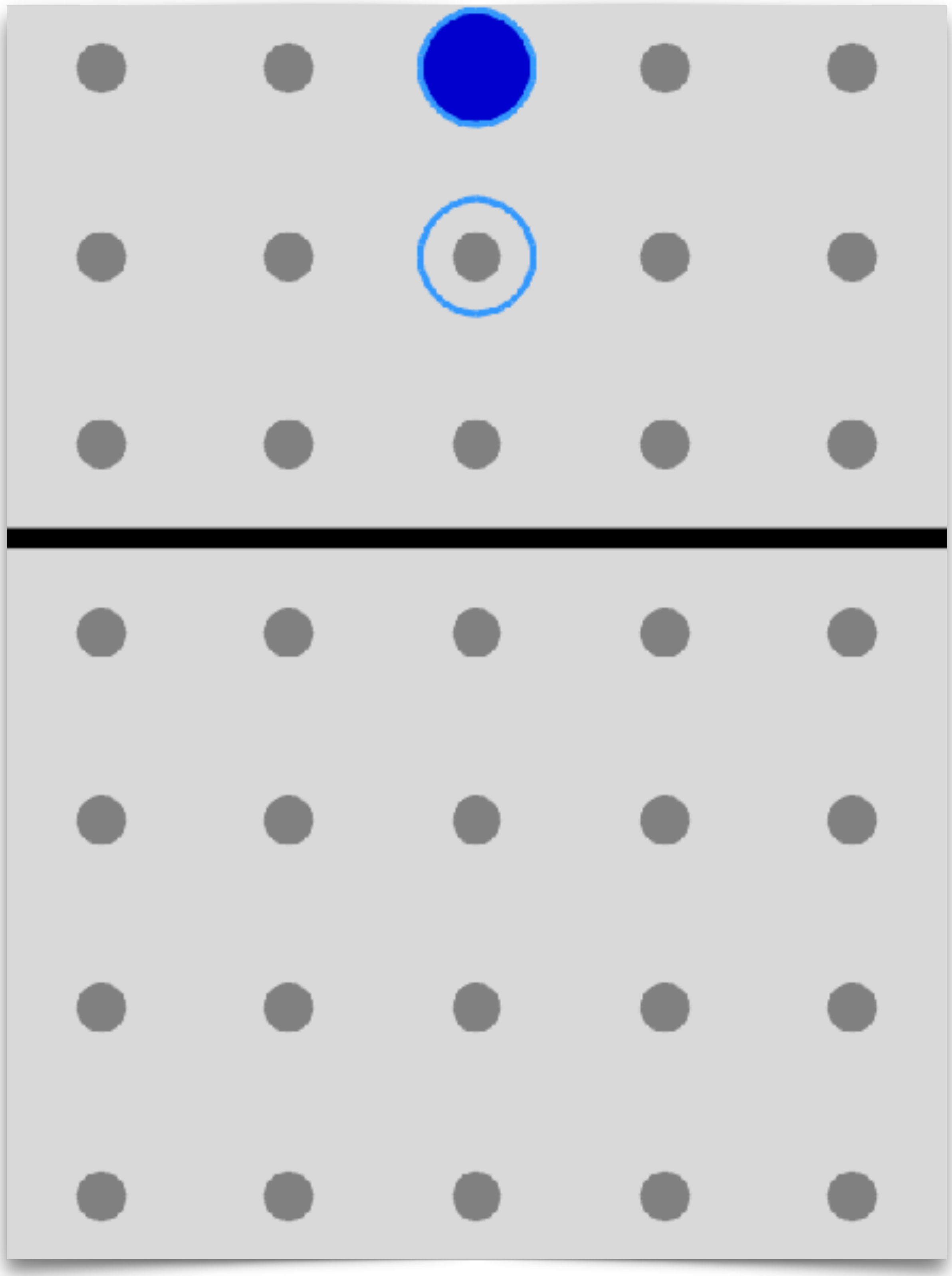


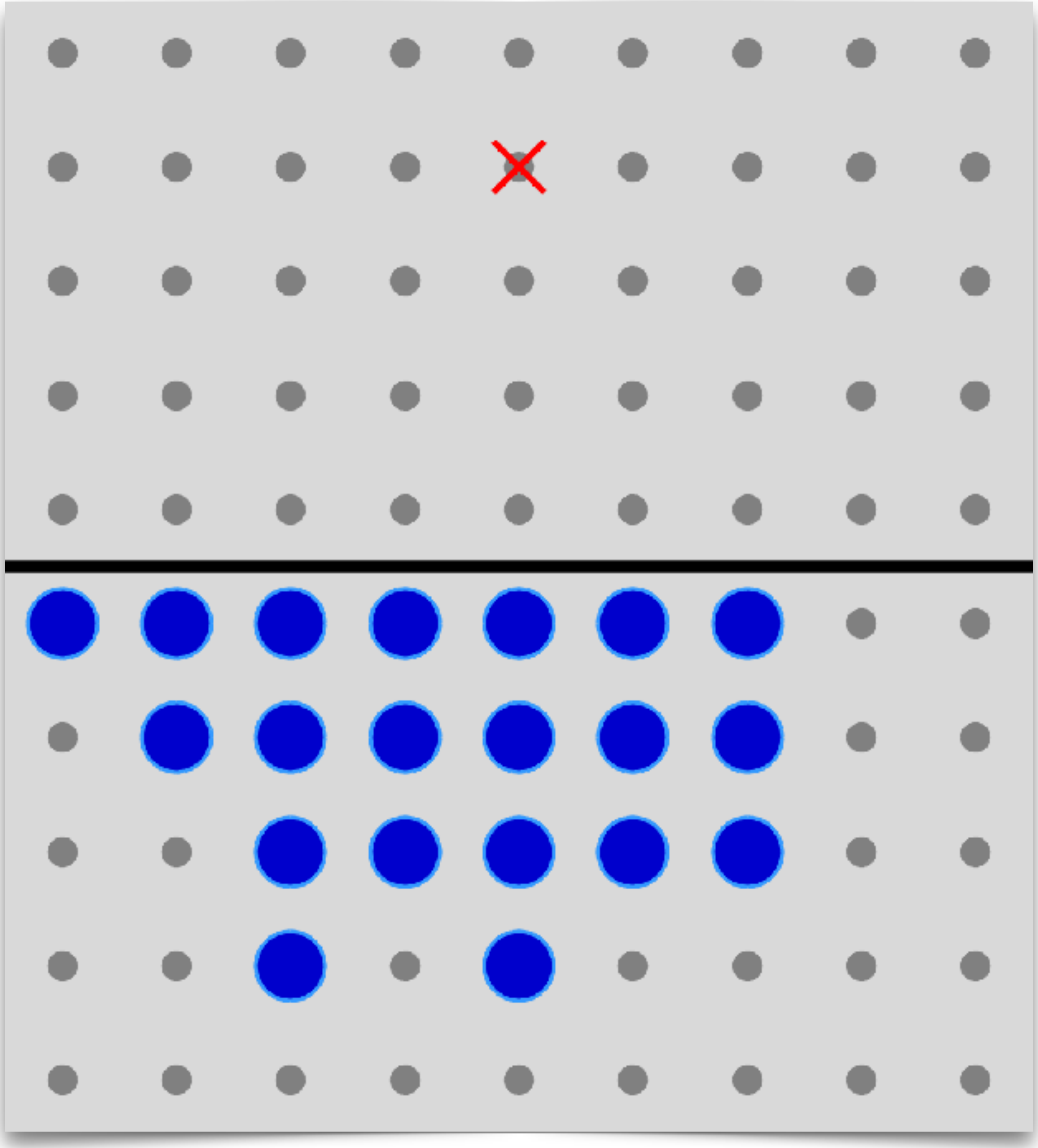
Level 3



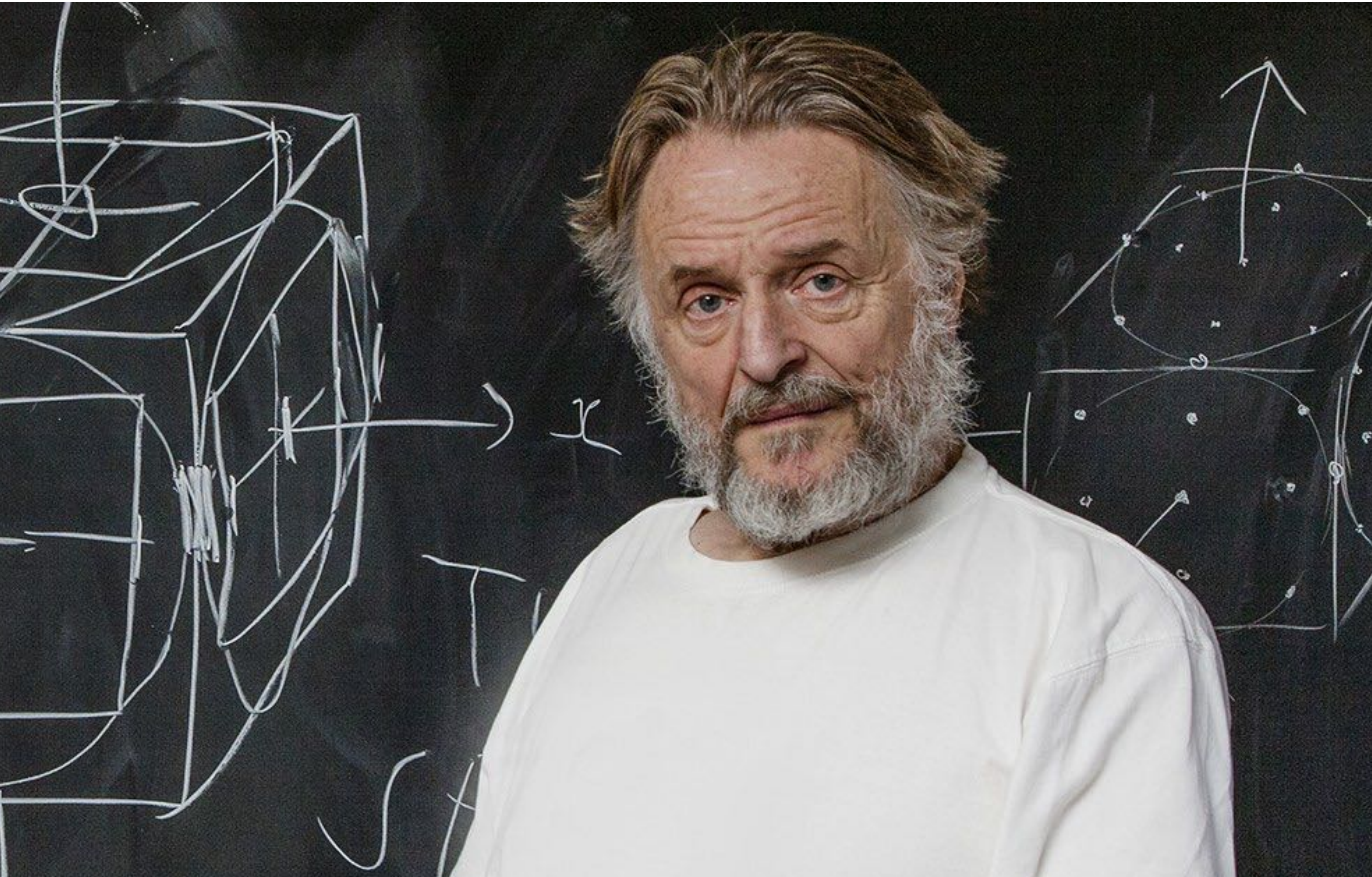
Minimum number of pegs required

Level 1	2 pegs
Level 2	4 pegs
Level 3	8 pegs
Level 4	**20 pegs!**

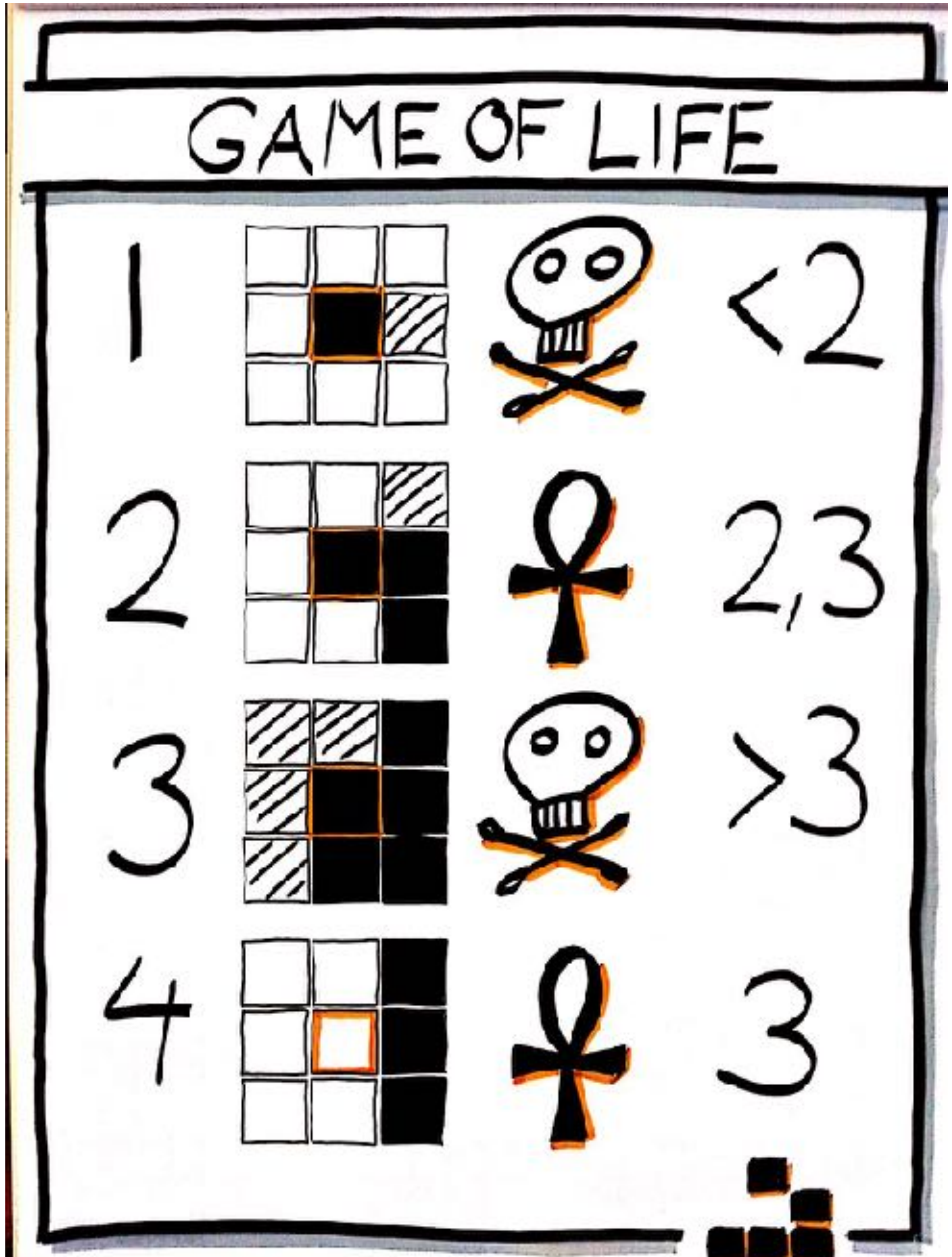
Level 4



John Horton Conway
