MATHEMATICAL PHYSICS II
MATH2071 · EPIPHANY · 2026

Electromagnetism and Special Relativity

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Contents

Preliminaries

These notes assume familiarity with vector calculus, in particular Gauss' theorem and Stokes' theorem, as covered in Mathematical Methods II (MATH2811). Two appendices collect background material on vector calculus and on Dirac's delta function. A third appendix contains derivations used in Section 2.2.

Conventions,

- Bold symbols (e.g. $\mathbf{E}$, $\mathbf{B}$, $\mathbf{J}$, $\mathbf{A}$, $\mathbf{x}$) denote three-dimensional vectors.
- We use Einstein summation for repeated indices.
- Latin indices $(a, b, i, j, \dots)$ run from 1 to 3.
- Greek indices $(\alpha, \beta, \mu, \nu, \dots)$ run from 0 to 3.
- For partial derivatives we write $\partial_\mu \equiv \frac{\partial}{\partial x^\mu}$ and $\partial'_\mu \equiv \frac{\partial}{\partial x^{\mu'}}$.
- Unless stated otherwise, all fields are smooth and we freely commute partial derivatives.
- For the three-dimensional Levi–Civita symbol we take $\varepsilon_{123} = 1$.

These notes are intended to be self-contained. For additional reading, the following textbooks are excellent references,

- D. J. Griffiths, *Introduction to Electrodynamics*.
- E. M. Purcell and D. J. Morin, *Electricity and Magnetism*.
- J. D. Jackson, *Classical Electrodynamics*.

§1 Introduction

These lectures broadly follow the historical development of electromagnetism, a theory for the electric and magnetic fields, sourced by charge density ρ and current density $\mathbf{J}$. We begin with time-independent configurations, where the subject naturally splits into electrostatics and magnetostatics and where $\mathbf{E}$ and $\mathbf{B}$ can be treated as distinct fields governed by separate laws.

We then turn to electrodynamics, where this separation breaks down. Time dependence ties electric and magnetic phenomena together through induction, and Maxwell's completion of the classical laws shows that $\mathbf{E}$ and $\mathbf{B}$ are intrinsically coupled components of a single theory. In this sense, Maxwell's theory,

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}\,, \tag{1.0.1}$$

$$\nabla \times \mathbf{E} = -\partial_t \mathbf{B}\,, \tag{1.0.2}$$

$$\nabla \cdot \mathbf{B} = 0\,, \tag{1.0.3}$$

$$\nabla \times \mathbf{B} = \mu_0\,\mathbf{J} + \mu_0\,\varepsilon_0\,\partial_t \mathbf{E}\,, \tag{1.0.4}$$

was the first great example of unification in physics. It also brings into view a new organising principle, *gauge symmetry*. We will repeatedly see that gauge freedom is not merely a calculational convenience, but a structural feature of the theory. After identifying suitable degrees of freedom, we will connect with the first part of the module and derive these equations from an action principle.

Finally, we refine this understanding by recognising that electromagnetism fits naturally within special relativity. The relativistic viewpoint clarifies how electric and magnetic fields transform between inertial frames and makes the unity of the theory manifest. It also motivates an efficient formulation in terms of spacetime tensors, most notably the electromagnetic field tensor $F_{\mu\nu}$, which packages $\mathbf{E}$ and $\mathbf{B}$ into a single Lorentz-covariant object and leads naturally to a compact statement of Maxwell's equations.

§2 Electric Charge and its Transport Properties

§2.1 Charge and Current Density

Electric charge is a real number assigned to particles, characterising how they interact with the electromagnetic field. Unlike mass, charge can be positive or negative; a neutral particle does not couple to electromagnetic fields. The SI unit is the Coulomb (C), which is a large unit. Note that the electron has charge $q = -e$ with $e = 1.6 \cdot 10^{-19}\,\mathrm{C}$, while quarks carry charges $q = 2e/3$ or $q = -e/3$.

In a point-particle description, charge is concentrated at isolated points. In many applications, however, it is convenient to treat charge as "smeared" along a wire, spread over a surface, or distributed smoothly in space. To describe a time-dependent distribution of charge we introduce the *charge density* $\rho(t, \mathbf{x})$, which appears as a source in Maxwell's equations. By definition, ρ is charge per unit volume, so the total charge contained in a region $V \subset \mathbb{R}^3$ at time t is,

$$Q_V(t) = \int_V \rho(t, \mathbf{x})\, d^3\mathbf{x}\,. \tag{2.1.1}$$

Example 2.1.1. *Point particles*
Consider a point particle of charge q located at $\mathbf{y}(t)$. The corresponding charge density $\rho(t, \mathbf{x})$ must satisfy,

$$\int_V \rho(t, \mathbf{x})\, d^3\mathbf{x} = \begin{cases} q & \text{if } \mathbf{y}(t) \in V \\ 0 & \text{if } \mathbf{y}(t) \notin V \end{cases}\,. \tag{2.1.2}$$

Using the defining property of Dirac's delta (Appendix B), we can write $\rho(t, \mathbf{x}) = q\,\delta(\mathbf{x} - \mathbf{y}(t))$. For N particles at positions $\mathbf{y}_i(t)$ with charges q_i, this generalises to,

$$\rho(t, \mathbf{x}) = \sum_{i=1}^{N} q_i\, \delta(\mathbf{x} - \mathbf{y}_i(t))\,. \tag{2.1.3}$$

Electric current is the transport of charge. The *current density* $\mathbf{J}(t, \mathbf{x})$ is defined so that, for an oriented surface S, the rate at which charge crosses S is,

$$I_S(t) = \int_S \mathbf{J}(t, \mathbf{x}) \cdot d\mathbf{S}\,. \tag{2.1.4}$$

As with ρ, one can describe currents confined to curves or surfaces, as well as currents flowing throughout three-dimensional space; we discuss these cases in more detail in the next subsection.

Example 2.1.2. *Fluid flow*
A charged fluid is described by its charge density $\rho(t, \mathbf{x})$ and by a velocity field $\mathbf{v}(t, \mathbf{x})$. The associated current density is $\mathbf{J}(t, \mathbf{x}) = \rho(t, \mathbf{x})\, \mathbf{v}(t, \mathbf{x})$.

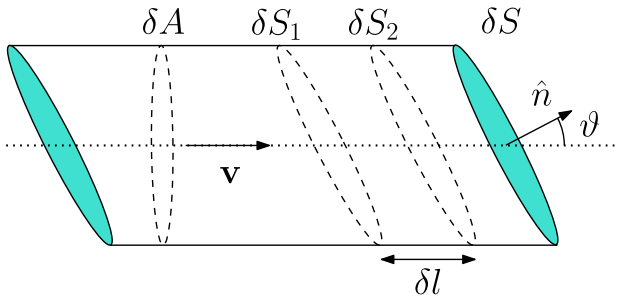


Figure 1: Flow of charged fluid through a small oriented surface element $\delta\mathbf{S}$.

To see this, consider the geometry in Figure 1. Let $\delta\mathbf{S} = \hat{\mathbf{n}}\,\delta S$ be a small surface element with unit normal $\hat{\mathbf{n}}$, and let ϑ be the angle between $\mathbf{v}$ and $\hat{\mathbf{n}}$. In a short time δt, the fluid travels a distance $\delta l = |\mathbf{v}|\,\delta t$ along the direction of $\mathbf{v}$. The amount of fluid between surfaces δS_1 and δS_2 occupies a thin slab of volume,

$$\delta V = (\delta S \cos\vartheta)\,\delta l\,,$$

and crosses $\delta\mathbf{S}$. Note that we used that $\delta A = \delta S\cos\vartheta$ is the area of the projection of the surface element perpendicular to the flow. The charge carried by this volume is $\delta Q = \rho\,\delta V$, hence,

$$\frac{\delta Q}{\delta t} = \rho\,\mathbf{v}\cdot\delta\mathbf{S}\,.$$

Comparing with the definition (2.1.4) *gives $\mathbf{J} = \rho\,\mathbf{v}$.*

Example 2.1.3. *The electromagnetic field exerts a force density $\mathbf{f}_{\rm L} = \rho\,\mathbf{E} + \mathbf{J}\times\mathbf{B}$ on a charge density ρ and current density $\mathbf{J}$.*

To show this result, consider a small fluid element of volume δV. The total charge inside the element can be approximated by $\delta a = \rho\,\delta V$ resulting in a Lorentz force

$$\delta\mathbf{F}_{\rm L} = \delta q\,(\mathbf{E} + \mathbf{v}\times\mathbf{B})\ ,$$

where $\mathbf{v}$ is the fluid velocity of the distribution. Substituting the expression for δq we have,

$$\begin{aligned}\delta\mathbf{F}_{\rm L} &= \rho\,(\mathbf{E} + \mathbf{v}\times\mathbf{B})\,\delta V\\ &= (\rho\,\mathbf{E} + \mathbf{J}\times\mathbf{B})\,\delta V\,,\end{aligned}$$

where in the second line we used $\mathbf{J} = \rho\,\mathbf{v}$, as we showed in Example 2.1.2. The last line proves the desired result.

A fundamental property is that electric charge can neither be destroyed nor created, it can only move from one region to another[1]. This statement of *local charge conservation*

[1]Electrically charged matter is described by complex fields for which internal rotations in the complex plane that are constant in space and time constitute a symmetry. The continuity equation we will discuss is nothing but the conservation law of the relevant Noether charge.

can be phrased as follows. Let V be an arbitrary volume with closed boundary $S = \partial V$. The rate of change of charge in V is equal to minus the net current flowing out through S,

$$\frac{d}{dt} Q_V(t) = -I_{S=\partial V}(t)\,. \tag{2.1.5}$$

If V_F denotes the full volume of the system under consideration (for example $V_F = \mathbb{R}^3$ for an infinite system), then $Q_{V_F}(t)$ is the total charge. For an isolated system with no charge exchange with the environment we impose $I_{\partial V_F}(t) = 0$.

Using (2.1.1), we have,

$$\frac{d}{dt} Q_V(t) = \int_V \partial_t \rho(t, \mathbf{x})\, d^3\mathbf{x}\,. \tag{2.1.6}$$

Using (2.1.4) and Gauss' theorem,

$$I_{S=\partial V}(t) = \oint_{\partial V} \mathbf{J}(t, \mathbf{x}) \cdot d\mathbf{S} = \int_V \nabla \cdot \mathbf{J}(t, \mathbf{x})\, d^3\mathbf{x}\,, \tag{2.1.7}$$

where we used Gauss' theorem to replace the surface integral by a volume integral of the divergence. Because charge conservation should not depend on the choice of volume V, combining these expressions yields the *continuity equation*,

$$\boxed{\partial_t \rho(t, \mathbf{x}) + \nabla \cdot \mathbf{J}(t, \mathbf{x}) = 0\,.} \tag{2.1.8}$$

Example 2.1.4. *Current density of charged particles Returning to Example 2.1.1, the natural expression for the current density is,*

$$\mathbf{J}(t, \mathbf{x}) = \sum_{i=1}^{N} q_i\, \dot{\mathbf{y}}_i(t)\, \delta(\mathbf{x} - \mathbf{y}_i(t))\,. \tag{2.1.9}$$

One can check directly that (2.1.3) *and* (2.1.9) *satisfy* (2.1.8)*. Using the chain rule,*

$$\partial_t \rho(t, \mathbf{x}) = \sum_{i=1}^{N} q_i\, \partial_t \delta(\mathbf{x} - \mathbf{y}_i(t)) = -\sum_{i=1}^{N} q_i\, \dot{\mathbf{y}}_i(t) \cdot \nabla \delta(\mathbf{x} - \mathbf{y}_i(t))\,, \tag{2.1.10}$$

while,

$$\nabla \cdot \mathbf{J}(t, \mathbf{x}) = \sum_{i=1}^{N} q_i\, \dot{\mathbf{y}}_i(t) \cdot \nabla \delta(\mathbf{x} - \mathbf{y}_i(t))\,. \tag{2.1.11}$$

The two contributions cancel, as required. Here $(\dot{\mathbf{y}}_i)_j$ *denotes the* j*th component of the velocity of the* i*th particle.*

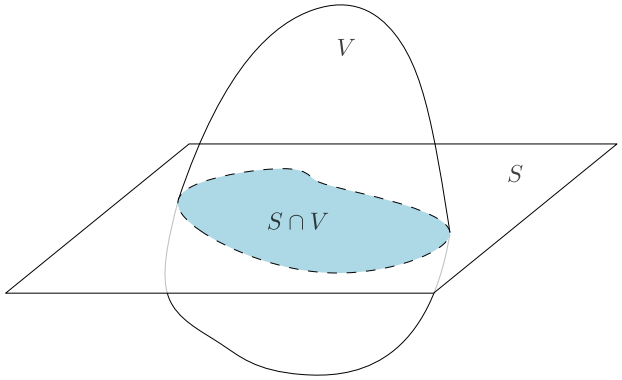


Figure 2: A three-dimensional volume V intersecting a charged surface S.

§2.2 Charge on Lower Dimensional Surfaces

As noted in Section 2.1, charge and electric currents may be confined to curves or surfaces. Maxwell's equations, however, are written in terms of three-dimensional densities. In this section we relate lower-dimensional descriptions to the corresponding three-dimensional distributions.

We first consider charge distributed on a two-dimensional surface S with parameters s_1 and s_2, so that points on the surface are given by an embedding $\mathbf{x}(s_1, s_2)$. The surface charge density $\sigma(t, s_1, s_2)$ is defined so that the total charge on a patch $S' \subset S$ is,

$$Q_{S'}(t) = \int_{S'} \sigma(t, s_1, s_2)\, dS\,, \tag{2.2.1}$$

where,

$$dS = |\partial_{s_1}\mathbf{x}(s_1, s_2) \times \partial_{s_2}\mathbf{x}(s_1, s_2)|\; ds_1\, ds_2\,.$$

The corresponding charge density from the three-dimensional perspective is a distribution supported on S,

$$\rho(t, \mathbf{x}) = \int_S \delta(\mathbf{x} - \mathbf{x}(s_1, s_2))\, \sigma(t, s_1, s_2)\, dS\,. \tag{2.2.2}$$

The delta function ensures that $\rho(t, \mathbf{x})$ vanishes for $\mathbf{x} \notin S$, and this form is precisely what reproduces the surface integral definition of charge when integrated over any test volume.

To check (2.2.2), let V be a volume that intersects S as in Figure 2. Then,

$$Q_V(t) = \int_V \rho(t, \mathbf{x})\, d^3\mathbf{x} = \int_S \left(\int_V \delta(\mathbf{x} - \mathbf{x}(s_1, s_2))\, d^3\mathbf{x} \right) \sigma(t, s_1, s_2)\, dS \tag{2.2.3}$$

$$= \int_{S\cap V} \sigma(t, s_1, s_2)\, dS = Q_{S\cap V}(t)\,, \tag{2.2.4}$$

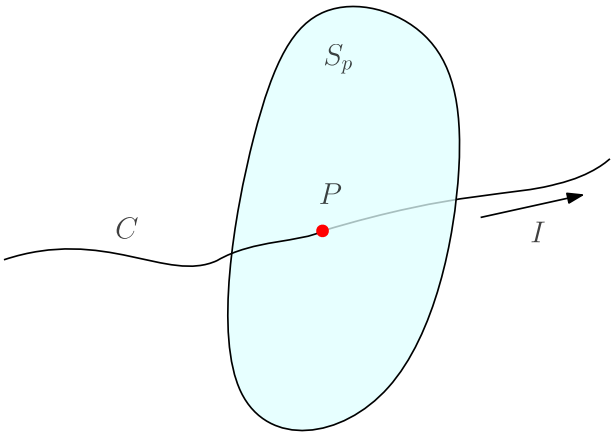


Figure 3: Intersection between a surface and a curve carrying current.

which is exactly what we expect from (2.2.1). In going from the first line to the second we used the basic property of the delta function, namely that,

$$\int_V \delta(\mathbf{x} - \mathbf{x}(s_1, s_2))\, d^3\mathbf{x} = \begin{cases} 1 & \text{if } \mathbf{x}(s_1, s_2) \in V \\ 0 & \text{if } \mathbf{x}(s_1, s_2) \notin V\,, \end{cases}$$

so the set of points contributing to the integral is precisely $S \cap V$.

As in the surface case, charge may also be confined to a one-dimensional curve C with parameter s and embedding $\mathbf{x}(s)$. The linear charge density $\eta(t, s)$ is defined by,

$$Q_{C'}(t) = \int_{C'} \eta(t, s)\, dl\,, \tag{2.2.5}$$

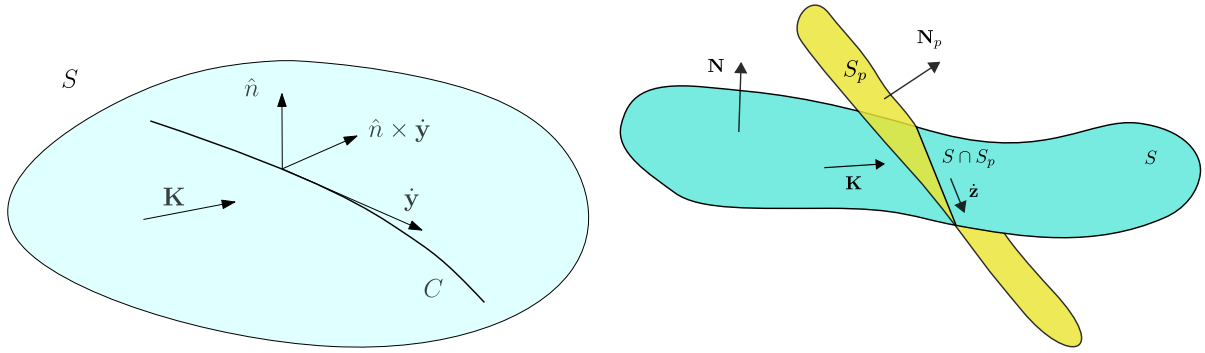
for a segment $C' \subset C$, where $dl = |\dot{\mathbf{x}}(s)|\, ds$. The associated three-dimensional charge density is,

$$\rho(t, \mathbf{x}) = \int_C \delta\left(\mathbf{x} - \mathbf{x}(s)\right)\, \eta(t, s)\, dl\,. \tag{2.2.6}$$

For current densities, the geometry is slightly more delicate. Consider a current $I(t, s)$ flowing along a curve C parameterised by $\mathbf{r}(s)$. From the three-dimensional perspective, we would like a current density whose flux through a surface S_p receives contributions only from the intersection points $P = C \cap S_p$, as shown in Figure 3. As shown in Appendix C, the correct expression is,

$$\mathbf{J}(t, \mathbf{x}) = \int_C \delta(\mathbf{x} - \mathbf{r}(s))\, I(t, s)\, \dot{\mathbf{r}}(s)\, ds\,. \tag{2.2.7}$$

Finally, we consider a surface current density $\mathbf{K}(t, \lambda_1, \lambda_2)$ flowing on a surface S parameterised by $\mathbf{r}(\lambda_1, \lambda_2)$. Since charge does not escape from the surface, $\mathbf{K}$ is everywhere tangent to S and therefore orthogonal to the normal vector $\mathbf{N} = \partial_{\lambda_1}\mathbf{r} \times \partial_{\lambda_2}\mathbf{r}$. From the perspective of the surface, it is natural to define the flux of current through a curve C that



(a) Current flux through a curve.

(b) Flux of surface current through a transverse surface.

Figure 4: Current flux for surface current densities.

lies entirely in S, as shown in Figure 4a. If $C \subset S$ and $\mathbf{y}(\lambda) = \mathbf{r}(\lambda_1(\lambda), \lambda_2(\lambda))$ parameterises C, then the tangent vector $\dot{\mathbf{y}}$ is orthogonal[2] to $\mathbf{N}$.

After defining the unit normal $\hat{\mathbf{n}}$ to the surface, the current flux across C can be written as,

$$I_C(t) = \int_C \mathbf{K}(t, \lambda_1(\lambda), \lambda_2(\lambda)) \cdot (\hat{\mathbf{n}} \times \dot{\mathbf{y}}(\lambda))\, d\lambda\,. \tag{2.2.8}$$

This mirrors the usual definition of flux through a two-dimensional surface: here $\hat{\mathbf{n}} \times \dot{\mathbf{y}}$ plays the role of a normal "line element" along C. Note that $\hat{\mathbf{n}} \perp \dot{\mathbf{y}}$, so $|\hat{\mathbf{n}} \times \dot{\mathbf{y}}| = |\dot{\mathbf{y}}|$.

From the three-dimensional perspective, we want a current density whose flux through a surface S_p reproduces the surface flux when S_p intersects S along the curve $C = S \cap S_p$, as in Figure 4b. As shown in Appendix C, the appropriate distributional current density is,

$$\mathbf{J}(t, \mathbf{x}) = \int_S \delta(\mathbf{x} - \mathbf{r}(s_1, s_2))\ \mathbf{K}(t, s_1, s_2)\, dS\,, \tag{2.2.9}$$

which indeed reduces to the correct expression when integrated over S_p.

[2]To see this, write $\dot{\mathbf{y}} = \dot{\lambda}_1\, \partial_{\lambda_1}\mathbf{r} + \dot{\lambda}_2\, \partial_{\lambda_2}\mathbf{r}$. Then $\dot{\mathbf{y}} \cdot \mathbf{N} = \dot{\lambda}_1\, (\partial_{\lambda_1}\mathbf{r} \cdot \mathbf{N}) + \dot{\lambda}_2\, (\partial_{\lambda_2}\mathbf{r} \cdot \mathbf{N}) = 0$ because $\mathbf{N} = \partial_{\lambda_1}\mathbf{r} \times \partial_{\lambda_2}\mathbf{r}$ and $\mathbf{a} \cdot (\mathbf{a} \times \mathbf{b}) = 0$ for any $\mathbf{a}, \mathbf{b}$.

§3 Static Fields

In the next few sections we will study charge and field configurations that are *time independent*. For this to be consistent, the charge and current densities sourcing the electromagnetic field must have vanishing time derivatives. However, charge and current density are not independent quantities. The charge continuity equation (2.1.8) implies that any static current configuration must satisfy,

$$\nabla \cdot \mathbf{J} = 0\,. \tag{3.0.1}$$

Assuming we have somehow achieved such a steady flow of current, Maxwell's equations equations (1.0.1) to (1.0.4) decouple the electric from the magnetic field into two separate sets. In particular, the electric field is sourced purely by the charge density,

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}\,, \tag{3.0.2}$$

$$\nabla \times \mathbf{E} = 0\,, \tag{3.0.3}$$

while the magnetic field is sourced by the electric current density,

$$\nabla \cdot \mathbf{B} = 0\,, \tag{3.0.4}$$

$$\nabla \times \mathbf{B} = \mu_0\, \mathbf{J}\,. \tag{3.0.5}$$

These equations show that, in the absence of time dependence, one can treat electric and magnetic fields as two largely independent entities. Historically this is indeed how they were understood for a long time. Allowing time dependence ties them together into a single theory, electromagnetism, one of the first and most striking examples of unification in physics.

Before studying these static systems in more detail, it is useful to understand some general features that they share. Both are captured by a set of differential equations of the form,

$$\begin{aligned} \nabla \cdot \mathbf{F} &= D\,, \\ \nabla \times \mathbf{F} &= \mathbf{C}\,, \end{aligned} \tag{3.0.6}$$

for some given scalar field D and vector field $\mathbf{C}$. Assuming $\mathbf{F}$ is smooth, the second equation implies the compatibility condition $\nabla \cdot \mathbf{C} = 0$, due to (A.0.2). For the electric field this requirement is satisfied trivially because (3.0.3) sets $\mathbf{C} = 0$. For the magnetic field, on the other hand, it becomes a non-trivial constraint. Taking the divergence of (3.0.5) requires $\nabla \cdot \mathbf{J} = 0$, which is precisely what the continuity equation enforces in a static configuration.

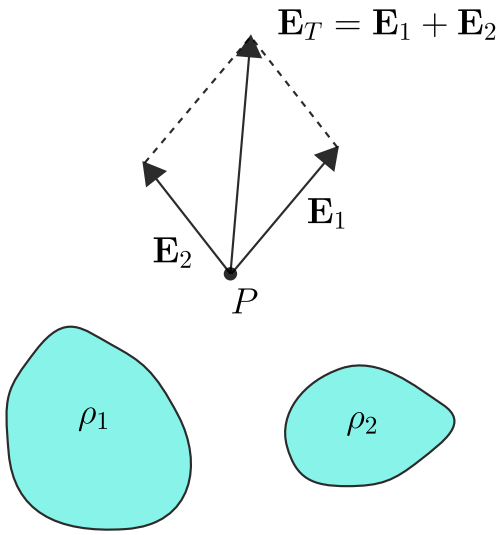


Figure 5: Illustration of superposition of static electric fields.

A natural mathematical question is whether a system of equations like (3.0.6) is enough to uniquely fix $\mathbf{F}$. In general, it is not. Indeed, given a solution $\mathbf{F}_0$, we can always add to it a vector field $\mathbf{W}$ which satisfies,

$$\nabla \cdot \mathbf{W} = 0\,, \tag{3.0.7}$$

$$\nabla \times \mathbf{W} = 0\,, \tag{3.0.8}$$

and still obtain a solution with the same scalar and vector sources. In fact, we can construct large families of such fields by writing $\mathbf{W} = \nabla\phi$. Due to (A.0.3), the second equation is then automatically satisfied, while the first reduces to the condition that ϕ is harmonic, $\nabla^2\phi = 0$. One such function is, for example, $\phi(x, y, z) = x\,y\,z$. The ingredient we are missing in order to pin down a unique solution is *boundary conditions* for the field $\mathbf{F}$. We will not prove the general statement here, since we will see explicitly how it works in the following sections, when we discuss the scalar and vector potentials.

Theorem 3.0.1. *Helmholtz theorem: If the divergence and the curl of a vector field* $\mathbf{F}$ *are specified, and if they both go to zero faster than* $1/r^2$ *at infinity and if* $\mathbf{F}$ *goes to zero as well, then* $\mathbf{F}$ *is uniquely fixed.*

Before moving on to examine our static configurations in more detail, it is important to appreciate the linearity of the equations we are dealing with. The linearity of Maxwell's equations mirrors the experimental observation that electric and magnetic fields satisfy a *superposition principle*. For example, consider the charge distribution ρ_1 sourcing the electric field $\mathbf{E}_1$ and the charge distribution ρ_2 sourcing $\mathbf{E}_2$. A highly non-obvious fact is that when we put together the charge distributions to create the total $\rho_T = \rho_1 + \rho_2$, the resulting electric field is the vector sum $\mathbf{E}_T = \mathbf{E}_1 + \mathbf{E}_2$, as illustrated in Figure 5. Mathematically, this is reflected in the linearity of Maxwell's equations. Similar considerations apply for the magnetic field.

§4 Electrostatics

In this section we focus on the electrostatic Maxwell equations (3.0.2) and (3.0.3). We begin by deriving the integral form of *Gauss' law*, and then compute the electric field for a number of charge distributions with simple geometries. A particularly instructive example is an infinite plane with uniform surface charge density. This system shows that surface charge densities are naturally treated as discontinuities in the electric field. We treat the more general case in the following subsection.

These examples help develop intuition for how an electric field is sourced by a static charge distribution. More fundamentally, we introduce the scalar potential and encounter the simplest form of gauge symmetry. This also provides a convenient way to study the asymptotic behaviour far from sources. Finally, we relate the work required to assemble a charge distribution to the energy stored in the electric field.