(1937–**2020**)



Conway's Game of Life



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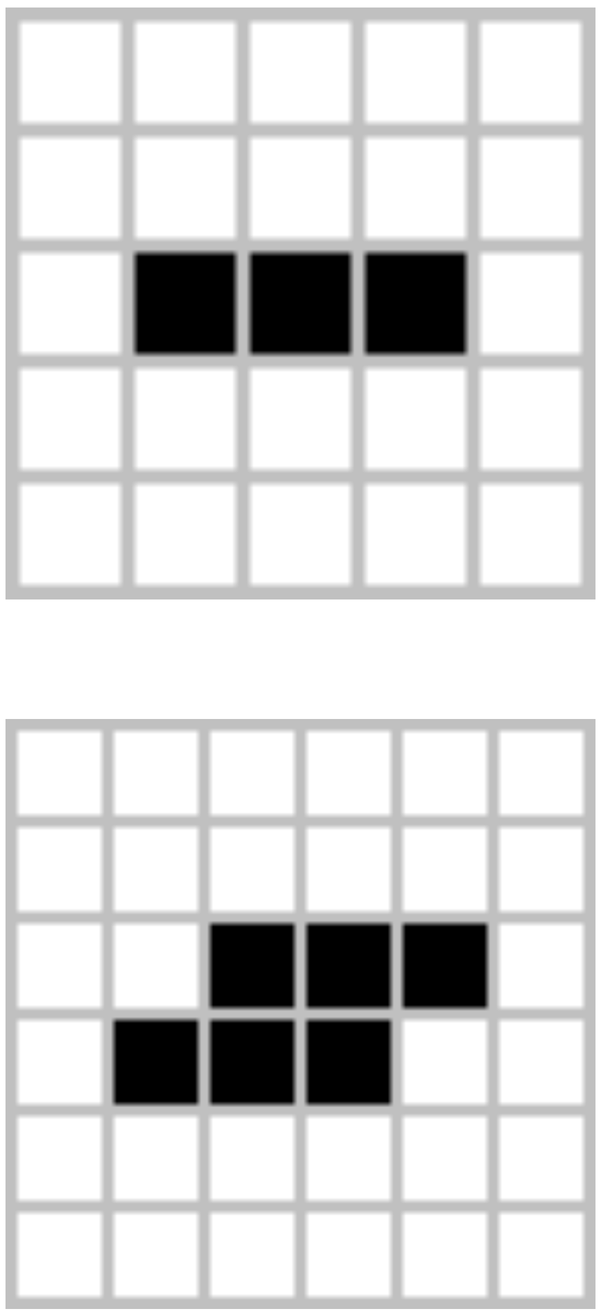
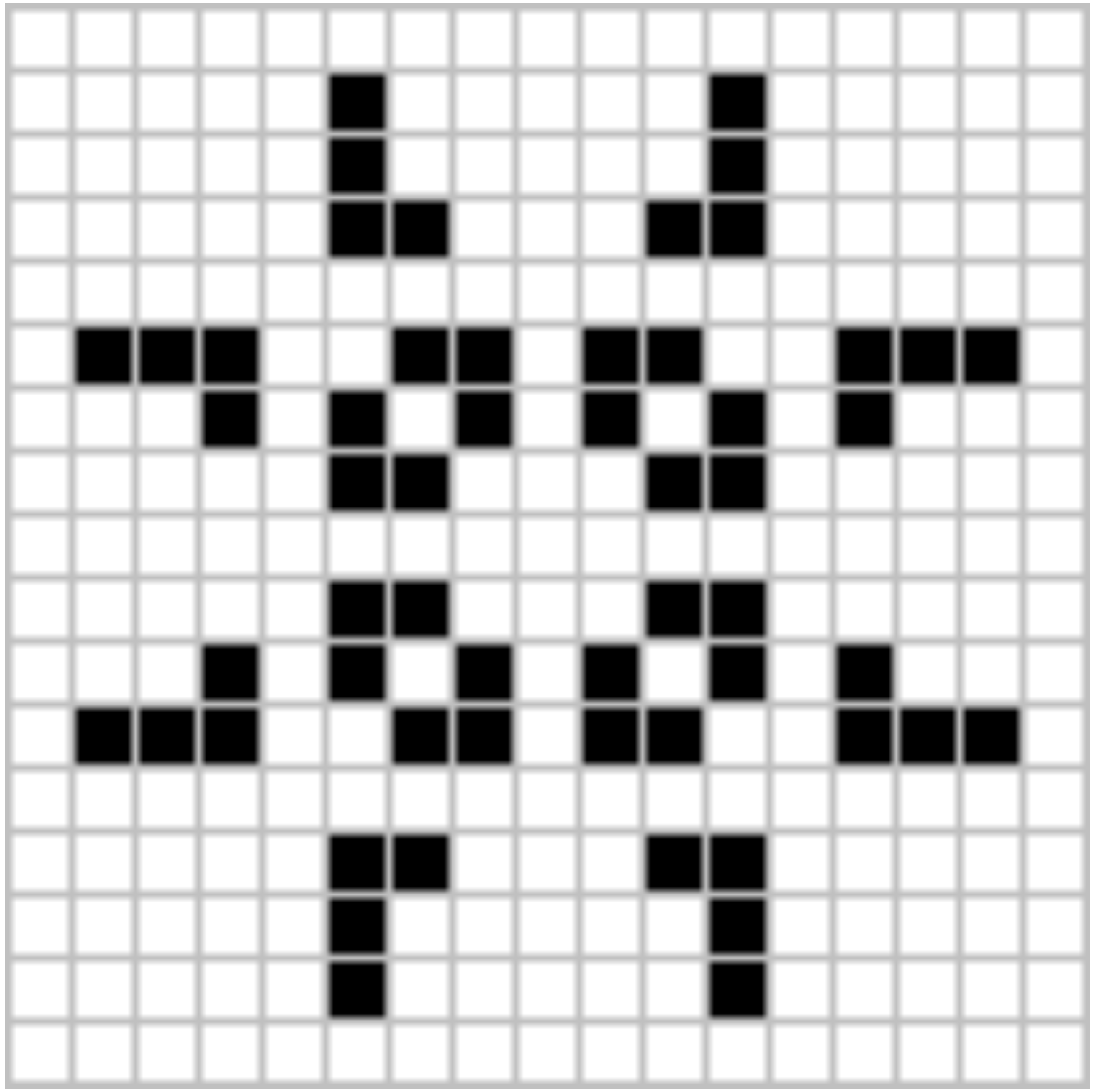
Conway's Game of Life





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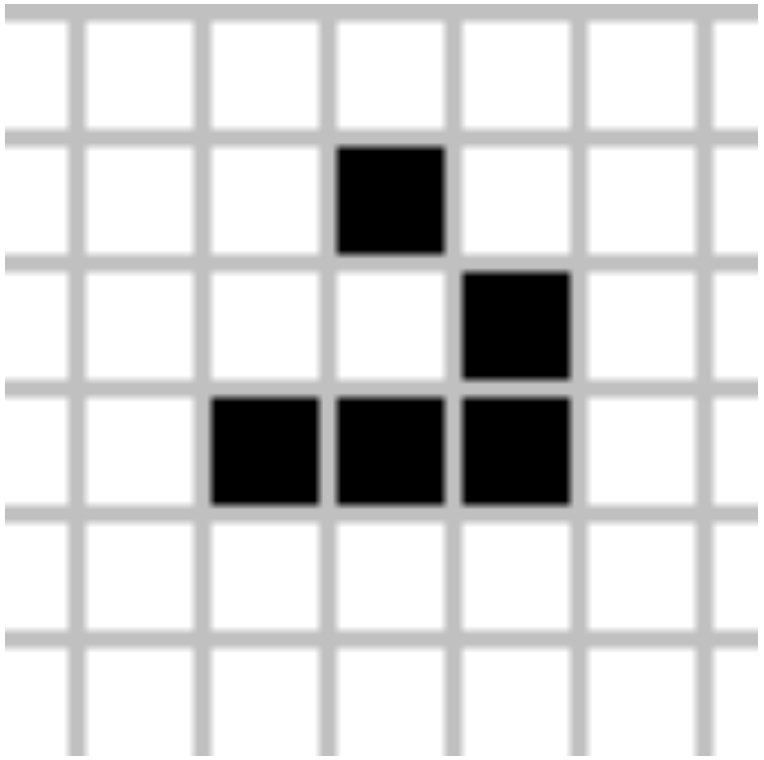