§4.1 Gauss' law

The first of Maxwell's equations (1.0.1) (or, in the electrostatic case, (3.0.2)) fixes the divergence of the electric field. This is the *differential* form of Gauss' law. A great deal of intuition comes from integrating this equation over a volume region V,

$$\int_V \nabla \cdot \mathbf{E}\, d^3\mathbf{x} = \frac{1}{\varepsilon_0} \int_V \rho\, d^3\mathbf{x}\,. \tag{4.1.1}$$

The integral on the right-hand side is simply the total charge enclosed by V,

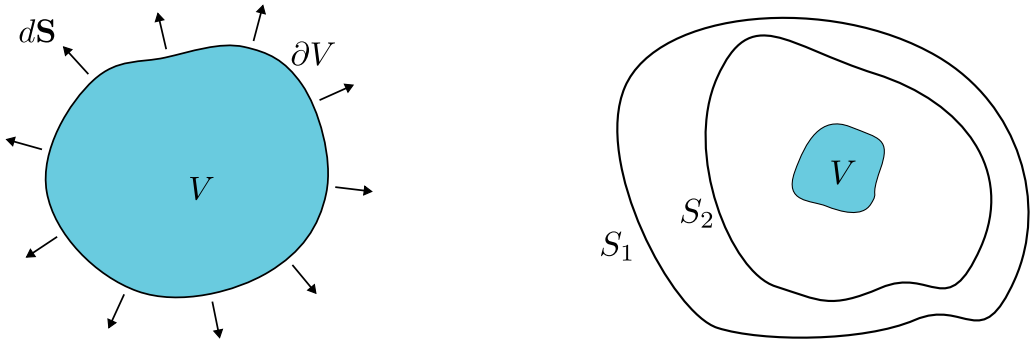
$$Q_V \equiv \int_V \rho\, d^3\mathbf{x}\,. \tag{4.1.2}$$

Applying Gauss' theorem to the left-hand side converts the volume integral into a surface integral over the closed boundary $S = \partial V$,

$$\boxed{\int_{\partial V} \mathbf{E} \cdot d\mathbf{S} = \frac{Q_V}{\varepsilon_0}\,.} \tag{4.1.3}$$

This is the *integral* form of Gauss' law. When applying Gauss' theorem, the oriented area element $d\mathbf{S}$ is defined using the outward-pointing unit normal to ∂V, as illustrated in Figure 6a.

Remark 4.1.1. Gauss' law immediately constrains the *net* flux of $\mathbf{E}$ through a closed surface. If $Q_V > 0$, then the flux $\int_{\partial V} \mathbf{E} \cdot d\mathbf{S}$ is positive, so (on average) field lines exit the surface. If $Q_V < 0$, the flux is negative and the field lines enter the surface.



(a) Orientation of boundary surface when applying Gauss' theorem.

(b) Electric field flux independent of surface deformations.

Figure 6: Gauss' theorem illustrations.

Remark 4.1.2. Gauss' law also shows that the flux depends only on the charge enclosed, not on the detailed shape of the surface. In particular, if we deform ∂V continuously without crossing any charge, the enclosed charge Q_V is unchanged and so is the flux. This is illustrated in Figure 6b. Since the charge density is non-zero only inside the volume V, both surfaces S_1 and S_2 enclose the same charge, and therefore the flux through either surface must be Q_V/ε_0.

The remarks above are often enough to determine $\mathbf{E}$ in situations with a high degree of symmetry. In more general cases we will develop a more systematic formalism (involving Green's functions). Even in the symmetric examples, however, one should keep in mind that Gauss' law (4.1.3) is equivalent only to (3.0.2). A full electrostatic solution must also satisfy (3.0.3) (together with appropriate boundary conditions).

§4.2 The point particle - Coulomb's law

In this section we study the electric field produced by a charged point particle, the simplest electrostatic source one can consider. The result is Coulomb's law, which you have likely seen before, but it is instructive to see how it follows directly from Maxwell's equations.

Given the spherical symmetry of the problem, the integral form of Gauss' law will be particularly useful for determining the electric field. It is therefore natural to work in spherical coordinates (r, θ, ϕ) with the corresponding basis vectors $\hat{\mathbf{e}}_r$, $\hat{\mathbf{e}}_\theta$ and $\hat{\mathbf{e}}_\phi$. Assuming that the charge sits at the origin, it is reasonable to expect that the electric field has only a radial component, depending only on the radial coordinate r. More concretely, we make the ansatz,

$$\mathbf{E}(\mathbf{r}) = E(r)\,\hat{\mathbf{e}}_r\,, \tag{4.2.1}$$

and then use Gauss' law to determine the function $E(r)$. Note that this ansatz satisfies (3.0.3) identically, so we will not have to consider that equation further.

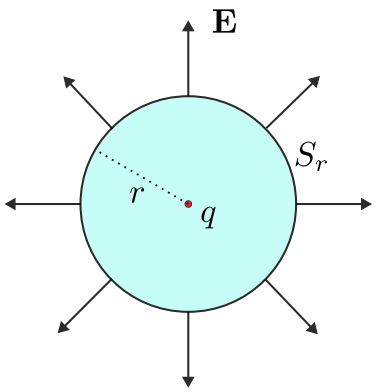


Figure 7: A sphere S_r of radius r as a surface choice in Gauss' law.

The natural choice of surface S in Gauss' law is a sphere S_r of radius r centred at the origin, where the point charge is located. In that case,

$$\int_{S_r} \mathbf{E}(\mathbf{r}) \cdot d\mathbf{S} = \frac{q}{\varepsilon_0} \,. \tag{4.2.2}$$

To evaluate the integral, we note that the oriented surface element is,

$$d\mathbf{S} = \hat{\mathbf{e}}_r \, r^2 \, \sin\theta \, d\theta \, d\phi = \hat{\mathbf{e}}_r \, r^2 \, d\Omega \,, \tag{4.2.3}$$

where $d\Omega$ is the integration measure on the unit sphere. The flux is therefore

$$\int_{S_r} \mathbf{E}(\mathbf{r}) \cdot d\mathbf{S} = \int_{S_r} E(r) \, r^2 \, d\Omega = E(r) \, r^2 \int_{S_r} d\Omega = 4\pi \, E(r) \, r^2 \,. \tag{4.2.4}$$

Putting everything together, we find,

$$\mathbf{E}(\mathbf{r}) = \frac{q}{4\pi\varepsilon_0} \, \frac{1}{r^2} \, \hat{\mathbf{e}}_r = \frac{q}{4\pi\varepsilon_0} \, \frac{\mathbf{r}}{r^3} \,, \tag{4.2.5}$$

where we used that the position vector can always be written as $\mathbf{r} = r \, \hat{\mathbf{e}}_r$. If the point charge is placed at $\mathbf{r}_0$ instead of the origin, the corresponding electric field is obtained by translation,

$$\mathbf{E}(\mathbf{r}) = \frac{q}{4\pi\varepsilon_0} \, \frac{\mathbf{r} - \mathbf{r}_0}{|\mathbf{r} - \mathbf{r}_0|^3} \,, \tag{4.2.6}$$

which is Coulomb's law for the electric field of a charged point particle.

While deriving (4.2.6) from the integral form of Gauss' law is satisfying, it is also useful to connect back to the differential form (3.0.2) and verify that (4.2.6) solves,

$$\nabla \cdot \mathbf{E}(\mathbf{r}) = \frac{q}{\varepsilon_0} \, \delta(\mathbf{r} - \mathbf{r}_0) \,, \tag{4.2.7}$$

since, according to Example 2.1.1, the charge density of a point particle is $\rho(\mathbf{r}) = q \, \delta(\mathbf{r} - \mathbf{r}_0)$.

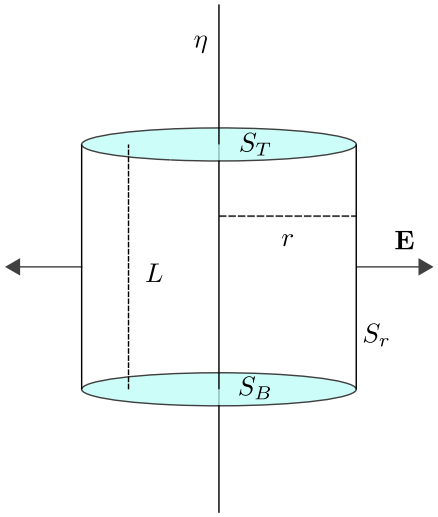


Figure 8: Gauss' law applied to an infinite charged wire.

§4.3 The infinite charged wire

The next geometry we examine is an infinite straight line C carrying a uniform linear charge density η. This configuration has cylindrical symmetry because it is invariant under translations along the line, a consequence of η being constant, and under rotations about it. It is therefore natural to work in cylindrical coordinates (r, ϕ, z) with basis vectors $\hat{\mathbf{e}}_r$, $\hat{\mathbf{e}}_\phi$ and $\hat{\mathbf{e}}_z$, and to place the charged line on the z-axis, i.e. at $r = 0$. We note that if η depended on z, the rotational symmetry would remain, but translation invariance along the line would be lost.

By symmetry, the electric field must point radially and depend only on the distance to the line. We therefore make the ansatz,

$$\mathbf{E}(\mathbf{r}) = E(r)\,\hat{\mathbf{e}}_r\,, \tag{4.3.1}$$

where "radial" is understood in the cylindrical sense. As before, this ansatz satisfies (3.0.3) identically. A convenient Gaussian surface is a cylinder S of length L and radius r, coaxial with the line of charge, as shown in Figure 8.

The charge enclosed by the cylinder comes entirely from the line, so $Q_V = \eta L$. The integral form of Gauss' law then gives,

$$\int_S \mathbf{E}(\mathbf{r}) \cdot d\mathbf{S} = \frac{\eta L}{\varepsilon_0}\,. \tag{4.3.2}$$

To evaluate the flux, decompose the surface as $S = S_r \cup S_T \cup S_B$, where S_T and S_B are the top and bottom discs (of radius r) and S_r is the cylindrical side. The end caps do not contribute, since their outward normal is parallel to $\hat{\mathbf{e}}_z$, which is orthogonal to $\mathbf{E}$. For the side surface we use,

$$d\mathbf{S}_r = \hat{\mathbf{e}}_r\, r\, d\phi\, dz\,, \tag{4.3.3}$$

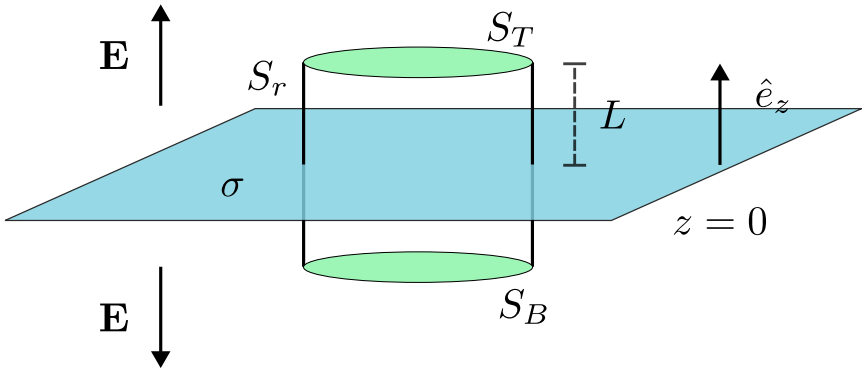


Figure 9: Electric field generated by a uniformly charged infinite plane.

and hence,

$$\int_S \mathbf{E}(\mathbf{r}) \cdot d\mathbf{S} = \int_{S_r} \mathbf{E}(\mathbf{r}) \cdot d\mathbf{S}_r = r\, E(r) \int_0^{2\pi} \int_0^L dz\, d\phi = 2\pi\, L\, r\, E(r)\,. \tag{4.3.4}$$

Combining these results, we obtain the electric field of an infinite charged wire,

$$\mathbf{E}(\mathbf{r}) = \frac{\eta}{2\pi\varepsilon_0} \frac{1}{r}\, \hat{\mathbf{e}}_r\,. \tag{4.3.5}$$

§4.4 The infinite charged surface

The final highly symmetric geometry we will examine is an infinite plane carrying a uniform surface charge density σ. The relevant symmetries are translations in the two directions parallel to the plane. A convenient coordinate system is therefore Cartesian coordinates (x, y, z) with basis vectors $\hat{\mathbf{e}}_x$, $\hat{\mathbf{e}}_y$ and $\hat{\mathbf{e}}_z$.

The symmetry of the configuration suggests that the electric field is everywhere normal to the plane, which we take to lie at $z = 0$. Moreover, reflecting the system through the plane sends $z \to -z$ while leaving the charge distribution unchanged; since the electric field is a vector, it must transform as $\mathbf{E}(-z) = -\mathbf{E}(z)$. A natural ansatz consistent with these symmetries is,

$$\mathbf{E}(z) = \begin{cases} E(z)\, \hat{\mathbf{e}}_z & \text{if } z > 0\,, \\ -E(-z)\, \hat{\mathbf{e}}_z & \text{if } z < 0\,. \end{cases} \tag{4.4.1}$$

It is straightforward to check that (3.0.3) is satisfied identically by construction.

A natural Gaussian surface for this geometry is a cylinder S whose axis is perpendicular to the charged plane and whose midpoint lies at $z = 0$. Choosing the cylinder to have total length $2L$, its two bases S_T and S_B (each of area A) are located at $z = L$ and $z = -L$, respectively We have included Figure 9 for illustration.

The intersection of the cylinder with the charged plane is a disk of area A, so the total enclosed charge comes entirely from this disk and equals $Q_V = \sigma A$. Gauss' law therefore gives,

$$\int_S \mathbf{E} \cdot d\mathbf{S} = \frac{\sigma A}{\varepsilon_0} \,. \tag{4.4.2}$$

We decompose the flux as,

$$\int_S \mathbf{E} \cdot d\mathbf{S} = \int_{S_r} \mathbf{E} \cdot d\mathbf{S_r} + \int_{S_T} \mathbf{E} \cdot d\mathbf{S_T} + \int_{S_B} \mathbf{E} \cdot d\mathbf{S_B} \,, \tag{4.4.3}$$

where S_r denotes the cylindrical side surface. The first term vanishes because $\mathbf{E}$ is parallel to the cylinder axis, whereas $d\mathbf{S}_r$ is orthogonal to it. Taking the outward orientation into account, the oriented area elements on the two bases are,

$$\begin{aligned} d\mathbf{S_T} &= \hat{\mathbf{e}}_z \, dS_T \,, \\ d\mathbf{S_B} &= -\hat{\mathbf{e}}_z \, dS_B \,. \end{aligned}$$

The corresponding surface integrals are,

$$\int_{S_T} \mathbf{E} \cdot d\mathbf{S_T} = \int_{S_T} \mathbf{E}(z = L) \cdot \hat{\mathbf{e}}_z \, dS_T = E(L) \int_{S_T} dS_T = E(L)\, A \,, \tag{4.4.4}$$

$$\int_{S_B} \mathbf{E} \cdot d\mathbf{S_B} = -\int_{S_B} \mathbf{E}(z = -L) \cdot \hat{\mathbf{e}}_z \, dS_B = E(L) \int_{S_B} dS_B = E(L)\, A \,. \tag{4.4.5}$$

Putting everything together fixes the function in the ansatz (4.4.1) to be a constant,

$$E(z) = \frac{\sigma}{2\varepsilon_0} \,. \tag{4.4.6}$$

The fact that the electric field is constant for $z \neq 0$ is far from obvious until one applies Gauss' law.