Conway’s Game of Life

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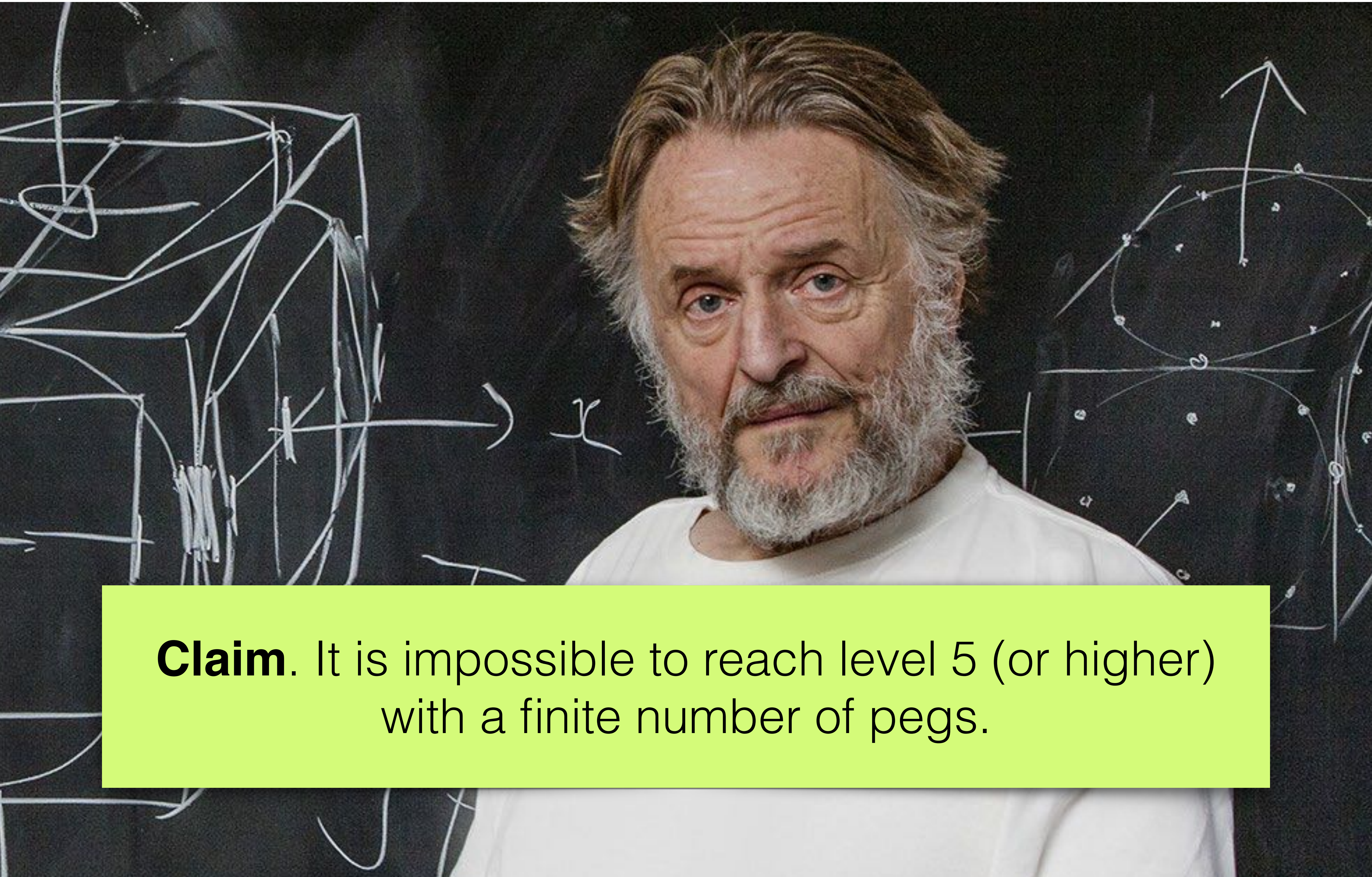
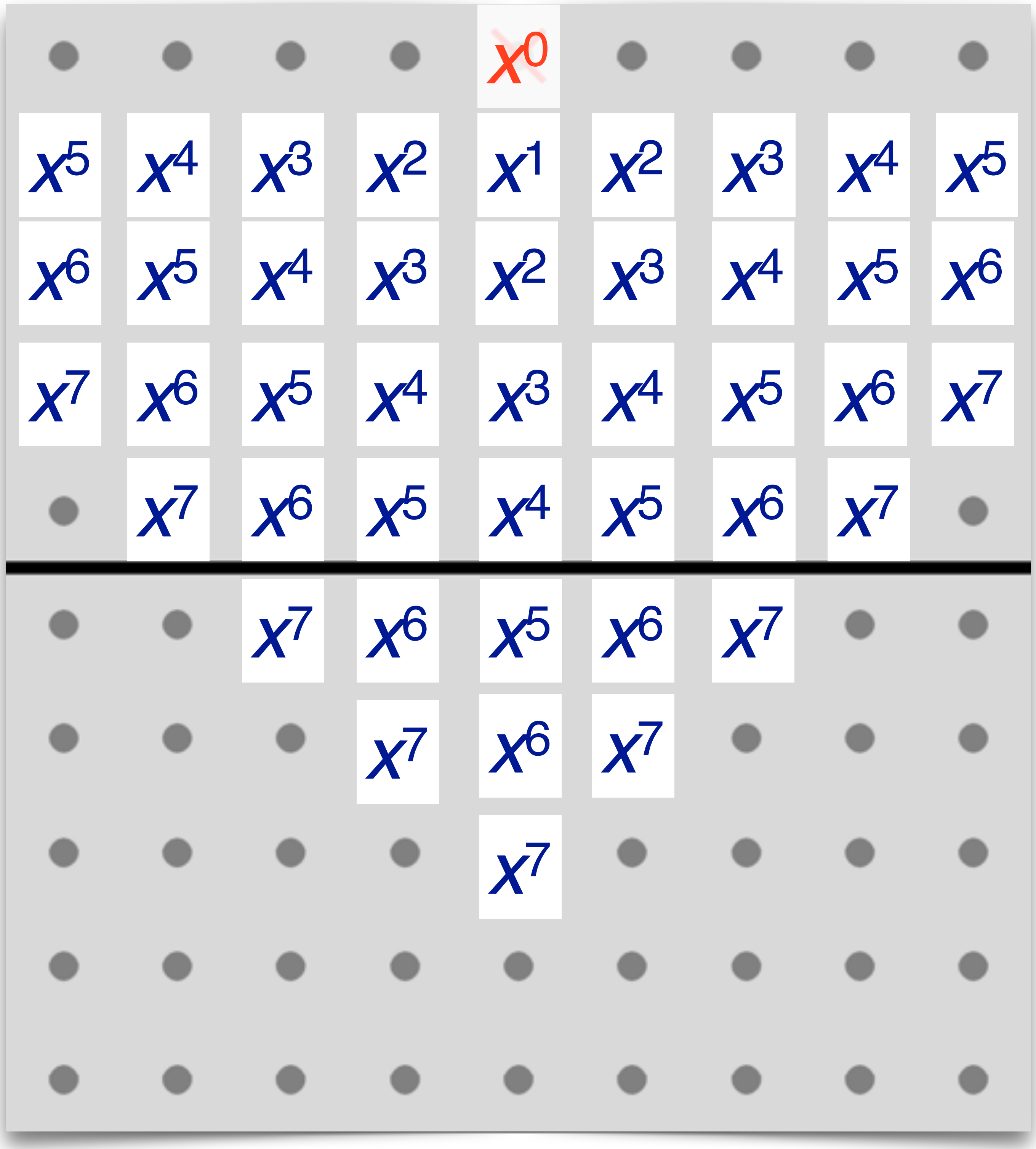


Claim. It is impossible to reach level 5 (or higher) with a finite number of pegs.

Proof?

1. Create a polynomial to encode the board position.

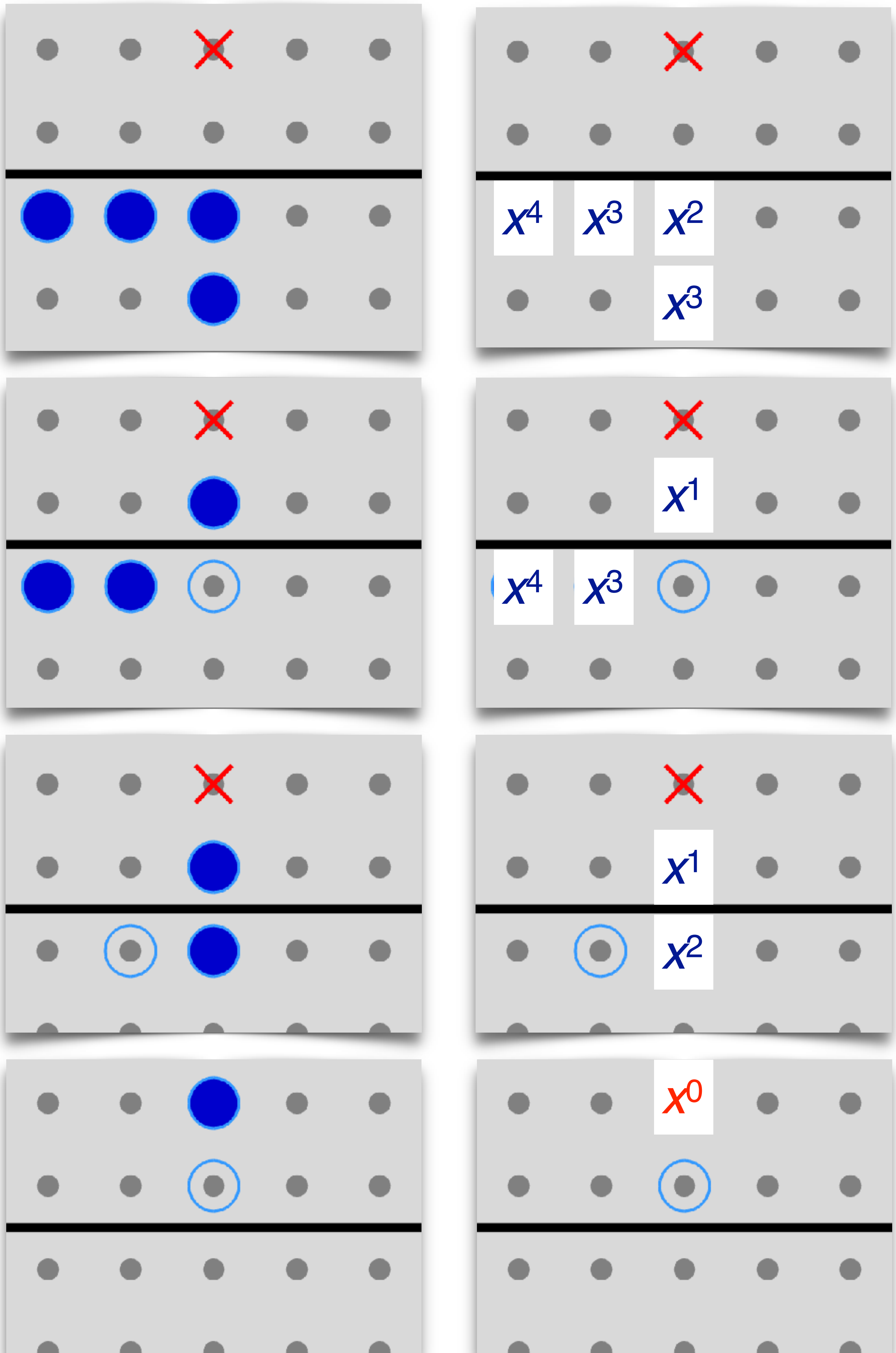


Label target hole 1 (= x^0).

Label all other holes by x^d, where d is "taxicab distance" from target.

To encode a position, add the terms for the corresponding pegs.

e.g.