A key point is what happens at the location of the charged plane itself. The solution (4.4.1) implies the jump condition,

$$\mathbf{E}(z = 0^+) - \mathbf{E}(z = 0^-) = \frac{\sigma}{\varepsilon_0} \hat{\mathbf{e}}_z \Rightarrow \left.\begin{array}{l} \hat{\mathbf{e}}_z \cdot (\mathbf{E}(z = 0^+) - \mathbf{E}(z = 0^-)) = \frac{\sigma}{\varepsilon_0} \,, \\ \hat{\mathbf{e}}_z \times (\mathbf{E}(z = 0^+) - \mathbf{E}(z = 0^-)) = 0 \end{array}\right\} \,. \tag{4.4.7}$$

This shows that the component normal to the surface is discontinuous, with a jump fixed by the surface charge density. The second equation states that any tangential component would be continuous; in the present example it is trivially satisfied since the field is everywhere normal to the plane.

It is instructive to see why this discontinuity is unavoidable directly from the differential form of Gauss' law (3.0.2). Since symmetry implies $\mathbf{E} = E_z(z)\, \hat{\mathbf{e}}_z$, equation (3.0.2) becomes,

$$\partial_z E_z(z) = \frac{1}{\varepsilon_0} \rho(x, y, z) \,, \tag{4.4.8}$$

with ρ the three-dimensional charge density. Using (2.2.2) with the surface parametrisation $\mathbf{x}(s_1, s_2) = (s_1, s_2, 0)$, we find,

$$\rho(x, y, z) = \int_{\mathbb{R}^2} \sigma\, \delta(x - s_1)\, \delta(y - s_2)\, \delta(z)\, ds_1\, ds_2 \tag{4.4.9}$$

$$= \sigma\, \delta(z) \int_{\mathbb{R}^2} \delta(x - s_1)\, \delta(y - s_2)\, ds_1\, ds_2 \tag{4.4.10}$$

$$= \sigma\, \delta(z)\,, \tag{4.4.11}$$

as expected. The charge density is proportional to a delta function supported at the surface. Gauss' law therefore takes the form,

$$\partial_z E_z(z) = \frac{\sigma}{\varepsilon_0}\, \delta(z)\,. \tag{4.4.12}$$

Integrating from $z = -\epsilon$ to $z = \epsilon$ gives precisely the discontinuity already obtained above,

$$\int_{-\epsilon}^{\epsilon} \partial_z E_z(z)\, dz = \frac{\sigma}{\varepsilon_0} \int_{-\epsilon}^{\epsilon} \delta(z)\, dz \Rightarrow E_z(\epsilon) - E_z(-\epsilon) = \frac{\sigma}{\varepsilon_0}\,. \tag{4.4.13}$$

In evaluating the integral on the right-hand side we used the defining property of the Dirac delta, discussed in Appendix B.

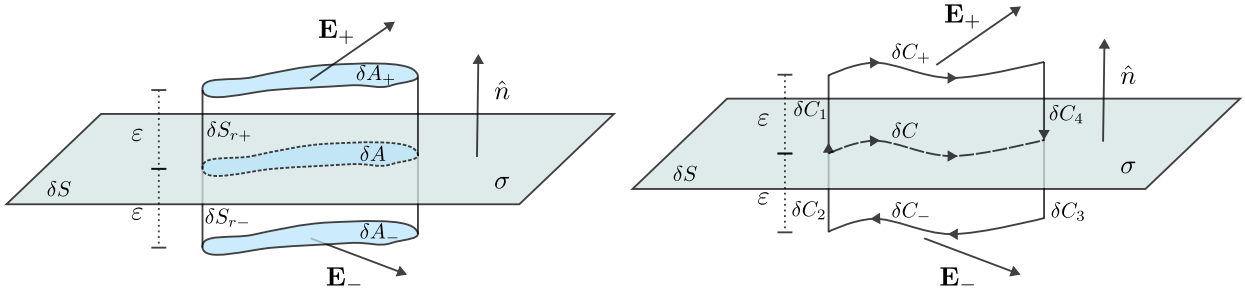
This argument shows that such jump conditions are far more general than the flat surface considered here. In the next section we discuss the general case; the underlying idea is the same as in this example.

§4.5 Surface Charge as an Interface

This subsection is non-examinable.

In the previous section we studied the electric field generated by an infinite plane carrying a uniform surface charge density. The key lesson was that the component of $\mathbf{E}$ normal to the surface is discontinuous. We traced this discontinuity back to the Dirac delta appearing in (2.2.2), which relates the surface charge density σ to the three-dimensional charge density ρ. Here we derive the corresponding jump conditions for a *general* surface S carrying a surface charge density σ, which may vary along the surface and, more generally, in time.

To study the discontinuity of the component of $\mathbf{E}$ normal to S, we use a pillbox-type Gaussian surface adapted to an arbitrary patch of the interface. Consider a small section $\delta S \subset S$ with unit normal vector $\hat{\mathbf{n}}$, as shown in Figure 10a. For ease of illustration the patch is drawn as flat, but the argument does not rely on this.



(a) Integration over a Gaussian surface. (b) Integration over a loop.

Figure 10: Surface and loop integrations used to derive boundary conditions for the electric field.

To construct the Gaussian surface, choose a small area $\delta A \subset \delta S$ and extrude each point of δA by ε along $\hat{\mathbf{n}}$. The collection of all resulting trajectories forms a (generalised) cylinder V_+ with bases δA and δA_+, and side δS_+. Repeating the construction by extruding δA in the opposite direction produces a second cylinder V_- with bases δA and δA_-, and side δS_-.

Gauss' law for the volume $V = V_+ \cup V_-$ reads,

$$\int_{\partial V} \mathbf{E} \cdot d\mathbf{S} = \frac{1}{\varepsilon_0} Q_V \,, \tag{4.5.1}$$

where ∂V is the boundary of V. The only charge inside V comes from the surface charge contained in δA, so,

$$Q_V = \int_{\delta A} \sigma \, dS \,. \tag{4.5.2}$$

The flux of the electric field can be decomposed as,

$$\int_{\partial V} \mathbf{E} \cdot d\mathbf{S} = \int_{\delta A_+} \mathbf{E} \cdot d\mathbf{S} + \int_{\delta S_+} \mathbf{E} \cdot d\mathbf{S} + \int_{\delta A_-} \mathbf{E} \cdot d\mathbf{S} + \int_{\delta S_-} \mathbf{E} \cdot d\mathbf{S} \,. \tag{4.5.3}$$

Remark 4.1.2 suggests that this flux should be independent of ε, including in the $\varepsilon \to 0$ limit. By construction, the charge Q_V is also independent of ε.

To evaluate the flux in this limit, note that the side surfaces δS_+ and δS_- shrink to zero area as $\varepsilon \to 0$. Provided the electric field remains finite (as we expect in the absence of further singular sources), the second and fourth terms in (4.5.3) therefore vanish. In the same limit, both bases δA_+ and δA_- converge to δA, with orientations $\hat{\mathbf{n}}$ and $-\hat{\mathbf{n}}$, respectively. It follows that,

$$\begin{aligned} \lim_{\varepsilon \to 0^+} \int_{\delta A_+} \mathbf{E} \cdot d\mathbf{S} &= \int_{\delta A} \hat{\mathbf{n}} \cdot \mathbf{E}_+ \, dS \,, \\ \lim_{\varepsilon \to 0^+} \int_{\delta A_-} \mathbf{E} \cdot d\mathbf{S} &= -\int_{\delta A} \hat{\mathbf{n}} \cdot \mathbf{E}_- \, dS \,, \end{aligned} \tag{4.5.4}$$

where $\mathbf{E}_+$ is the limit of the electric field as we approach δS from the side toward which $\hat{\mathbf{n}}$ points, and $\mathbf{E}_-$ is the corresponding limit from the opposite side.

Putting everything together, and using that δA was arbitrary, Gauss' law (4.5.3) yields the discontinuity condition,

$$\int_{\delta A} \hat{\mathbf{n}} \cdot (\mathbf{E}_+ - \mathbf{E}_-)\, dS = \frac{1}{\varepsilon_0} \int_{\delta A} \sigma\, dS \Rightarrow \tag{4.5.5}$$

$$\hat{\mathbf{n}} \cdot (\mathbf{E}_+ - \mathbf{E}_-) = \frac{\sigma}{\varepsilon_0}\,, \tag{4.5.6}$$

for the normal component, as promised.

Beyond the normal component, we would also like to understand what happens to the components of $\mathbf{E}$ tangent to the surface. In the planar example these components vanished by symmetry and were therefore trivially continuous. In the general case we need an argument that does not rely on symmetry.

Instead of an area δA, consider a curve δC lying within the surface δS. Repeating the extrusion construction along the unit normal $\hat{\mathbf{n}}$ produces a surface δS_l bounded by the loop $\delta C_l = \delta C_1 \cup \delta C_2 \cup \delta C_- \cup \delta C_3 \cup \delta C_4 \cup \delta C_+$, as shown in Figure 10b. Integrating both sides of (3.0.3) over δS_l and applying Stokes' theorem gives,

$$\int_{\delta S_l} (\nabla \times \mathbf{E}) \cdot d\mathbf{S} = 0 \Rightarrow$$

$$\int_{\delta C_l} \mathbf{E} \cdot d\mathbf{r} = 0 \Rightarrow$$

$$\int_{\delta C_1} \mathbf{E} \cdot d\mathbf{l} + \int_{\delta C_2} \mathbf{E} \cdot d\mathbf{l} + \int_{\delta C_-} \mathbf{E} \cdot d\mathbf{l} + \int_{\delta C_3} \mathbf{E} \cdot d\mathbf{l} + \int_{\delta C_4} \mathbf{E} \cdot d\mathbf{l} + \int_{\delta C_+} \mathbf{E} \cdot d\mathbf{l} = 0\,, \tag{4.5.7}$$

with orientations as in Figure 10b. Taking the $\varepsilon \to 0$ limit, the segments δC_i for $i = 1, \ldots, 4$ shrink to zero and their contributions vanish. In the same limit, δC_+ and δC_- both approach the curve δC with opposite orientations, and we obtain,

$$\int_{\delta C} (\mathbf{E}_+ - \mathbf{E}_-) \cdot d\mathbf{r} = 0\,. \tag{4.5.8}$$

Since this holds for any curve $\delta C \subset \delta S$, we can also write the local condition,

$$\hat{\mathbf{n}}_{\parallel} \cdot (\mathbf{E}_+ - \mathbf{E}_-) = 0\,, \tag{4.5.9}$$

for any vector $\hat{\mathbf{n}}_{\parallel}$ tangent to the surface δS. An equivalent, and often more practical, way of expressing this condition is in terms of the unit normal,

$$\hat{\mathbf{n}} \times (\mathbf{E}_+ - \mathbf{E}_-) = 0\,. \tag{4.5.10}$$

We have therefore shown that the component of the electric field parallel to the surface is continuous.

§4.6 The Electrostatic Potential

So far we have been able to determine the electric field in a number of highly symmetric electrostatic setups. In each case we could write down an ansatz for $\mathbf{E}$ that automatically satisfied the curl-free condition (3.0.3). It is worth pausing to understand what this equation implies more generally.

By Stokes' theorem, (3.0.3) implies that the line integral of the electric field around any closed loop C vanishes,

$$\oint_C \mathbf{E} \cdot d\mathbf{r} = 0\,. \tag{4.6.1}$$

Equivalently, the line integral of $\mathbf{E}$ along a curve connecting two fixed endpoints $\mathbf{r}_1$ and $\mathbf{r}_2$ depends only on the endpoints and not on the particular path. Since the force on a particle of charge q is $\mathbf{F} = q\,\mathbf{E}$, this means that the work,

$$W_{12} = \int_{C_{12}} \mathbf{F} \cdot d\mathbf{r} = q \int_{C_{12}} \mathbf{E} \cdot d\mathbf{r} \tag{4.6.2}$$

required to move the particle from $\mathbf{r}_1$ to $\mathbf{r}_2$ along any path C_{12} is path independent.

In addition, the identity (A.0.3) shows that (3.0.3) is automatically satisfied if we write the electric field as a gradient,

$$\mathbf{E} = -\nabla\varphi\,, \tag{4.6.3}$$

for some scalar function φ, called the *scalar* (or *electrostatic*) potential. In fact, one can prove the converse as well. Under appropriate regularity assumptions, any curl-free field can be written locally in this form. Substituting (4.6.3) into Gauss' law (3.0.2) gives a single scalar equation for φ,

$$\nabla^2\varphi = -\frac{\rho}{\varepsilon_0}\,, \tag{4.6.4}$$

which is often considerably simpler to analyse than the original coupled system.

We should now make the following remarks,

Remark 4.6.1. The physically relevant quantity is the electric field $\mathbf{E}$, not the potential φ. Notice that $\mathbf{E}$ is invariant under shifts of φ by a constant. We usually fix this constant by choosing a convention, for example by demanding that φ vanishes at infinity (when this is compatible with the physical setup).

Remark 4.6.2. For a particle of charge q moving in an electric field $\mathbf{E}$, the potential energy is $V(\mathbf{x}) = q\,\varphi(\mathbf{x})$. The corresponding Lagrangian is,

$$L = \frac{1}{2} m\,\dot{\mathbf{x}}^2 - q\,\varphi(\mathbf{x})\,. \tag{4.6.5}$$

Remark 4.6.3. With the boundary condition that $\mathbf{E}$ vanishes at infinity, the equation for φ is linear and the potential obeys a superposition principle. If $\rho = \rho_1 + \rho_2$, then (after fixing the same reference for the additive constant) $\varphi = \varphi_1 + \varphi_2$.

At this point we can appreciate that introducing the scalar potential is a very useful way to simplify Maxwell's equations. However, as we will later see, it has a more fundamental meaning than a mere computational trick. Moreover, Remark 4.6.1 is the simplest example of what we will later call a *gauge symmetry*.

We now consider an instructive example of solving (4.6.4) for a point charge.

Example 4.6.4. *Scalar potential for point particle*
We solve (4.6.4) *for a particle carrying electric charge q. If the particle is at rest at the origin, Example 2.1.1 gives the charge density $\rho(\mathbf{x}) = q\,\delta(\mathbf{x})$, and the potential must satisfy,*

$$\nabla^2\varphi(\mathbf{x}) = -\frac{q}{\varepsilon_0}\,\delta(\mathbf{x})\,. \tag{4.6.6}$$

Spherical symmetry suggests using spherical coordinates, with φ depending only on the radial coordinate r.

We first solve the equation away from the origin, where the delta-function source vanishes. In this region we have,

$$\frac{1}{r^2}\partial_r\left(r^2\,\partial_r\varphi(r)\right) = 0\,. \tag{4.6.7}$$

whose general solution is,

$$\varphi(r) = \frac{a}{r} + b\,, \tag{4.6.8}$$

with constants of integration a and b. Following Remark 4.6.1 and choosing $\varphi \to 0$ as $r \to \infty$ fixes $b = 0$. The remaining constant a is determined by requiring that (4.6.6) *holds globally, including at the origin.*

To do this, integrate both sides of (4.6.6) *over a ball B_R of radius R. The integral of the right-hand side is $-q/\varepsilon_0$, since the origin lies inside B_R. For the left-hand side we use Gauss' theorem to convert the volume integral over B_R into a surface integral over the sphere S_R of radius R. Since $\nabla\varphi = -\frac{a}{r^2}\,\hat{\mathbf{e}}_r$, we find,*

$$\int_{B_R}\nabla^2\varphi\,d^3\mathbf{x} = \int_{S_R}\nabla\varphi\cdot d\mathbf{S} = -\frac{a}{R^2}\int_{S_R}\hat{\mathbf{e}}_r\cdot d\mathbf{S} = -\frac{a}{R^2}\,R^2\int_{S_R}d\Omega = -4\pi a\,. \tag{4.6.9}$$

Here we used $d\mathbf{S} = R^2\,\hat{\mathbf{e}}_r\,d\Omega$, where $d\Omega$ is the area element on the unit sphere. Equating both sides gives $a = \frac{q}{4\pi\varepsilon_0}$, and therefore,

$$\varphi(\mathbf{x}) = \frac{q}{4\pi\varepsilon_0}\,\frac{1}{|\mathbf{x}|}\,. \tag{4.6.10}$$

If the particle is located at $\mathbf{x}_0$ *instead of the origin, the equation becomes,*

$$\nabla^2\varphi(\mathbf{x}) = -\frac{q}{\varepsilon_0}\,\delta(\mathbf{x}-\mathbf{x}_0)\,, \tag{4.6.11}$$

and by translation invariance we obtain,

$$\varphi(\mathbf{x}) = \frac{q}{4\pi\varepsilon_0}\,\frac{1}{|\mathbf{x}-\mathbf{x}_0|}\,. \tag{4.6.12}$$

The resulting electric field is,

$$\mathbf{E}(\mathbf{x}) = -\nabla\varphi(\mathbf{x}) = -\frac{q}{4\pi\varepsilon_0}\,\nabla\left(\frac{1}{|\mathbf{x}-\mathbf{x}_0|}\right) = \frac{q}{4\pi\varepsilon_0}\,\frac{\mathbf{x}-\mathbf{x}_0}{|\mathbf{x}-\mathbf{x}_0|^3}\,, \tag{4.6.13}$$

in agreement with (4.2.6).

§4.7 The General Charge Density

In Example 4.6.4 of the previous section we solved a seemingly simple problem. As we will see in this section, that calculation contains the basic ingredient needed to write down the solution for a general charge density $\rho(\mathbf{x})$. At a technical level, equations (4.6.11) and (4.6.12) tell us that we have found a function,

$$\boxed{G(\mathbf{x},\mathbf{x}_0) = -\frac{1}{4\pi}\,\frac{1}{|\mathbf{x}-\mathbf{x}_0|}\,,} \tag{4.7.1}$$

which satisfies,

$$\boxed{\nabla^2 G(\mathbf{x},\mathbf{x}_0) = \delta(\mathbf{x}-\mathbf{x}_0)\,.} \tag{4.7.2}$$

A function $G(\mathbf{x},\mathbf{x}_0)$ satisfying (4.7.2) is important enough to have its own name, it is a *Green's function* for the Laplace operator.

Given a charge distribution $\rho(\mathbf{x})$ that decays "sufficiently fast" at infinity, the scalar potential $\varphi(\mathbf{x})$ solving (4.6.4) is,

$$\boxed{\begin{aligned}\varphi(\mathbf{x}) &= -\frac{1}{\varepsilon_0}\int \rho(\mathbf{x}')\,G(\mathbf{x},\mathbf{x}')\,d^3\mathbf{x}' \\ &= \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}\,d^3\mathbf{x}'\,.\end{aligned}} \tag{4.7.3}$$

The fall-off condition ensures that the integral converges and, for this choice of G, that φ vanishes at infinity.

To check that (4.7.3) indeed solves (4.6.4), we apply the Laplace operator with respect to $\mathbf{x}$,

$$\begin{aligned}\nabla^2\varphi(\mathbf{x}) &= -\frac{1}{\varepsilon_0}\,\nabla^2\left(\int \rho(\mathbf{x}')\,G(\mathbf{x},\mathbf{x}')\,d^3\mathbf{x}'\right)\\ &= -\frac{1}{\varepsilon_0}\int \rho(\mathbf{x}')\,\nabla^2 G(\mathbf{x},\mathbf{x}')\,d^3\mathbf{x}'\\ &= -\frac{1}{\varepsilon_0}\int \rho(\mathbf{x}')\,\delta(\mathbf{x}-\mathbf{x}')\,d^3\mathbf{x}' = -\frac{1}{\varepsilon_0}\,\rho(\mathbf{x})\,.\end{aligned} \tag{4.7.4}$$

From the second line to the third we used the defining property (4.7.2) of the Green's function; in the last step we used the defining property of the Dirac delta.

Given the solution (4.7.3) for the scalar potential, we can compute the electric field as,

$$\begin{aligned}\mathbf{E}(\mathbf{x}) = -\nabla\varphi(\mathbf{x}) &= \frac{1}{\varepsilon_0}\int \rho(\mathbf{x}')\,\nabla G(\mathbf{x},\mathbf{x}')\,d^3\mathbf{x}'\\ &= -\frac{1}{4\pi\varepsilon_0}\int \rho(\mathbf{x}')\,\nabla\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right)\,d^3\mathbf{x}'\\ &= \frac{1}{4\pi\varepsilon_0}\int \rho(\mathbf{x}')\,\frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3}\,d^3\mathbf{x}'\,.\end{aligned} \tag{4.7.5}$$

It is striking that a single function G yields the solution for an arbitrary source. From a physicist's point of view, however, the structure of (4.7.5) is also natural. It is precisely a continuous version of the superposition of point-charge fields in (4.2.6).

Example 4.7.1. *System of N charged particles at rest*
We now apply the general result for the scalar potential to the configuration of N stationary point charges from Example 2.1.1. The charge density is,

$$\rho\,(\mathbf{x}') = \sum_{i=1}^{N} q_i\,\delta\,(\mathbf{x}'-\mathbf{y}_i)\ . \tag{4.7.6}$$

Substituting this into (4.7.3) *gives,*

$$\begin{aligned}\varphi(\mathbf{x}) &= \frac{1}{4\pi\varepsilon_0}\int\left(\sum_{i=1}^{N} q_i\,\delta\,(\mathbf{x}'-\mathbf{y}_i)\right)\frac{1}{|\mathbf{x}-\mathbf{x}'|}\,d^3\mathbf{x}'\\ &= \frac{1}{4\pi\varepsilon_0}\sum_{i=1}^{N} q_i\int\frac{\delta\,(\mathbf{x}'-\mathbf{y}_i)}{|\mathbf{x}-\mathbf{x}'|}\,d^3\mathbf{x}'\\ &= \frac{1}{4\pi\varepsilon_0}\sum_{i=1}^{N}\frac{q_i}{|\mathbf{x}-\mathbf{y}_i|}\,,\end{aligned} \tag{4.7.7}$$

which, as expected, is a simple superposition of the single-particle result (4.6.12). *The corresponding electric field is,*

$$\mathbf{E}(\mathbf{x}) = -\nabla\varphi(\mathbf{x}) = \frac{1}{4\pi\varepsilon_0}\sum_{i=1}^{N} q_i \frac{\mathbf{x}-\mathbf{y}_i}{|\mathbf{x}-\mathbf{y}_i|^3}\,, \tag{4.7.8}$$

which can also be obtained directly from (4.7.5) *by evaluating the integral for this charge density.*

Note 4.7.2

There is an important subtlety regarding the Green's function. The defining equation (4.7.2) must be supplemented with boundary conditions in order to determine $G(\mathbf{x},\mathbf{x}_0)$ uniquely. The expression (4.7.1) corresponds to boundary conditions where the potential vanishes at infinity. Indeed, taking $|\mathbf{x}|$ large in the second line of (4.7.3) gives $\varphi(\mathbf{x}) \to 0$ as $|\mathbf{x}| \to \infty$.

There is a class of important problems involving conducting surfaces S, where one instead imposes $\varphi = 0$ on S. When S is not at infinity, the Green's function differs from (4.7.1). The general solution is still given by the first line of (4.7.3), but with the Green's function appropriate to the chosen boundary conditions.

Before closing this section, we will write the general solutions for the scalar potential for the cases of surface and linear charge densities. In fact, we can plug in the expressions (2.2.2) and (2.2.6) in the solution (4.7.3) to obtain,

$$\varphi(\mathbf{x}) = \frac{1}{4\pi\,\varepsilon_0}\int_S \frac{\sigma(s_1,s_2)}{|\mathbf{x}-\mathbf{x}'(s_1,s_2)|}\,dS\,, \tag{4.7.9}$$

$$\varphi(\mathbf{x}) = \frac{1}{4\pi\,\varepsilon_0}\int_C \frac{\eta(s)}{|\mathbf{x}-\mathbf{x}'(s_1,s_2)|}\,dl\,. \tag{4.7.10}$$

with electric fields,

$$\mathbf{E}(\mathbf{x}) = \frac{1}{4\pi\,\varepsilon_0}\int_S \sigma(s_1,s_2)\,\frac{\mathbf{x}-\mathbf{x}'(s_1,s_2)}{|\mathbf{x}-\mathbf{x}'(s_1,s_2)|^3}\,dS\,, \tag{4.7.11}$$

$$\mathbf{E}(\mathbf{x}) = \frac{1}{4\pi\,\varepsilon_0}\int_C \eta(s)\,\frac{\mathbf{x}-\mathbf{x}'(s_1,s_2)}{|\mathbf{x}-\mathbf{x}'(s_1,s_2)|^3}\,dl\,. \tag{4.7.12}$$

§4.8 The Multipole Expansion (Long Distance Behaviour)

In this section we study the scalar potential and electric field far from their sources, as illustrated in Figure 11. We assume that the charge density $\rho(\mathbf{x})$ is localised inside a

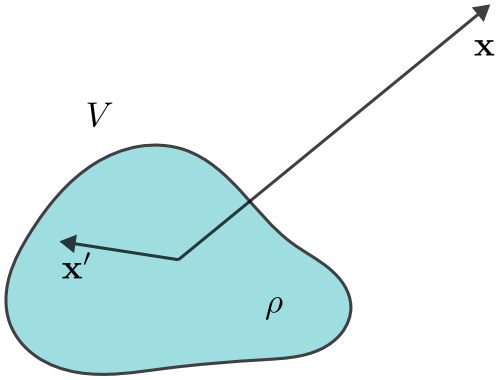


Figure 11: Behaviour at distances far away from the sources.

bounded region $V \subset \mathbb{R}^3$, so $\rho(\mathbf{x}) = 0$ outside V. In the integral expression (4.7.3), the integration variable $\mathbf{x}'$ therefore ranges only over V. We are interested in the behaviour of the potential at observation points satisfying $|\mathbf{x}| \gg |\mathbf{x}'|$ for all $\mathbf{x}' \in V$.