$$p(x) = x^2 + 2x^3 + x^4$$

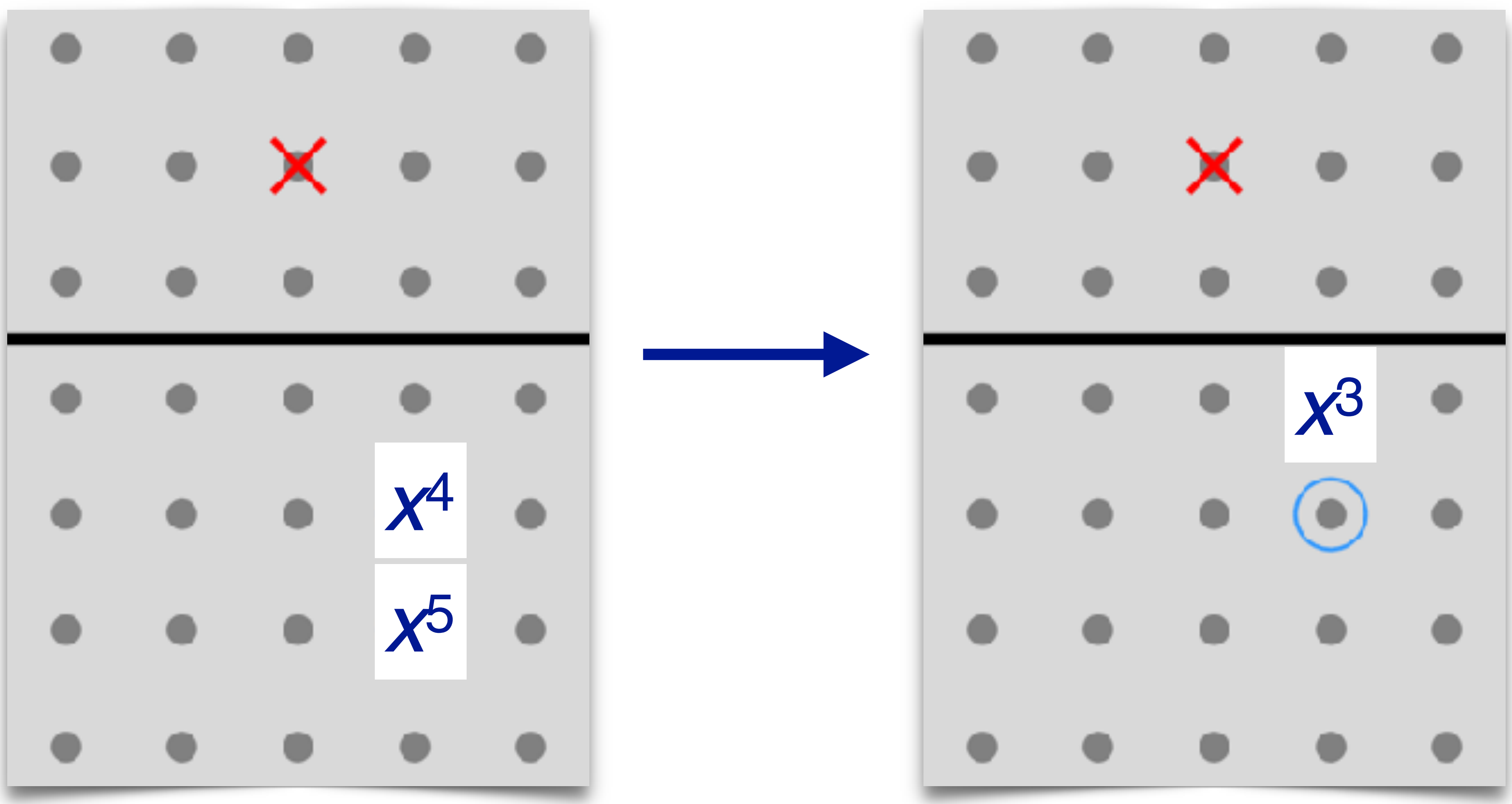
$$p(x) = x + x^3 + x^4$$

$$p(x) = x + x^2$$

$$p(x) = 1$$

2. Analyse the effect of the possible moves.

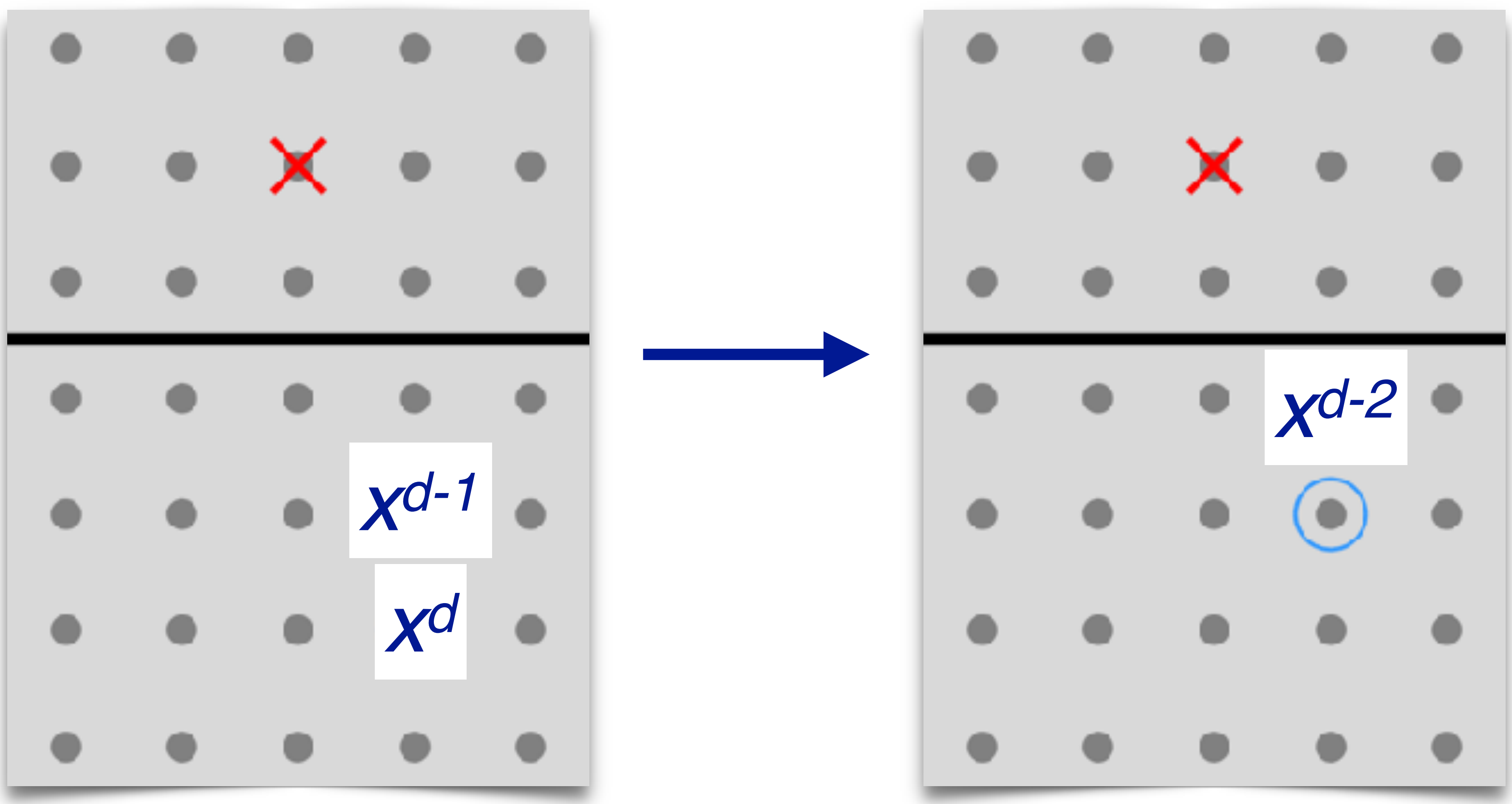
a) “Positive jump” - towards target.



$$p_{\mathrm{new}}(x) - p_{\mathrm{old}}(x) = x^3 - x^4 - x^5$$

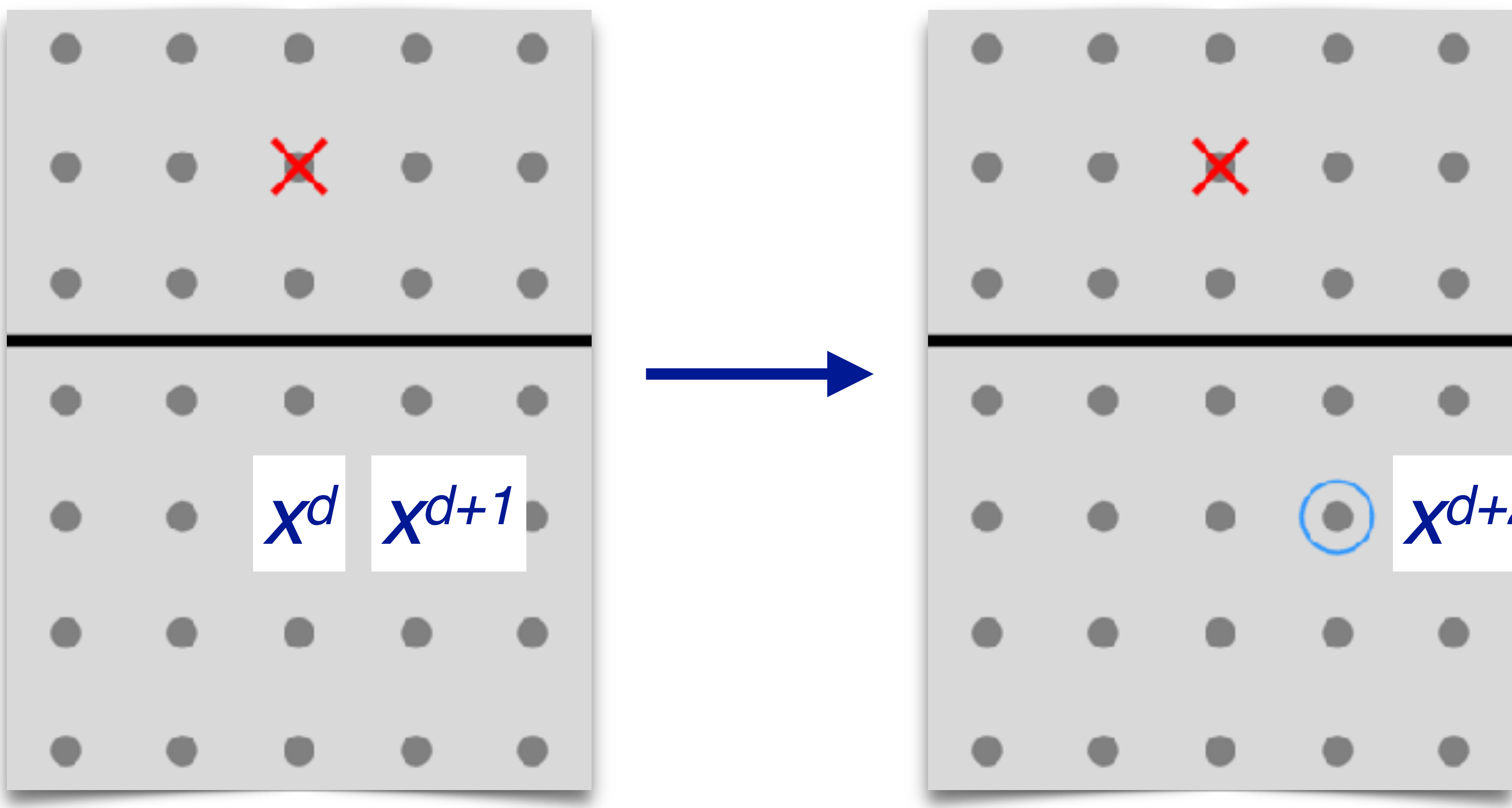
2. Analyse the effect of the possible moves.

a) "Positive jump" - towards target.



$$p_{\text{new}}(x) - p_{\text{old}}(x) = x^{d-2} - x^{d-1} - x^{d}$$
$$= x^{d-2}(1 - x - x^2)$$

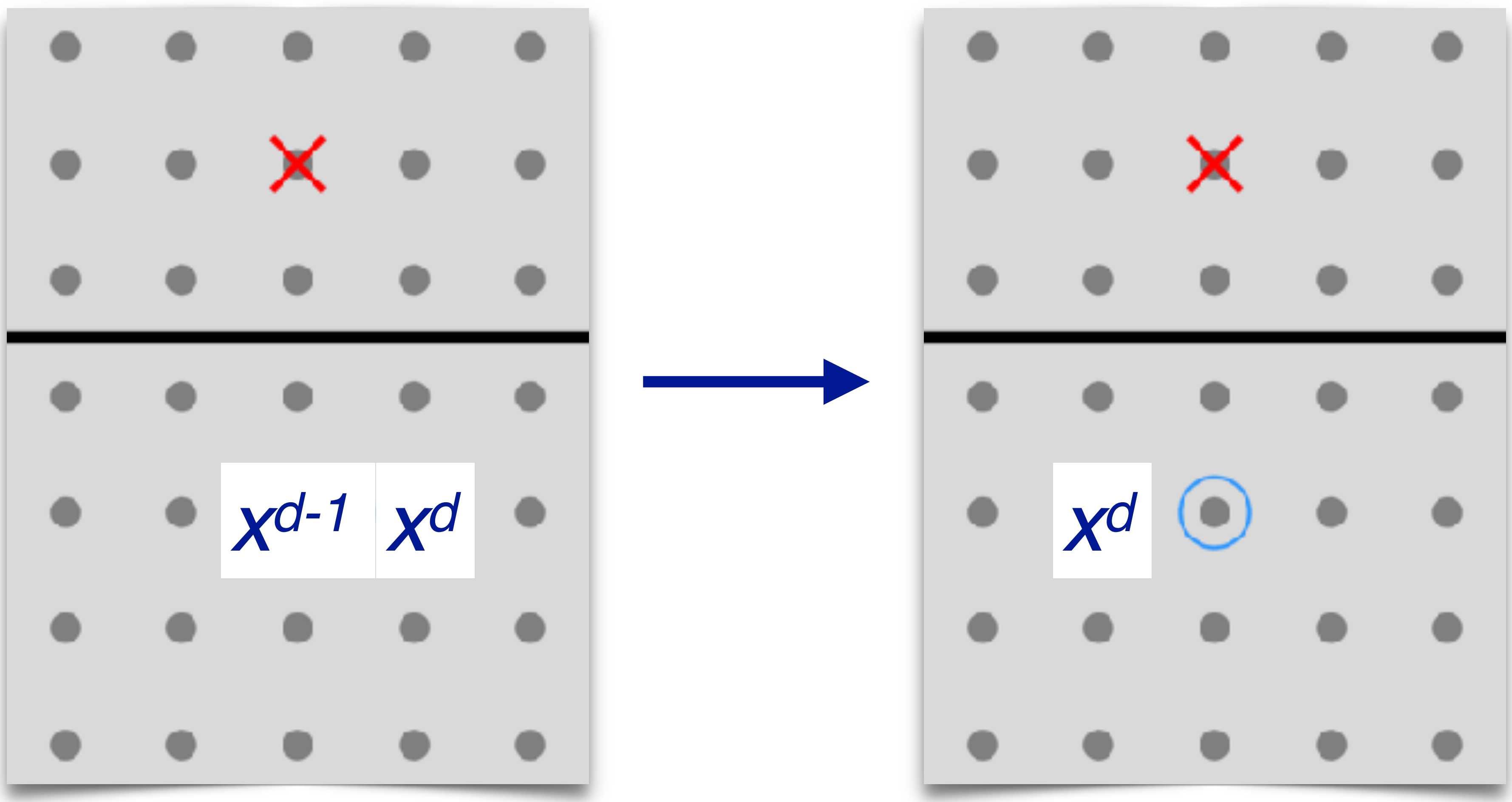
b) “Negative jump” - away from target.