The key ingredient is an asymptotic expansion of the Green's-function kernel,

$$\begin{aligned} \frac{1}{|\mathbf{x}-\mathbf{x}'|} &= \frac{1}{|\mathbf{x}|} - \mathbf{x}'\cdot\nabla\frac{1}{|\mathbf{x}|} + \frac{1}{2}\left(\mathbf{x}'\cdot\nabla\right)^2\frac{1}{|\mathbf{x}|} + \cdots \\ &= \frac{1}{|\mathbf{x}|} + \frac{\mathbf{x}\cdot\mathbf{x}'}{|\mathbf{x}|^3} + \cdots , \end{aligned} \tag{4.8.1}$$

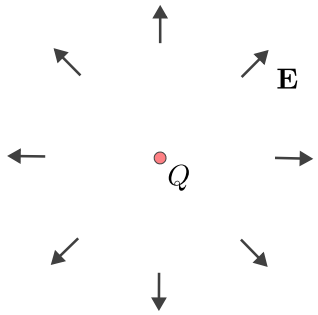
which is simply the Taylor expansion in $\mathbf{x}'$, with derivatives acting on $\mathbf{x}$. Keeping only the first few terms and substituting into the general solution (4.7.3) gives,

$$\begin{aligned} \varphi(\mathbf{x}) &= \frac{1}{4\pi\varepsilon_0}\frac{1}{|\mathbf{x}|}\int_V \rho(\mathbf{x}')\,d^3\mathbf{x}' + \frac{1}{4\pi\varepsilon_0}\frac{1}{|\mathbf{x}|^3}\,\mathbf{x}\cdot\int_V \rho(\mathbf{x}')\,\mathbf{x}'\,d^3\mathbf{x}' + \cdots \\ &= \frac{1}{4\pi\varepsilon_0}\frac{Q}{|\mathbf{x}|} + \frac{1}{4\pi\varepsilon_0}\frac{\mathbf{x}\cdot\mathbf{p}}{|\mathbf{x}|^3} + \cdots . \end{aligned} \tag{4.8.2}$$

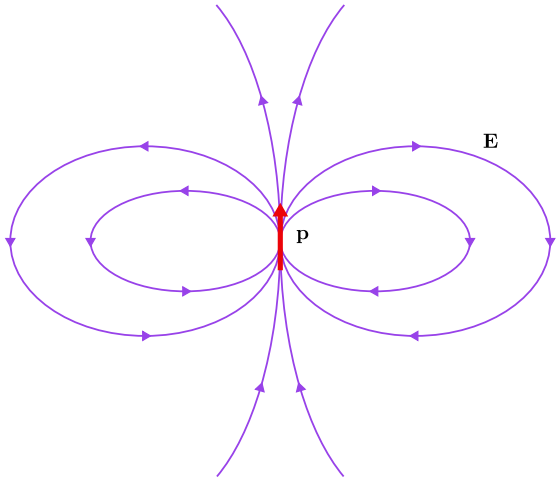
In this expansion we have defined,

$$\boxed{\begin{aligned} Q &= \int_V \rho(\mathbf{x}')\,d^3\mathbf{x}' , \\ \mathbf{p} &= \int_V \rho(\mathbf{x}')\,\mathbf{x}'\,d^3\mathbf{x}' , \end{aligned}} \tag{4.8.3}$$

as the total charge (the *monopole moment*) and the *dipole moment* of the charge distribution. Keeping higher-order terms in (4.8.1) would similarly define higher multipole moments. The general pattern is that higher moments contribute terms that fall off more rapidly far from the sources. The monopole potential scales as $1/|\mathbf{x}|$, while the dipole correction scales as $1/|\mathbf{x}|^2$.



(a) Electric field with $Q \neq 0$ and $\mathbf{p} = 0$.



(b) Electric field with $Q = 0$ and $\mathbf{p} \neq 0$.

Figure 12: Field lines of distributions with different dominant moments.

The corresponding electric field is obtained by taking the gradient,

$$\mathbf{E}(\mathbf{x}) = -\nabla\varphi(\mathbf{x}) = \frac{Q}{4\pi\varepsilon_0}\frac{\mathbf{x}}{|\mathbf{x}|^3} + \frac{1}{4\pi\varepsilon_0}\frac{1}{|\mathbf{x}|^3}\left(3\,\frac{\mathbf{p}\cdot\mathbf{x}}{|\mathbf{x}|^2}\,\mathbf{x} - \mathbf{p}\right) + \cdots . \tag{4.8.4}$$

The first term is the Coulomb field produced by the total charge Q, while the second is the leading correction due to the dipole moment.

Based on this general discussion, we can make the following remarks,

Remark 4.8.1. From the expansion (4.8.2), an observer far away from the sources first measures the net charge Q of the distribution. When $Q \neq 0$, the dipole moment $\mathbf{p}$ appears as a subleading distortion of the spherically symmetric monopole field.

Remark 4.8.2. The dipole moment depends on the choice of origin (and the same is true for all higher moments). In fact, under a translation of the coordinate system by $\mathbf{a}$, one finds $\mathbf{p} \mapsto \mathbf{p} + Q\,\mathbf{a}$. In particular, for a spherically symmetric charge distribution, choosing the origin at the centre of symmetry sets all multipole moments beyond Q to zero.

Remark 4.8.3. There are charge distributions for which either Q or $\mathbf{p}$ vanishes. The corresponding electric fields are illustrated in Figures 12a and 12b.

It is instructive to work out the moments explicitly for a simple example involving two particles,

Example 4.8.4. *We consider two point charges q_1 and q_2 located at positions $\mathbf{x}_1$ and $\mathbf{x}_2$ in a chosen coordinate system.*

The coordinate-independent quantity is their relative displacement $\mathbf{d} = \mathbf{x}_1 - \mathbf{x}_2$. The charge density is,

$$\rho(\mathbf{x}) = q_1\,\delta(\mathbf{x} - \mathbf{x}_1) + q_2\,\delta(\mathbf{x} - \mathbf{x}_2)\,. \tag{4.8.5}$$

The total charge is,

$$\begin{aligned} Q &= \int \rho(\mathbf{x})\, d^3\mathbf{x} = q_1 \int \delta(\mathbf{x}-\mathbf{x}_1)\, d^3\mathbf{x} + q_2 \int \delta(\mathbf{x}-\mathbf{x}_2)\, d^3\mathbf{x} \\ &= q_1 + q_2\,, \end{aligned} \tag{4.8.6}$$

as expected. The dipole moment (4.8.3) *is,*

$$\begin{aligned} \mathbf{p} &= \int \rho(\mathbf{x})\, \mathbf{x}\, d^3\mathbf{x} = q_1 \int \delta(\mathbf{x}-\mathbf{x}_1)\, \mathbf{x}\, d^3\mathbf{x} + q_2 \int \delta(\mathbf{x}-\mathbf{x}_2)\, \mathbf{x}\, d^3\mathbf{x} \\ &= q_1\, \mathbf{x}_1 + q_2\, \mathbf{x}_2 = q_1\, \mathbf{d} + (q_1+q_2)\, \mathbf{x}_2\,. \end{aligned} \tag{4.8.7}$$

We see explicitly that $\mathbf{p}$ *is independent of the choice of origin when the total charge vanishes: if* $q_1 = -q_2 = q$*, then* $\mathbf{p} = q\,\mathbf{d}$*.*

§4.9 Energy Stored in Electric Field

In this section we derive an expression for the energy stored in an electrostatic field configuration. Equivalently, we compute the work that must be done to assemble a given charge distribution. The idea is to imagine bringing charge in slowly from infinity. Each new "piece" of charge is moved against the electric forces generated by the charges that have already been deposited.

It is useful to start with a single test particle of charge q brought in from infinity. Suppose the particle is finally placed at $\mathbf{x}$, and let C be a curve along which we transport it, parametrised by $\mathbf{x}(s)$. We choose the parameter so that $\mathbf{x}(s = s_{\text{initial}})$ is at infinity and $\mathbf{x}(s = s_{\text{final}}) = \mathbf{x}$. The energy we must supply is the work done against the electric force,

$$\begin{aligned} W &= -\int_C \mathbf{F}(\mathbf{x}(s)) \cdot \dot{\mathbf{x}}(s)\, ds = -q \int_C \mathbf{E}(\mathbf{x}(s)) \cdot \dot{\mathbf{x}}(s)\, ds = q \int_C \nabla\varphi(\mathbf{x}(s)) \cdot \dot{\mathbf{x}}(s)\, ds \\ &= q \int_C \frac{d}{ds}\varphi(\mathbf{x}(s))\, ds = q\, \left(\varphi(\mathbf{x}(s_{\text{final}})) - \varphi(\mathbf{x}(s_{\text{initial}}))\right) \\ &= q\, \varphi(\mathbf{x})\,, \end{aligned} \tag{4.9.1}$$

where we assumed that the scalar potential vanishes at infinity, i.e. $\varphi(\mathbf{x}(s_{\text{initial}})) = 0$. This gives the familiar result that the energy of a charge placed at $\mathbf{x}$ equals q times the potential $\varphi(\mathbf{x})$ created by other charges.

Now consider a system of N stationary point charges q_i located at $\mathbf{x}_i$, $i = 1, \ldots, N$. We will define,

$$U_{i,j} = q_j\, \varphi_i(\mathbf{x}_j) = \frac{1}{4\pi\varepsilon_0} \frac{q_j\, q_i}{|\mathbf{x}_i - \mathbf{x}_i|}, \tag{4.9.2}$$

to be the potential of the particle i due to the field created by the particle j. Avoiding the double counting that would occur if we simply summed the pairwise interaction energies over all ordered pairs (i, j), the total energy of the system can be written as,

$$U = \sum_{i=1}^{N} \sum_{j>1}^{N} U_{i,j} \,. \tag{4.9.3}$$

Notice that we have the symmetry $U_{i,j} = U_{j,i}$ that allows us to write,

$$U = \frac{1}{2} \sum_{i=1}^{N} \sum_{j \neq i} U_{i,j} = \frac{1}{2} \sum_{i=1}^{N} \sum_{j \neq i} q_i \, \varphi_j(\mathbf{x}_i) = \frac{1}{2} \sum_{i=1}^{N} q_i \, \varphi \left(\mathbf{x}_i \right) \,, \tag{4.9.4}$$

where we have defined,

$$\varphi \left(\mathbf{x}_i \right) = \sum_{j \neq i} \varphi_j(\mathbf{x}_i) \,, \tag{4.9.5}$$

to be the potential at $\mathbf{x}_i$ due to all $N - 1$ particles, except for the i-th one.

It is now straightforward to pass from the discrete sum to a continuous charge distribution,

$$U_{el} = \frac{1}{2} \int_{\mathbb{R}^3} \rho(\mathbf{x}) \, \varphi(\mathbf{x}) \, d^3\mathbf{x} \,. \tag{4.9.6}$$

This form is convenient, but we would like an expression entirely in terms of the electric field. Using Gauss' law (3.0.2) to write $\rho = \varepsilon_0 \, \nabla \cdot \mathbf{E}$, we obtain,

$$U_{el} = \frac{\varepsilon_0}{2} \int_{\mathbb{R}^3} \varphi(\mathbf{x}) \, \nabla \cdot \mathbf{E}(\mathbf{x}) \, d^3\mathbf{x} \,. \tag{4.9.7}$$

Integrating by parts and assuming the fields fall off sufficiently fast at infinity (so that boundary terms vanish), we find,

$$\boxed{U_{el} = \frac{\varepsilon_0}{2} \int_{\mathbb{R}^3} |\mathbf{E}(\mathbf{x})|^2 \, d^3\mathbf{x} \,,} \tag{4.9.8}$$

where we used $\mathbf{E} = -\nabla\varphi$ from (4.6.3).

§5 Magnetostatics

It is now time to turn our attention to time-independent configurations of magnetic fields sourced by steady current densities. In analogy with electrostatics, we will derive an integral form of Ampère's law (3.0.5). This will clarify how magnetic fields are generated by currents and will allow us to compute the magnetic field in simple geometries.

Later, we will introduce the vector potential and encounter a richer version of gauge symmetry than in electrostatics. At a practical level, the vector potential provides a systematic way to solve magnetostatic problems for a general current density distribution. The resulting closed-form expression is the Biot–Savart law. We will then use it to develop a multipole expansion for the magnetic field far from its sources.

Starting from the Lorentz force, we will derive an expression for the force exerted on a small current distribution. By rewriting this force as a divergence, we will identify the potential energy of a magnetic dipole in an external magnetic field. The computation of the energy stored in a magnetic field will be postponed to the next section, since time-dependent effects need to be taken into account.

§5.1 Ampère's Law

The two equations of magnetostatics that will occupy us in this section are (3.0.4) and (3.0.5). As we discussed earlier, given appropriate boundary conditions, Helmholtz's theorem guarantees a unique solution. In magnetostatics the magnetic field is always divergence free, while its curl is fixed by the sources. In this sense, the roles of divergence and curl are reversed compared to electrostatics.

Remark 5.1.1. The condition $\nabla\cdot\mathbf{B} = 0$ says that, in the Maxwell theory as we have written it, there are no magnetic charges. From a purely mathematical point of view, however, nothing prevents us from postulating a sourced equation of the form,

$$\nabla \cdot \mathbf{B} = g\,\rho_m\,, \tag{5.1.1}$$

for a magnetic charge density ρ_m. It is an experimental fact that magnetic charges (monopoles) have not been observed (yet) in nature.

With Stokes' theorem in mind, the natural way to obtain an integral form of Ampère's law (3.0.5) is to integrate both sides over an open surface S with boundary $C = \partial S$. In doing so, it is important to remember that the orientation of the boundary C is determined by the orientation of S via the right-hand rule, as shown in Figure 13a.

Integrating both sides of (3.0.5) over S yields,

$$\int_S (\nabla \times \mathbf{B}) \cdot d\mathbf{S} = \mu_0 \int_S \mathbf{J} \cdot d\mathbf{S} = \mu_0\, I_S\,, \tag{5.1.2}$$

(a) The right-hand rule of orientation in Stokes' theorem.

(b) Ampère's law relating the current flux to a line integral of magnetic field.

Figure 13: Applying Stokes' theorem to Ampère's law.

where we used that the surface integral of the current density gives the electric current flow[3] through S. Applying Stokes' theorem to the left-hand side gives the *integral form of Ampère's law*,

$$\int_{\partial S} \mathbf{B} \cdot d\mathbf{r} = \mu_0 \, I_S \, . \tag{5.1.3}$$

Similarly to Gauss' law in electrostatics, this integral form will help us build intuition for the general behaviour of magnetic fields.

Remark 5.1.2. Magnetic fields are generated by currents according to the same right-hand rule that appears in Stokes' theorem. To see this, suppose that the electric current density $\mathbf{J}$ points in the same direction across the whole surface S. Without loss of generality, choose the orientation $d\mathbf{S}$ to point along $\mathbf{J}$, in the sense that the inner product of the unit normal $\hat{\mathbf{n}}$ with $\mathbf{J}$ is everywhere positive. This makes the current flux I_S positive, so the line integral of the resulting $\mathbf{B}$ must also be positive. Therefore $\mathbf{B}$ has to circulate in the direction of $d\mathbf{r}$ determined by the right-hand rule. The resulting rule is illustrated in Figure 13b.

Remark 5.1.3. The line integral of the magnetic field over a curve C depends only on the flux of the current density it encloses. This suggests that we are free to deform C as we wish, as long as the current passing through it remains unchanged.