$$p_{\text{new}}(x) - p_{\text{old}}(x) = x^{d+2} - x^{d+1} - x^d$$

$$= x^d(x^2 - x - 1)$$

c) “Neutral jump” - remain same distance from target.



$$p_{\text{new}}(x) - p_{\text{old}}(x) = -x^{d-1}$$

Positive jump:

$$p_{\mathrm{new}}(x) - p_{\mathrm{old}}(x) = x^{d-2}\left(1 - x - x^2\right)$$

Negative jump:

$$p_{\mathrm{new}}(x) - p_{\mathrm{old}}(x) = x^{d}\left(x^2 - x - 1\right)$$

Neutral jump:

$$p_{\mathrm{new}}(x) - p_{\mathrm{old}}(x) = -x^{d-1}$$

3. Choose a helpful value of *x*.

Choose $x = x_* > 0$ so total $p(x_*)$ never increases.

$$x_*^2 + x_* - 1 = 0$$

Positive jump:

$$p_{\text{new}}(x_*) - p_{\text{old}}(x_*) = -x_*^{d-2}\big(x_*^2 + x_* - 1\big) = 0$$

Negative jump:

$$\begin{aligned} p_{\text{new}}(x_*) - p_{\text{old}}(x_*) &= x_*^d\big(x_*^2 - x_* - 1\big) \\ &= x_*^d\big(x_*^2 + x_* - 1 - 2x_*\big) \\ &= -2x_*^{d+1} < 0 \end{aligned}$$

Neutral jump:

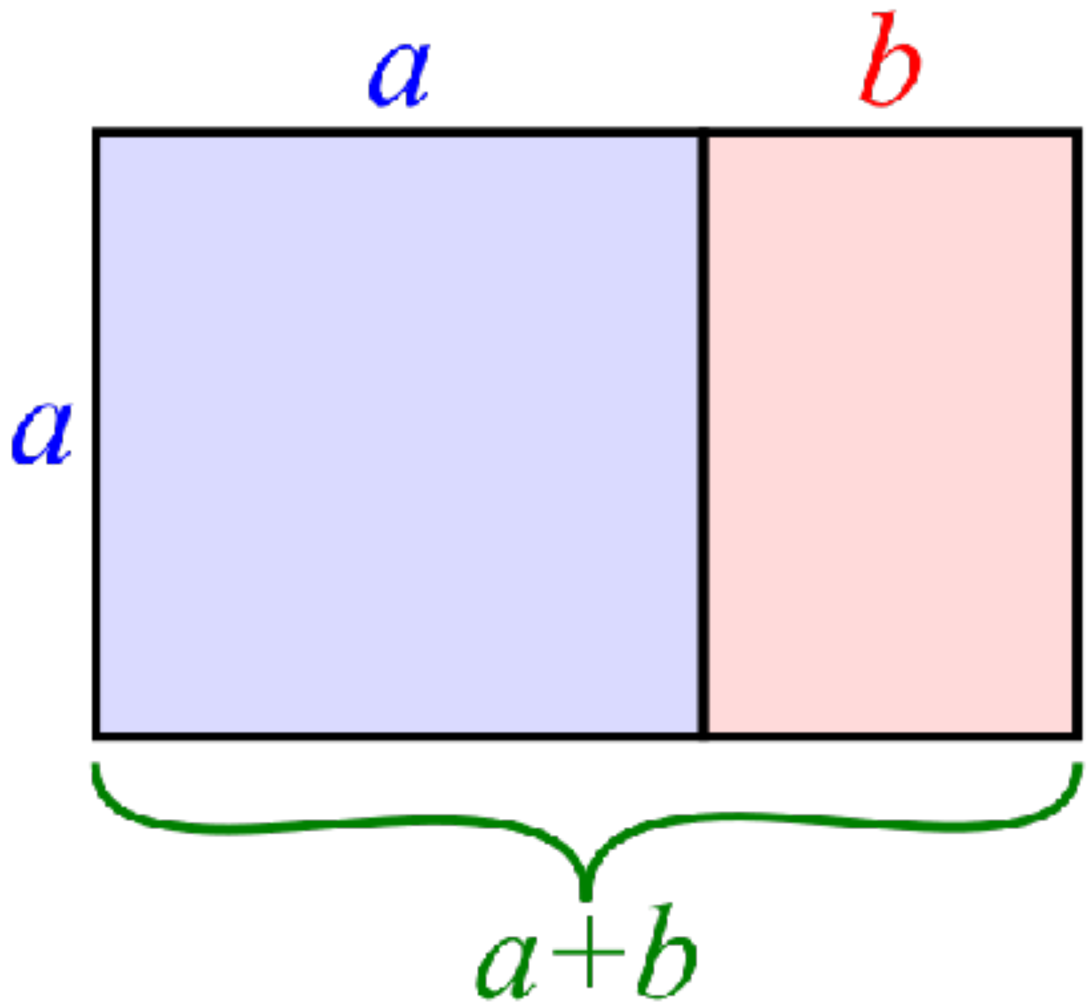
$$p_{\text{new}}(x_*) - p_{\text{old}}(x_*) = -x_*^{d-1} < 0$$

$$x_*^2 + x_* - 1 = 0 \quad \Longrightarrow \quad x = \frac{-1 \pm \sqrt{1+4}}{2}$$

Positive root:

$$x = \frac{\sqrt{5}-1}{2} = 0.618\ldots$$

Aside: the golden ratio



$$\frac{b}{a} = \frac{a}{a+b}$$

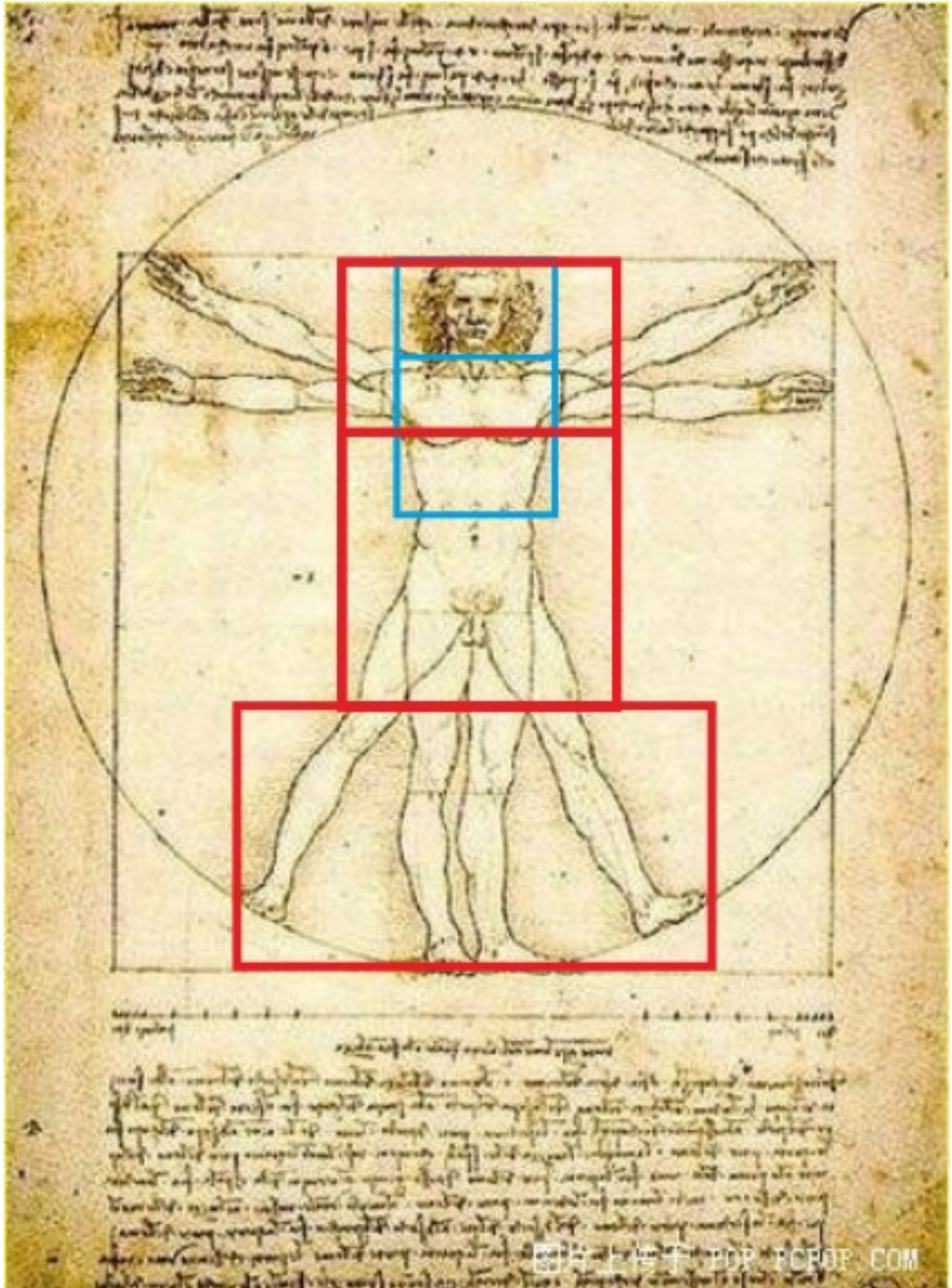
$$b(a+b) = a^2$$

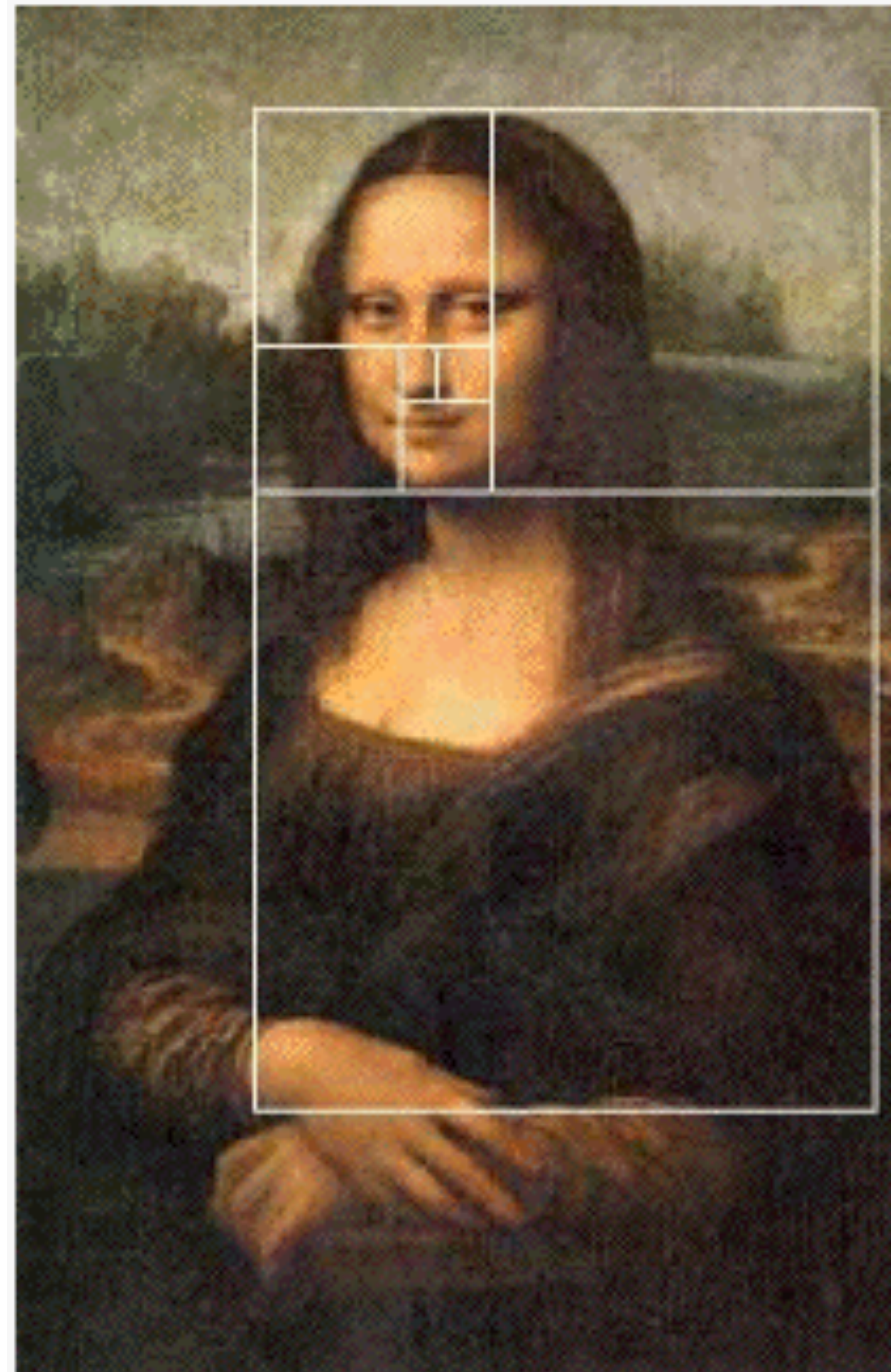
$$ab + b^2 = a^2$$

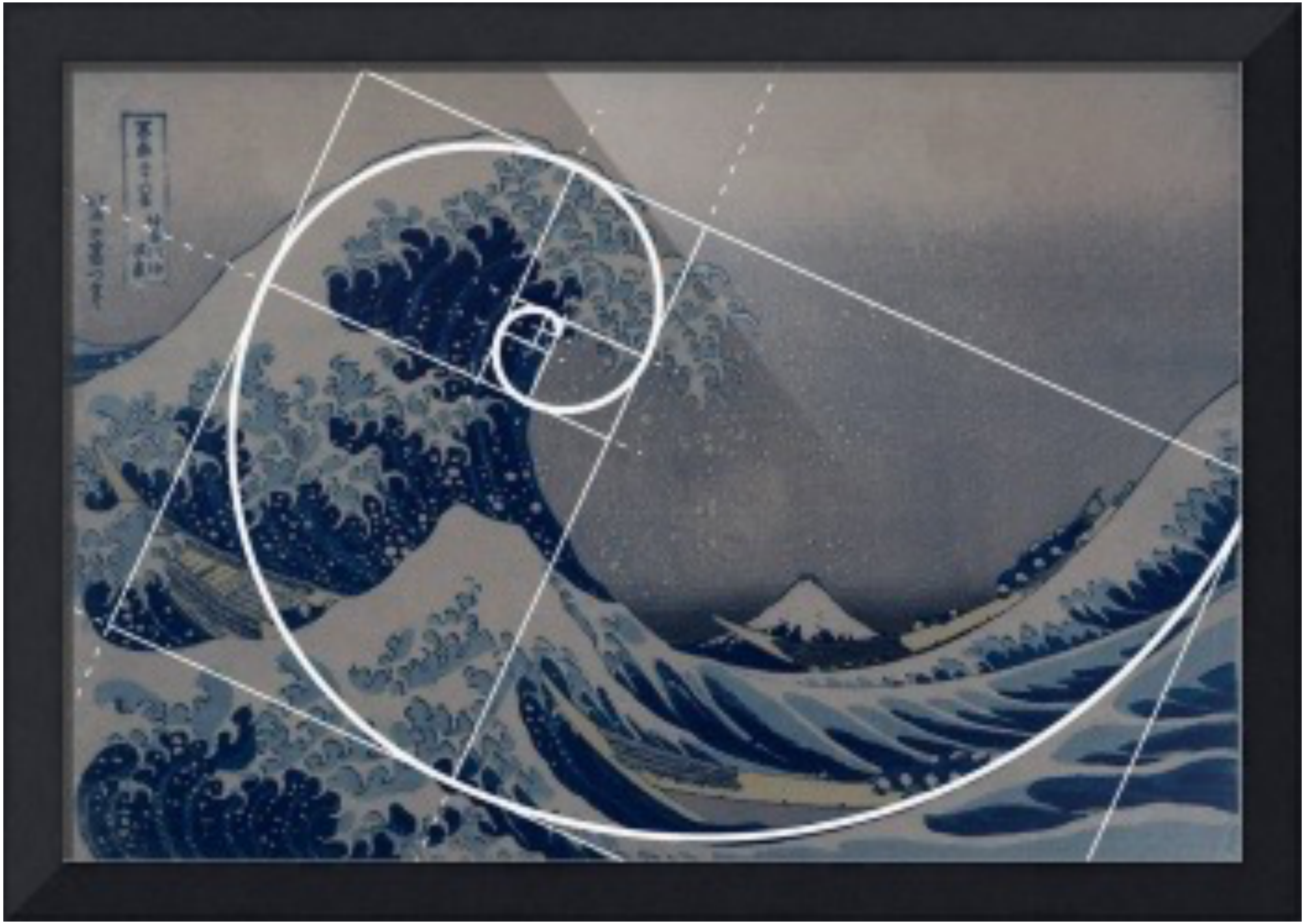
$$\frac{b}{a} + \left(\frac{b}{a}\right)^2 = 1$$

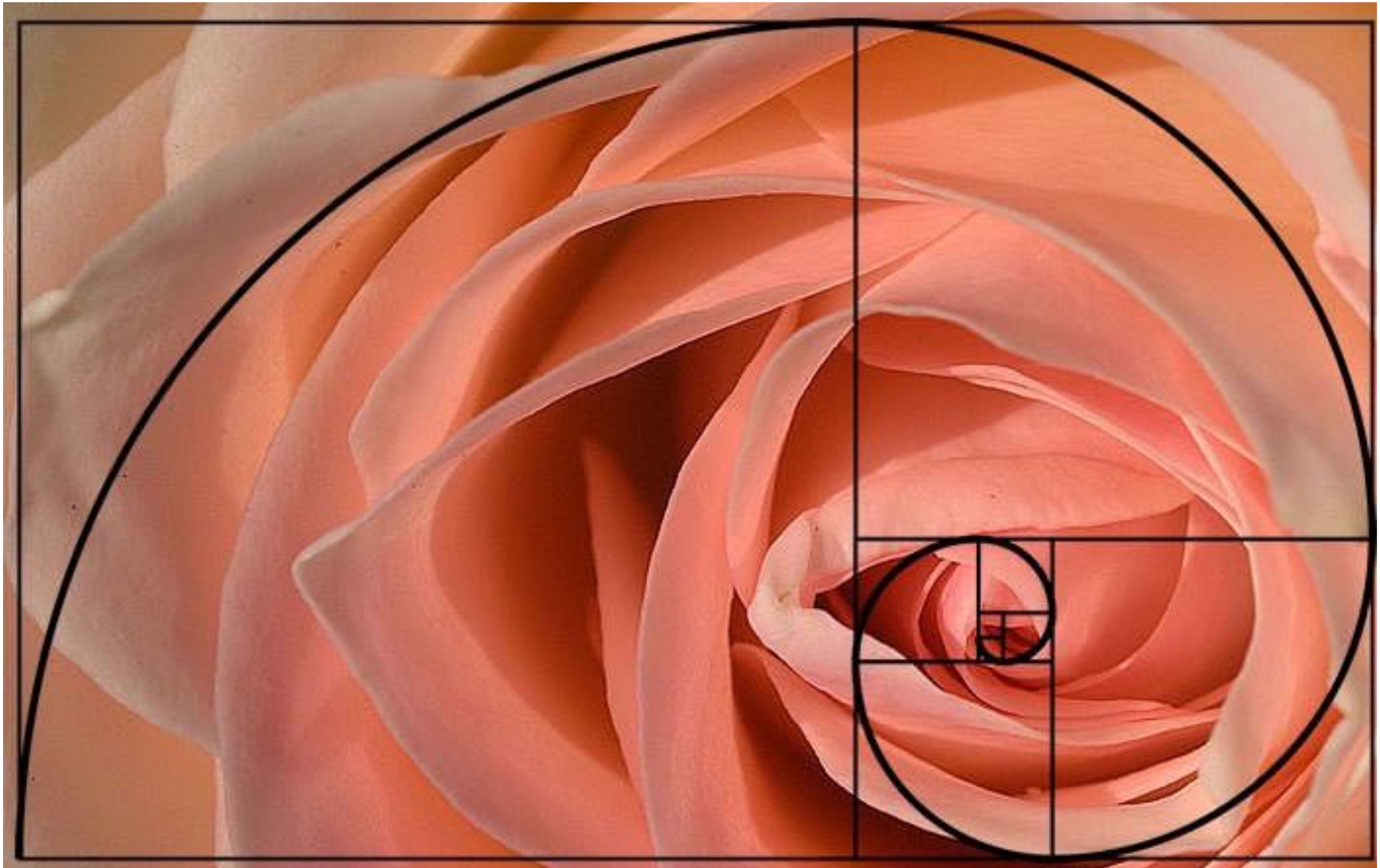
$$\left(\frac{b}{a}\right)^2 + \frac{b}{a} - 1 = 0$$

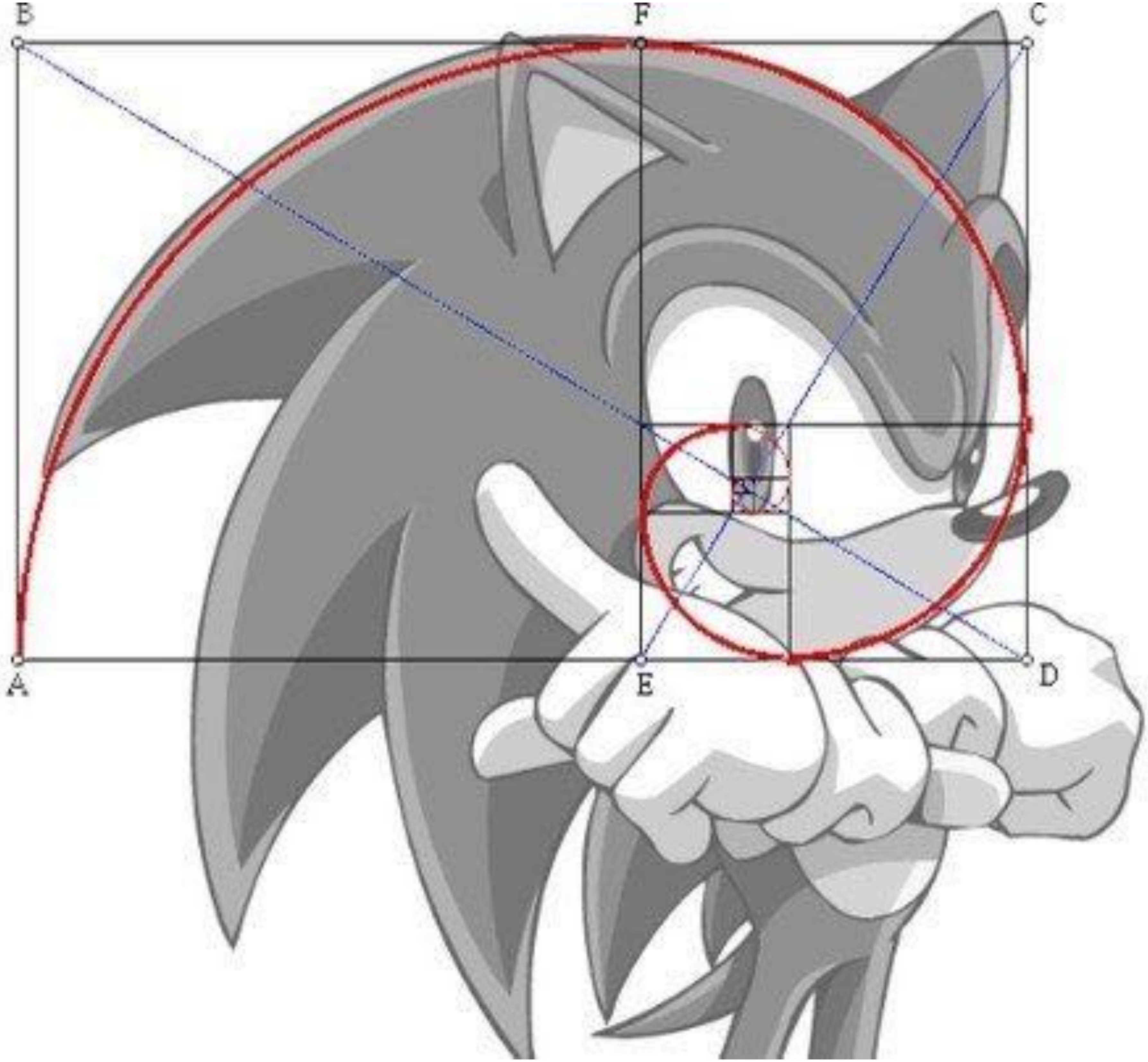
$$x_*^2 + x_* - 1 = 0$$







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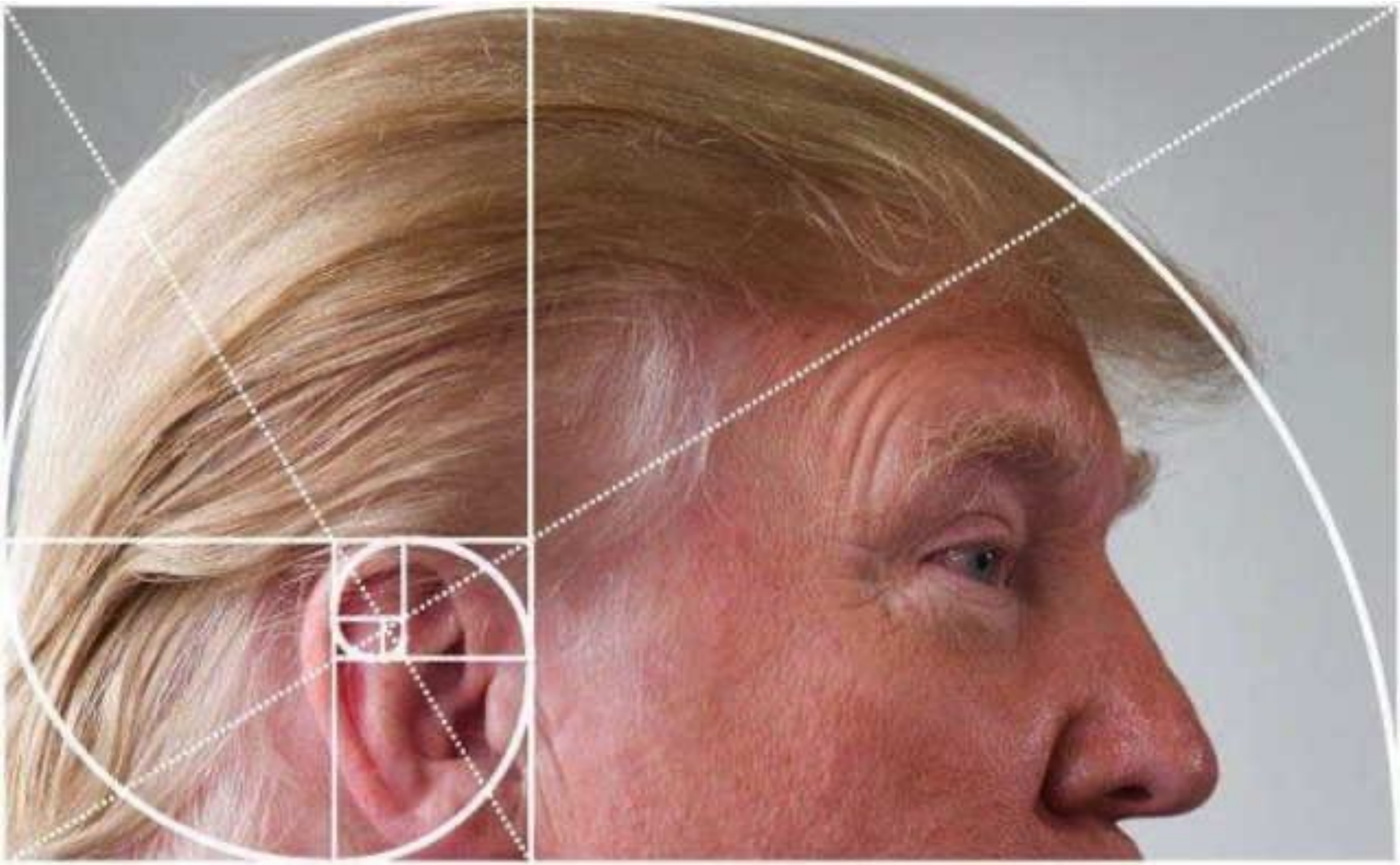
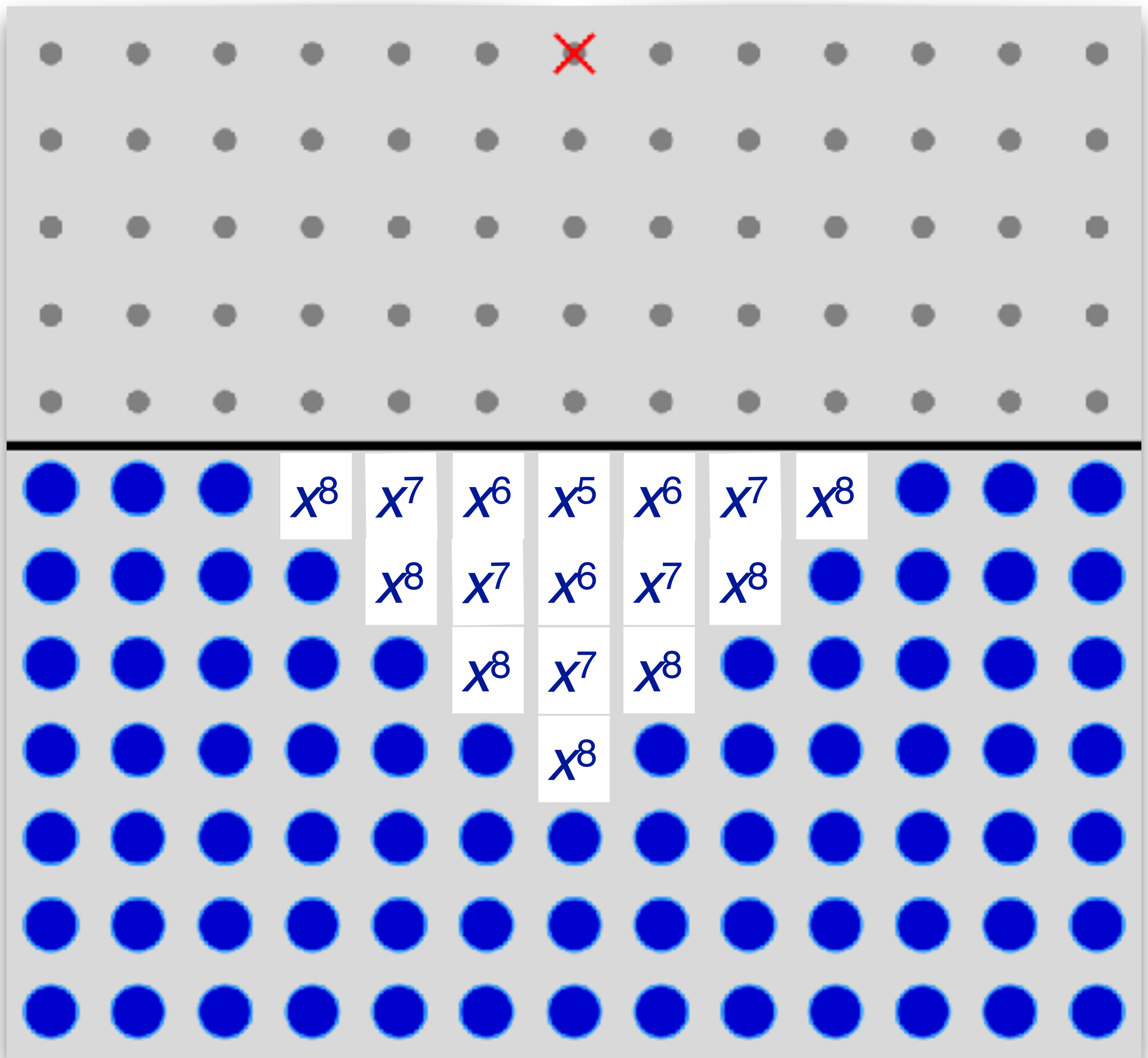


4. Calculate the value of an infinite starting position.



$$p(x) = x^5 + 3x^6 + 5x^7 + 7x^8 + \ldots$$

$$p(x) = x^5 + 3x^6 + 5x^7 + 7x^8 + \ldots$$

$$= x^5 \left(1 + 3x + 5x^2 + 7x^3 + \ldots\right)$$

$$S$$

“arithmetic-geometric series”

$$S = 1 + 3x + 5x^2 + 7x^3 + \ldots$$

$$xS = x + 3x^2 + 5x^3 + 7x^4 + \ldots$$

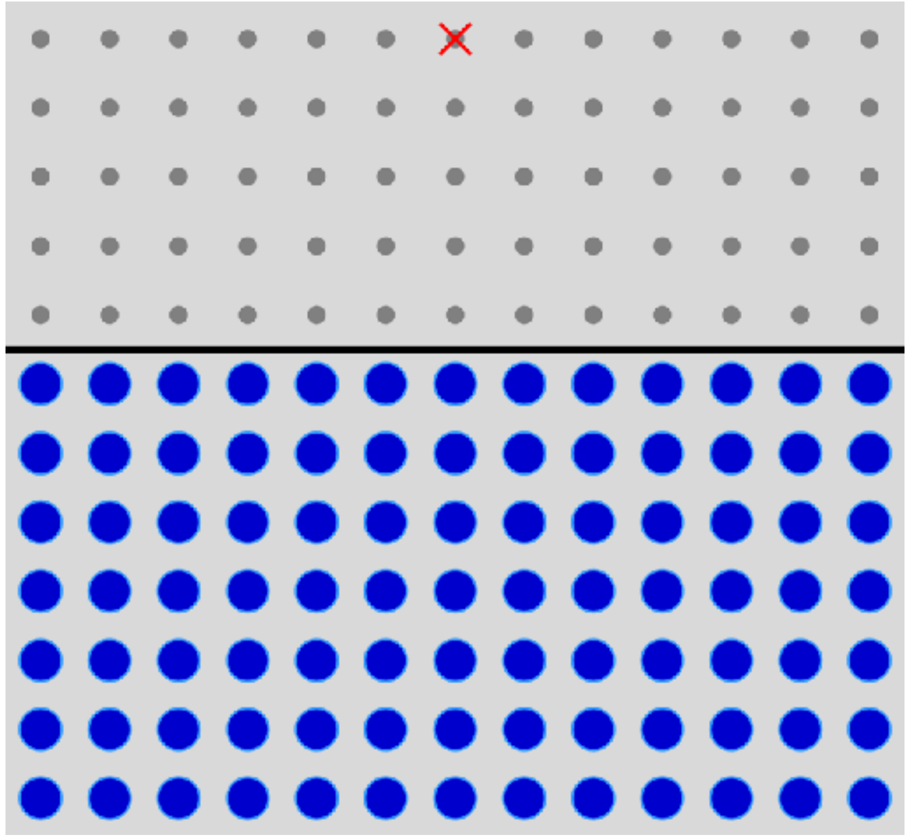
$$S - xS = (1-x)S = 1 + 2x + 2x^2 + 2x^3 + \ldots$$

$$= 1 + 2\left(x + x^2 + x^3 + \ldots\right)$$

$$= 1 + \frac{2x}{1-x}$$

$$= \frac{1+x}{1-x}$$

$$S = \frac{1+x}{(1-x)^2}$$



$$p(x) = x^5 S = \frac{x^5(1+x)}{(1-x)^2}$$

Now put in our value of *x*:

$$x_*^2 + x_* - 1 = 0 \quad \Longrightarrow \quad x_*(x_*+1) = 1$$

$$x_*^2 = 1 - x_*$$

So the infinite starting configuration has

$$p(x_*) = 1$$

5. Put everything together.

We need to reach the target $p(x_*) = 1$.

Any finite start must have $p(x_*) < 1$, since it has fewer pegs than the infinite configuration with $p(x_*) = 1$.