As in the case of Gauss' law, Ampère's law will allow us to determine $\mathbf{B}$ completely in simple geometries. Once again, we stress that the divergence free condition (3.0.4) needs to be taken into account simultaneously.

§5.2 The Infinitely Long Wire

As a first magnetostatic example, consider an infinitely long, straight wire carrying a steady current I. The system is invariant under rotations about the wire and under translations

[3] See the definition in equation (2.1.4).

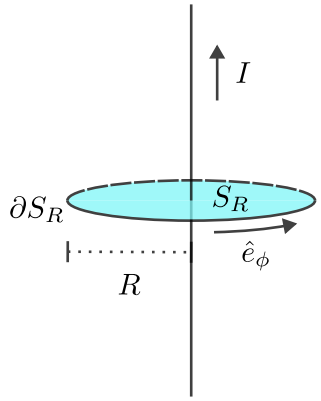


Figure 14: Geometry for an infinitely long straight wire.

along it. It is therefore natural to use cylindrical coordinates (r, ϕ, z) and place the wire on the z-axis, as shown in Figure 14.

The symmetry immediately constrains the form of $\mathbf{B}$. Translation invariance along the wire implies no z-dependence, while rotational invariance implies no ϕ-dependence. Moreover, the field lines must circle the wire, so the direction of $\mathbf{B}$ is azimuthal. We therefore make the ansatz,

$$\mathbf{B}(\mathbf{r}) = B(r)\,\hat{\mathbf{e}}_\phi\,. \tag{5.2.1}$$

It is straightforward to check that (5.2.1) satisfies the divergence-free condition (3.0.4).

To determine the function $B(r)$, apply Ampère's law (5.1.3) to a disc S_R of radius R orthogonal to the wire and centred on the z-axis. The current piercing this surface is simply $I_{S_R} = I$. On the boundary circle ∂S_R we have $r = R$ and,

$$d\mathbf{r} = R\,\hat{\mathbf{e}}_\phi\,d\phi\,, \tag{5.2.2}$$

so the line integral becomes,

$$\int_{\partial S_R} \mathbf{B}(\mathbf{r})\cdot d\mathbf{r} = \int_0^{2\pi} B(R)\,\hat{\mathbf{e}}_\phi\cdot(R\,\hat{\mathbf{e}}_\phi\,d\phi) = 2\pi R\,B(R)\,. \tag{5.2.3}$$

Ampère's law then gives $2\pi R\,B(R) = \mu_0 I$, and hence,

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0\,I}{2\pi r}\,\hat{\mathbf{e}}_\phi\,. \tag{5.2.4}$$

This result relied heavily on symmetry. Later, after introducing the vector potential and the Biot–Savart law, we will return to this example as a useful check of the general formalism.

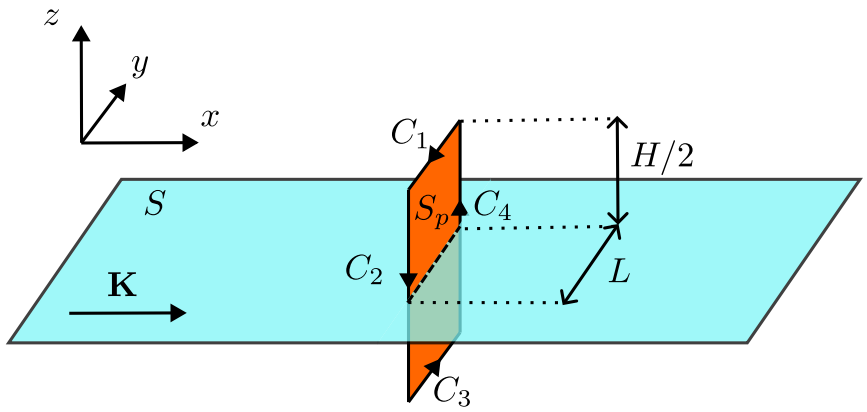


Figure 15: The infinite plane with constant surface current density.

§5.3 *Surface Currents*

We begin with the simplest magnetostatic example involving a surface current. An infinite plane S carrying a constant surface current density $\mathbf{K}$. The most obvious symmetries are translations parallel to the surface. In addition, there is a less obvious *discrete* symmetry, a 180° rotation about an axis lying in S and parallel to $\mathbf{K}$.

We work in Cartesian coordinates, taking S to be the plane $z = 0$ and choosing the surface current density to be $\mathbf{K} = K\,\hat{\mathbf{e}}_x$, as shown in Figure 15. From the example of the infinite wire in the previous section, we expect $\mathbf{B}$ to be orthogonal to $\mathbf{K}$, so it should lie in the span of $\hat{\mathbf{e}}_y$ and $\hat{\mathbf{e}}_z$. Translation invariance in the x and y directions implies that $\mathbf{B}$ can depend only on z, so we make the ansatz,

$$\mathbf{B}(\mathbf{x}) = B_y(z)\,\hat{\mathbf{e}}_y + B_z(z)\,\hat{\mathbf{e}}_z\,. \tag{5.3.1}$$

Under the 180° rotation mentioned above, $\hat{\mathbf{e}}_y \mapsto -\hat{\mathbf{e}}_y$ and $\hat{\mathbf{e}}_z \mapsto -\hat{\mathbf{e}}_z$, while $z \mapsto -z$. Invariance of the magnetic field therefore imposes,

$$B_y(-z) = -B_y(z), \qquad B_z(-z) = -B_z(z)\,. \tag{5.3.2}$$

Imposing the divergence-free condition (3.0.4) on (5.3.1) gives,

$$\partial_z B_z(z) = 0\,, \tag{5.3.3}$$

so B_z must be a constant. The only function that is both constant and odd is the trivial one, hence,

$$B_z(z) = 0\,. \tag{5.3.4}$$

The magnetic field therefore reduces to,

$$\mathbf{B}(\mathbf{x}) = B(z)\,\hat{\mathbf{e}}_y\,, \tag{5.3.5}$$

with $B(-z) = -B(z)$.

We now use the integral form of Ampère's law (5.1.3) to determine $B(z)$. Consider a rectangular surface S_p of height H and length L, with sides parallel to the z- and y-axes, intersecting the plane S as shown in Figure 15. To apply Ampère's law we need the current flux through S_p. As discussed in Section 2.2, we use (2.2.8) and integrate $\mathbf{K}$ over the intersection $S_p \cap S$, obtaining,

$$I_{S_p} = I_{S_p \cap S} = K\, L\,, \tag{5.3.6}$$

since $\mathbf{K}$ is orthogonal to $S_p \cap S$.

Next, compute the line integral around the boundary ∂S_p. Decompose $\partial S_p = C_1 \cup C_2 \cup C_3 \cup C_4$ with the orientations shown in Figure 15, and write,

$$\int_{\partial S_p} \mathbf{B}\cdot d\mathbf{r} = \int_{C_1} \mathbf{B}\cdot d\mathbf{r}_1 + \int_{C_2} \mathbf{B}\cdot d\mathbf{r}_2 + \int_{C_3} \mathbf{B}\cdot d\mathbf{r}_3 + \int_{C_4} \mathbf{B}\cdot d\mathbf{r}_4\,. \tag{5.3.7}$$

Along these segments we have,

$$d\mathbf{r}_1 = -\hat{\mathbf{e}}_y\, dy\,, \quad d\mathbf{r}_2 = -\hat{\mathbf{e}}_z\, dz\,, \quad d\mathbf{r}_3 = \hat{\mathbf{e}}_y\, dy\,, \quad d\mathbf{r}_4 = \hat{\mathbf{e}}_z\, dz\,. \tag{5.3.8}$$

The integrals along C_2 and C_4 vanish because $d\mathbf{r}$ is orthogonal to $\mathbf{B} \propto \hat{\mathbf{e}}_y$. For the remaining two segments,

$$\begin{aligned} \int_{C_1} \mathbf{B}\cdot d\mathbf{r}_1 &= -B(H/2) \int_{C_1} dy = -B(H/2)\, L\,, \\ \int_{C_3} \mathbf{B}\cdot d\mathbf{r}_3 &= B(-H/2) \int_{C_3} dy = B(-H/2)\, L\,. \end{aligned} \tag{5.3.9}$$

Ampère's law (5.1.3) then gives,

$$\begin{aligned} &-B(H/2)\, L + B(-H/2)\, L = \mu_0\, K\, L \Rightarrow \\ &B(H/2) = -\frac{\mu_0\, K}{2}\,, \end{aligned} \tag{5.3.10}$$

where we used that B is odd. Since H is arbitrary, this determines $B(z)$ for all $z \neq 0$, and we find,

$$\mathbf{B}(\mathbf{x}) = \begin{cases} -\frac{\mu_0\, K}{2}\, \hat{\mathbf{e}}_y, & z > 0 \\ \frac{\mu_0\, K}{2}\, \hat{\mathbf{e}}_y, & z < 0 \end{cases}\,, \tag{5.3.11}$$

showing that the field is constant on each side of the surface current.

The rest of this subsection is non-examinable.

As in the electrostatic example of an infinite uniformly charged plane, the field is discontinuous across the surface. Denoting by $\mathbf{B}_\pm$ the limits as $z \to 0^\pm$, the discontinuity is in the component parallel to the plane,

$$\mathbf{B}_+ - \mathbf{B}_- = -\mu_0 K \, \hat{\mathbf{e}}_y \,, \tag{5.3.12}$$

which can be written in a coordinate-free form as,

$$\boxed{\hat{\mathbf{n}} \times (\mathbf{B}_+ - \mathbf{B}_-) = \mu_0 \, \mathbf{K} \,,} \tag{5.3.13}$$

where $\hat{\mathbf{n}} = \hat{\mathbf{e}}_z$ is the unit normal to S. In contrast to the electric field, the component of $\mathbf{B}$ normal to the surface is continuous,

$$\boxed{\hat{\mathbf{n}} \cdot (\mathbf{B}_+ - \mathbf{B}_-) = 0 \,.} \tag{5.3.14}$$

These discontinuity conditions hold for a general surface S carrying a surface current density $\mathbf{K}$. The derivation parallels that of Section 4.5, but in magnetostatics the continuity properties are reversed, $\mathbf{B}$ has zero divergence but a sourced curl.

§5.4 The Vector Potential

In Section 4.6 we discussed the consequences of the curl-free nature of the electrostatic field which allows us to introduce a scalar potential. In magnetostatics the corresponding structural equation is instead the absence of magnetic charge, and this leads naturally to the introduction of the *vector potential*. The following remarks rely only on (1.0.3) and therefore remain true even in time-dependent electromagnetism.

Remark 5.4.1. By integrating both sides of equation (3.0.4) over a volume region V and using Gauss' theorem, we conclude that,

$$\boxed{\oint_S \mathbf{B} \cdot d\mathbf{S} = 0 \,,} \tag{5.4.1}$$

for any closed surface $S = \partial V$.

Remark 5.4.2. A useful consequence of (5.4.1) is that the magnetic flux through an open surface depends only on its boundary loop. Consider two open surfaces S_1 and S_2 with the same boundary curve C, $\partial S_1 = C = \partial S_2$, but with opposite induced orientations on C (as in Figure 16). The union $S_1 \cup S_2$ then forms a closed surface, so,

$$\oint_{S_1 \cup S_2} \mathbf{B} \cdot d\mathbf{S} = 0 \,. \tag{5.4.2}$$

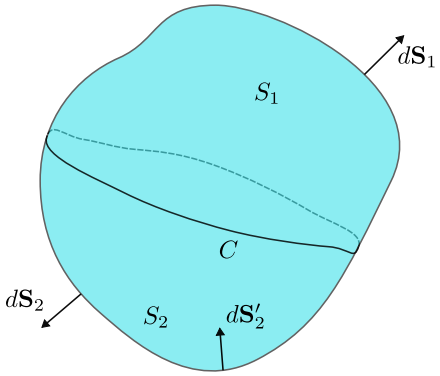


Figure 16: Magnetic flux through surfaces with a common boundary.

Decomposing the closed-surface integral gives,

$$\int_{S_1} \mathbf{B} \cdot d\mathbf{S}_1 + \int_{S_2} \mathbf{B} \cdot d\mathbf{S}_2 = 0 \Rightarrow$$
$$\int_{S_1} \mathbf{B} \cdot d\mathbf{S}_1 = -\int_{S_2} \mathbf{B} \cdot d\mathbf{S}_2 \,. \tag{5.4.3}$$

In other words, once the boundary orientation is fixed, the flux through any spanning surface is the same. This motivates defining the *magnetic flux through the loop* C as the flux through any surface with boundary C.

The condition (1.0.3) also suggests a stronger statement. Locally, a divergence-free vector field can be written as the curl of another vector field. This motivates the following definition,

$$\boxed{\mathbf{B} = \nabla \times \mathbf{A}\,,} \tag{5.4.4}$$

where $\mathbf{A}$ is called the *vector potential*. By construction, (5.4.4) automatically satisfies (1.0.3) because of property (A.0.2).

Remark 5.4.3. The vector potential is not uniquely determined by $\mathbf{B}$. Indeed, if $\chi(\mathbf{x})$ is any smooth scalar function, then the *gauge transformation*,

$$\boxed{\mathbf{A} \mapsto \mathbf{A}' = \mathbf{A} + \nabla\chi} \tag{5.4.5}$$

leaves the magnetic field unchanged,

$$\nabla \times \mathbf{A}' = \nabla \times \mathbf{A} + \nabla \times (\nabla\chi) = \nabla \times \mathbf{A} = \mathbf{B}\,,$$

using (A.0.3). This is the *gauge symmetry* of the vector potential.

Substituting (5.4.4) into Ampère's law (3.0.5) yields an equation determining $\mathbf{A}$ in terms of the current density,

$$\boxed{-\nabla^2\mathbf{A} + \nabla\,(\nabla \cdot \mathbf{A}) = \mu_0\,\mathbf{J}\,.} \tag{5.4.6}$$

To actually solve for $\mathbf{A}$, we will need to deal with the fact that $\mathbf{A}$ is not unique due to remark 5.4.3. This non-uniqueness is the magnetic analogue of the freedom to shift the scalar potential by a constant in electrostatics, but now it is a *local* freedom. We will therefore need to fix this freedom by imposing an additional *gauge fixing condition* or *gauge choice condition*,

$$\boxed{\mathcal{G}(\mathbf{A}) = 0\,,} \tag{5.4.7}$$

where $\mathcal{G}$ is a scalar function of the vector potential $\mathbf{A}$. We will see a specific gauge choice in the next subsection.