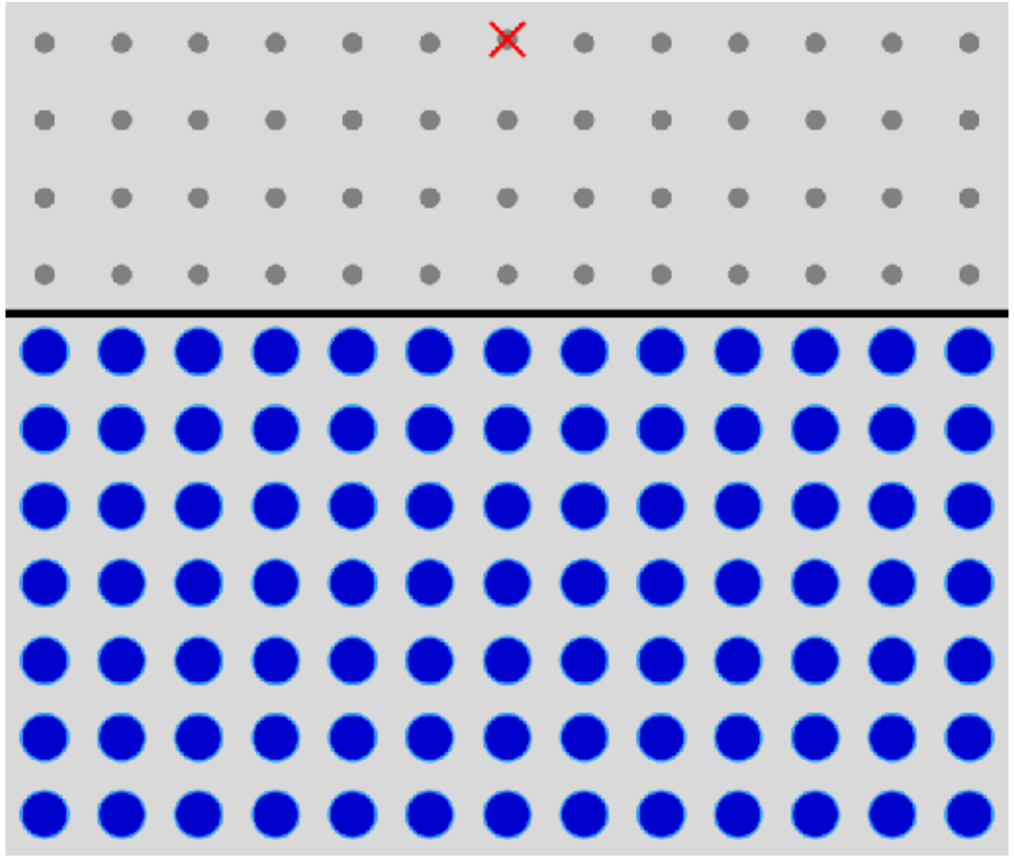
Since no move can increase $p(x_*)$, we can never reach level 5 with a finite number of pegs!

Q.E.D.

Why does this argument fail for levels 1, 2, 3, 4?

The infinite starting configuration has $p(x_*) > 1$.

e.g. level 4:

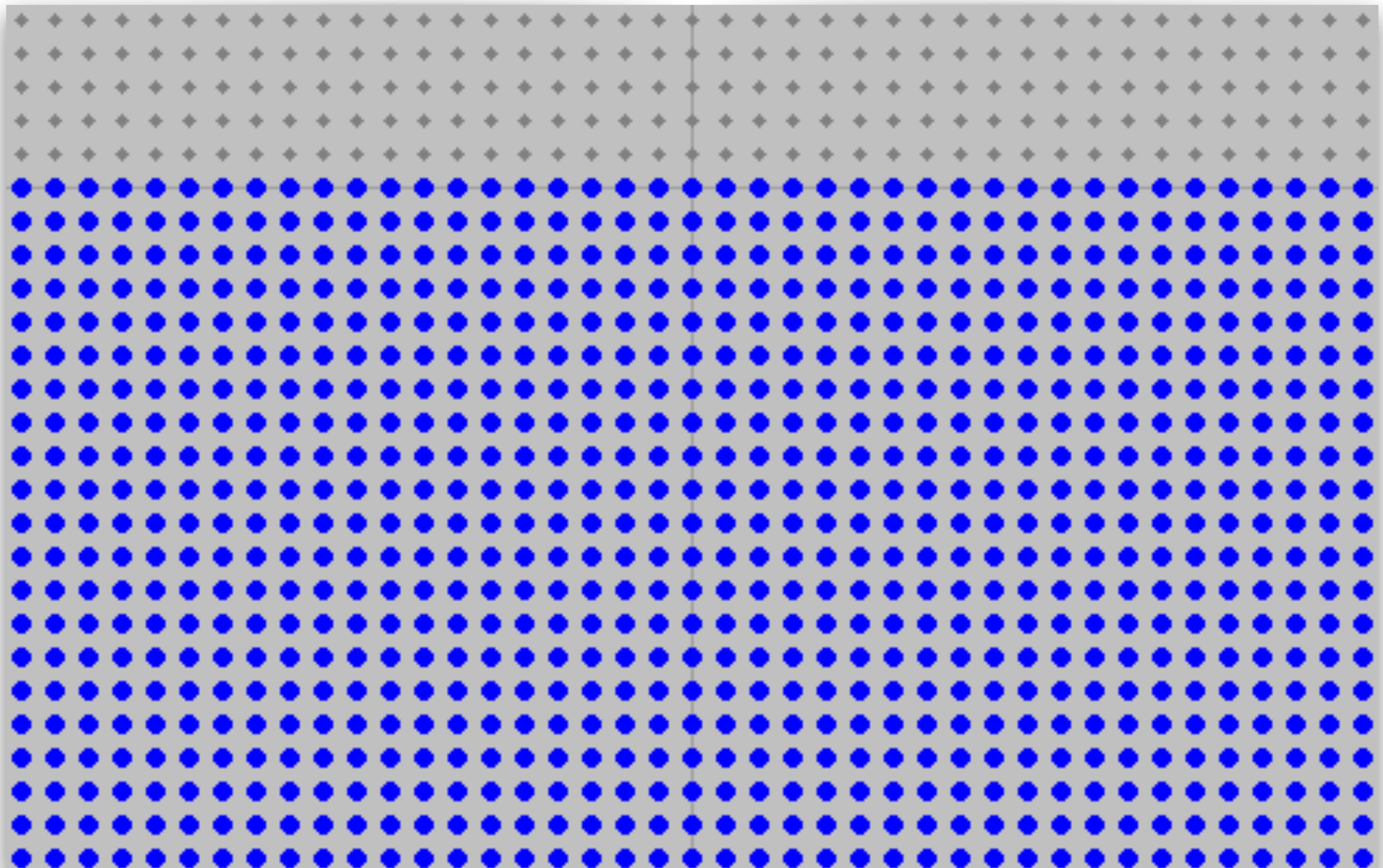


$$p(x) = x^4 S = \frac{x^4(1+x)}{(1-x)^2}$$

$$p(x_*) = \frac{1}{x_*} = 1.618\ldots$$

Infinite number of pegs

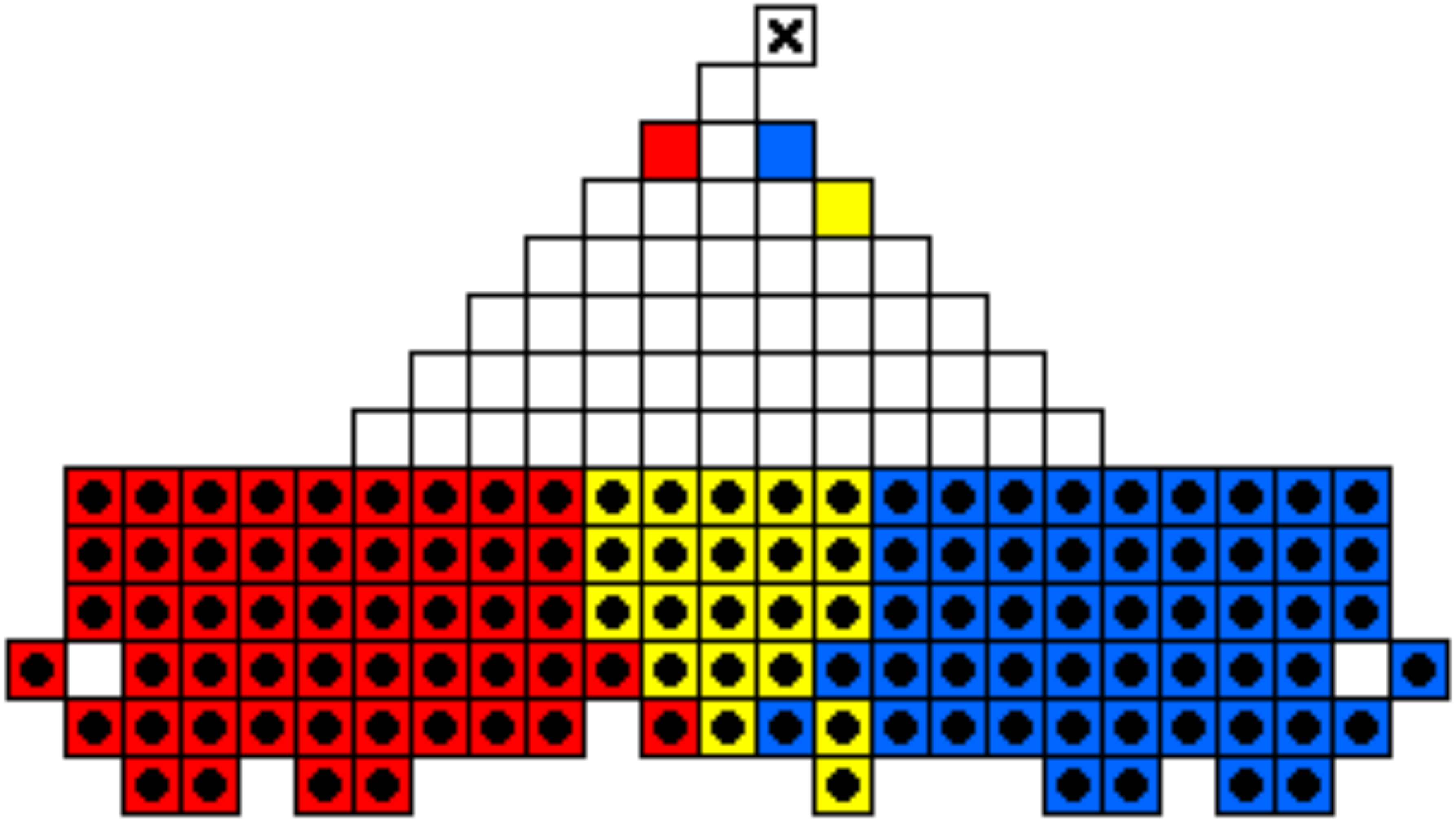
Infinite number of pegs



Simon Tatham & Geoff Taylor

Adding diagonal jumps

Maximum possible is **level 8** (needs **123 pegs**):



References

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Berlekamp, E. R., *Winning ways for your mathematical plays Vol 4 (Routledge)*

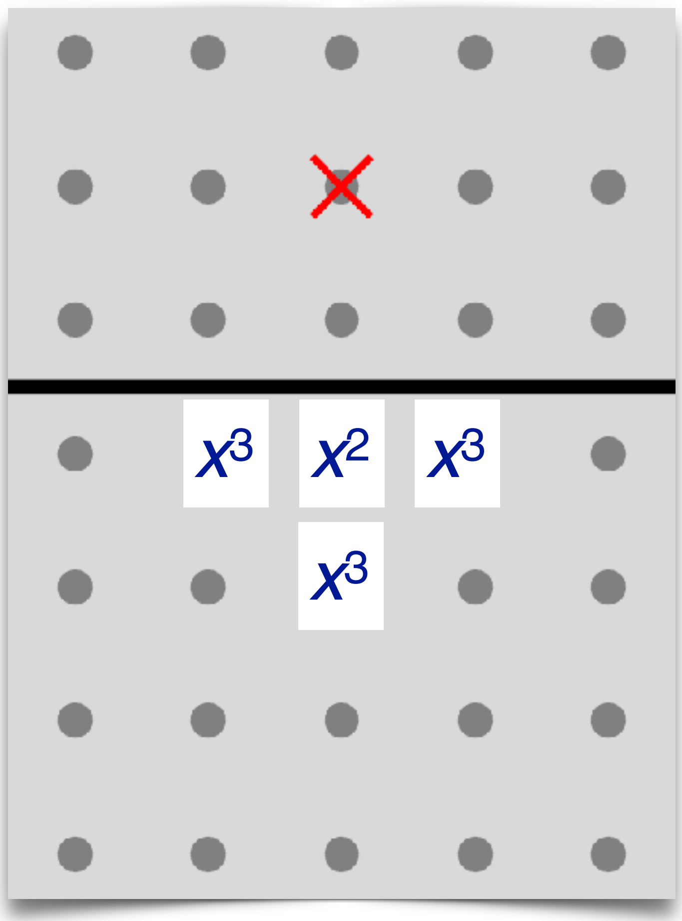
Tatham, S. & Taylor, G., *Reaching row five in solitaire army,* (http://www.chiark.greenend.org.uk/~sgtatham/solarmy/)

Bell, G. I., Hirschberg, D. & Guerrero-Garcia P., *The minimum size required of a solitaire army,* INTEGERS **G7**, 2007. http://www.gibell.net/pegsolitaire/army/index.html

A life in games: the playful genius of John Conway, Wired, http://www.wired.com/2015/09/life-games-playful-genius-john-conway/

Calculating minimum numbers of pegs…

e.g. level 2.



Largest possible values with

2 pegs: $p(x_*) = x_*^2 + x_*^3 = 0.618\ldots$

3 pegs: $p(x_*) = x_*^2 + 2x_*^3 = 0.854\ldots$

4 pegs: $p(x_*) = x_*^2 + 3x_*^3 = 1.090\ldots$

So minimum number of pegs is *at least* 4.