Remark 5.4.4. A classical particle of mass m and charge q experiences the Lorentz force $m\ddot{\mathbf{x}} = q\,\dot{\mathbf{x}} \times \mathbf{B}$. Using $\mathbf{B} = \nabla \times \mathbf{A}$, one can show that these equations follow from a variational principle with Lagrangian,

$$L = \frac{1}{2} m\,\dot{\mathbf{x}}^2 + q\,\dot{\mathbf{x}} \cdot \mathbf{A}(\mathbf{x}(t))\,. \tag{5.4.8}$$

Under the gauge transformation $\mathbf{A} \mapsto \mathbf{A} + \nabla\chi$, the Lagrangian changes by a total time derivative, $L \mapsto L + q\,\frac{d}{dt}\chi(\mathbf{x}(t))$, so the equations of motion are unchanged. Gauge invariance is therefore built directly into the variational description.

Remark 5.4.5. Suppose that the open surface S has $C = \partial S$ as its closed boundary. Using Stokes' theorem together with (5.4.4), we can express the flux of $\mathbf{B}$ through S as a line integral over $\mathbf{A}$,

$$\int_S \mathbf{B} \cdot d\mathbf{S} = \oint_C \mathbf{A} \cdot d\mathbf{r}\,. \tag{5.4.9}$$

This is often the most convenient way to compute magnetic fluxes in practice.

Remark 5.4.6. Given vanishing boundary conditions at infinity for the magnetic field, the vector potential is additive. If $\mathbf{B} = \mathbf{B}_1 + \mathbf{B}_2$ with corresponding potentials $\mathbf{A}_1$ and $\mathbf{A}_2$, then $\mathbf{A} = \mathbf{A}_1 + \mathbf{A}_2$ is a potential for $\mathbf{B}$ (up to a gauge transformation).

Note 5.4.7

All of the above remarks, except for equation (5.4.6), also hold in the time-dependent case, provided only that $\nabla \cdot \mathbf{B} = 0$. This is always true for the magnetic field. In contrast, the equation determining $\mathbf{A}$ from the sources changes once time dependence is included.

§5.5 Biot–Savart Law

In this section we derive an explicit expression for the magnetic field produced by a general *static* current density $\mathbf{J}(\mathbf{x})$. We do this by solving the vector potential equation (5.4.6)

for $\mathbf{A}$, and then computing $\mathbf{B}$ via (5.4.4). Throughout we assume that the current density decays sufficiently fast at infinity, in particular faster than $1/|\mathbf{x}|^2$ as $|\mathbf{x}| \to \infty$.

As we discussed in the previous section, solving (5.4.6) requires an appropriate choice of gauge. A convenient choice is the *Coulomb gauge*,

$$\mathcal{G}(\mathbf{A}) = \nabla \cdot \mathbf{A} = 0\,. \tag{5.5.1}$$

To see why this is always achievable, suppose $\tilde{\mathbf{A}}$ is any solution of (5.4.6) and define $\psi = \nabla \cdot \tilde{\mathbf{A}}$. By Remark 5.4.3 we may perform a gauge transformation,

$$\mathbf{A} = \tilde{\mathbf{A}} + \nabla\chi\,, \tag{5.5.2}$$

without changing the magnetic field. We obtain,

$$\nabla \cdot \mathbf{A} = \nabla \cdot \tilde{\mathbf{A}} + \nabla^2\chi = \psi + \nabla^2\chi\,, \tag{5.5.3}$$

so imposing the Coulomb gauge amounts to solving,

$$\nabla^2\chi = -\psi\,, \tag{5.5.4}$$

with boundary conditions such that χ vanishes at infinity. This is precisely the same Poisson problem that appears in electrostatics (see Section 4.7), after the identifications $\chi \leftrightarrow \varphi$ and $\psi \leftrightarrow \rho/\varepsilon_0$.

In Coulomb gauge, the vector potential equation (5.4.6) simplifies by imposing (5.5.1), and we obtain,

$$\nabla^2\mathbf{A} = -\mu_0\,\mathbf{J}\,. \tag{5.5.5}$$

In Cartesian coordinates this is the component-wise Poisson equation,

$$\nabla^2 A_i = -\mu_0\,J_i\,, \tag{5.5.6}$$

which is a multi-component generalisation of (4.6.4). Therefore, by the same Green's-function method used in (4.7.3), the solution for each component is,

$$A_i(\mathbf{x}) = \frac{\mu_0}{4\pi}\int_{\mathbb{R}^3} \frac{J_i(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}\,d^3\mathbf{x}'\,. \tag{5.5.7}$$

or, equivalently, in vector notation,

$$\boxed{\mathbf{A}(\mathbf{x}) = \frac{\mu_0}{4\pi}\int_{\mathbb{R}^3} \frac{\mathbf{J}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}\,d^3\mathbf{x}'\,,} \tag{5.5.8}$$

where the components of $\mathbf{A}$ pair with the components of $\mathbf{J}$ in the integrand when written in Cartesian coordinates.

Note 5.5.1

This might be the first time you encounter an integral producing a vector. A useful property is that given a function $f(\mathbf{x})$ and a constant vector $\mathbf{v}$, you can write,

$$\int_V f(\mathbf{x})\,\mathbf{v}\,d^3\mathbf{x} = \left(\int_V f(\mathbf{x})\,d^3\mathbf{x}\right)\,\mathbf{v}\,.$$

For example, if a vector field $\mathbf{C}(\mathbf{x})$ is decomposed in a Cartesian basis $\mathbf{C}(\mathbf{x}) = C_x(\mathbf{x})\,\hat{\mathbf{e}}_x + C_y(\mathbf{x})\,\hat{\mathbf{e}}_y + C_z(\mathbf{x})\,\hat{\mathbf{e}}_z$, then,

$$\int_V \mathbf{C}(\mathbf{x})\,d^3\mathbf{x} = \left(\int_V C_x(\mathbf{x})\,d^3\mathbf{x}\right)\,\hat{\mathbf{e}}_x + \left(\int_V C_y(\mathbf{x})\,d^3\mathbf{x}\right)\,\hat{\mathbf{e}}_y + \left(\int_V C_z(\mathbf{x})\,d^3\mathbf{x}\right)\,\hat{\mathbf{e}}_z\,.$$

However, decomposing in e.g. cylindrical coordinates basis $\mathbf{C}(\mathbf{x}) = C_r(\mathbf{x})\,\hat{\mathbf{e}}_r + C_\phi(\mathbf{x})\,\hat{\mathbf{e}}_\phi + C_z(\mathbf{x})\,\hat{\mathbf{e}}_z$, allows us to write only,

$$\int_V \mathbf{C}(\mathbf{x})\,d^3\mathbf{x} = \int_V C_r(\mathbf{x})\,\hat{\mathbf{e}}_r\,d^3\mathbf{x} + \int_V C_\phi(\mathbf{x})\,\hat{\mathbf{e}}_\phi\,d^3\mathbf{x} + \left(\int_V C_z(\mathbf{x})\,d^3\mathbf{x}\right)\,\hat{\mathbf{e}}_z\,,$$

since the only constant basis vector is $\hat{\mathbf{e}}_z$.

As a sanity check, we verify that (5.5.8) indeed satisfies Coulomb gauge provided $\mathbf{J}$ is divergence free, as it must be in a steady configuration. Taking the divergence with respect to $\mathbf{x}$ gives,

$$\begin{aligned}
\nabla_{\mathbf{x}}\cdot\mathbf{A}(\mathbf{x}) &= \frac{\mu_0}{4\pi}\,\nabla_{\mathbf{x}}\cdot\left(\int_{\mathbb{R}^3}\frac{\mathbf{J}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}\,d^3\mathbf{x}'\right)\\
&= \frac{\mu_0}{4\pi}\int_{\mathbb{R}^3}\mathbf{J}(\mathbf{x}')\cdot\nabla_{\mathbf{x}}\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right)\,d^3\mathbf{x}'\\
&= -\frac{\mu_0}{4\pi}\int_{\mathbb{R}^3}\mathbf{J}(\mathbf{x}')\cdot\nabla_{\mathbf{x}'}\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right)\,d^3\mathbf{x}'\\
&= \frac{\mu_0}{4\pi}\int_{\mathbb{R}^3}\left(\nabla_{\mathbf{x}'}\cdot\mathbf{J}(\mathbf{x}')\right)\,\frac{1}{|\mathbf{x}-\mathbf{x}'|}\,d^3\mathbf{x}'\,,
\end{aligned}\tag{5.5.9}$$

where in the last step we integrated by parts and used the assumed fall-off at infinity to discard the boundary term. For a steady current, $\nabla\cdot\mathbf{J} = 0$, so the final expression vanishes and (5.5.1) is satisfied.

We now compute the magnetic field by taking the curl of (5.5.8). Since $\mathbf{J}(\mathbf{x}')$ depends

only on the integration variable, the $\mathbf{x}$-curl acts only on the Green's-function kernel,

$$\begin{aligned}\mathbf{B}(\mathbf{x}) = \nabla \times_{\mathbf{x}} \mathbf{A}(\mathbf{x}) &= \frac{\mu_0}{4\pi}\int_{\mathbb{R}^3} \nabla\mathbf{x} \times \left(\frac{\mathbf{J}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}\right) d^3\mathbf{x}' \\ &= \frac{\mu_0}{4\pi}\int_{\mathbb{R}^3} \nabla_{\mathbf{x}}\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right) \times \mathbf{J}(\mathbf{x}')\, d^3\mathbf{x}' \\ &= \frac{\mu_0}{4\pi}\int_{\mathbb{R}^3} \frac{\mathbf{J}(\mathbf{x}')\times(\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^3}\, d^3\mathbf{x}' \,. \end{aligned} \tag{5.5.10}$$

This is the *Biot–Savart law* for a general current density,

$$\boxed{\mathbf{B}(\mathbf{x}) = \frac{\mu_0}{4\pi}\int_{\mathbb{R}^3} \frac{\mathbf{J}(\mathbf{x}')\times(\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^3}\, d^3\mathbf{x}' \,.} \tag{5.5.11}$$

A common special case is a current confined to a thin wire along a curve C, parametrised by $\mathbf{r}(s)$ and carrying current $I(s)$. In that case the volume current density can be written as (2.2.7). Substituting it into (5.5.11) and performing the $\mathbf{x}'$ integral using the Dirac delta gives,

$$\boxed{\mathbf{B}(\mathbf{x}) = \frac{\mu_0}{4\pi}\int_C I(s)\, \frac{\dot{\mathbf{r}}(s)\times(\mathbf{x}-\mathbf{r}(s))}{|\mathbf{x}-\mathbf{r}(s)|^3}\, ds \,.} \tag{5.5.12}$$

Example 5.5.2. *The infinitely long wire (revisited).*
We now recover the magnetic field of an infinitely long straight wire, previously obtained in Section 5.2, as a check of (5.5.12)*. Work in cylindrical coordinates and parametrise the wire by,*

$$\mathbf{r}(s) = s\,\hat{\mathbf{e}}_z \,, \qquad I(s) = I \,, \tag{5.5.13}$$

while the observation point is $\mathbf{x} = r\,\hat{\mathbf{e}}_r + z\,\hat{\mathbf{e}}_z$*. Then,*

$$\begin{aligned}\dot{\mathbf{r}}(s)\times(\mathbf{x}-\mathbf{r}(s)) &= \hat{\mathbf{e}}_z \times (r\,\hat{\mathbf{e}}_r + (z-s)\,\hat{\mathbf{e}}_z) = r\,\hat{\mathbf{e}}_z\times\hat{\mathbf{e}}_r = r\,\hat{\mathbf{e}}_\phi \,, \\ |\mathbf{x}-\mathbf{r}(s)|^2 &= (z-s)^2 + r^2 \,. \end{aligned} \tag{5.5.14}$$

Substituting into (5.5.12) *yields,*

$$\begin{aligned}\mathbf{B}(\mathbf{x}) &= \frac{\mu_0\, I}{4\pi}\left(\int_{-\infty}^{+\infty} \frac{r}{((z-s)^2+r^2)^{3/2}}\, ds\right)\hat{\mathbf{e}}_\phi \\ &\overset{s=z+r\tan\theta}{=} \frac{\mu_0\, I}{4\pi\, r}\left(\int_{-\pi/2}^{+\pi/2} \cos\theta\, d\theta\right)\hat{\mathbf{e}}_\phi \\ &= \frac{\mu_0\, I}{2\pi\, r}\,\hat{\mathbf{e}}_\phi \,, \end{aligned} \tag{5.5.15}$$

in agreement with our previous result.

§5.6 The Multipole Expansion

As in the electrostatic case discussed in Section 4.8, we can use the general solution for the vector potential (5.5.8) to study the magnetic field far away from its sources. We assume that the current density $\mathbf{J}(\mathbf{x}')$ is localised inside some bounded region V, and we evaluate the vector potential at observation points $\mathbf{x}$ satisfying $|\mathbf{x}| \gg |\mathbf{x}'|$ for all $\mathbf{x}' \in V$. Substituting the large-distance expansion (4.8.1) into (5.5.8) gives,

$$\mathbf{A}(\mathbf{x}) = \frac{\mu_0}{4\pi}\left(\frac{1}{|\mathbf{x}|}\int_V \mathbf{J}(\mathbf{x}')\,d^3\mathbf{x}' + \frac{1}{|\mathbf{x}|^3}\int_V \mathbf{J}(\mathbf{x}')\;(\mathbf{x}\cdot\mathbf{x}')\;d^3\mathbf{x}' + \cdots\right). \tag{5.6.1}$$

For later use it is convenient to write this expansion in components,

$$A_i(\mathbf{x}) = \frac{\mu_0}{4\pi}\left(\frac{1}{|\mathbf{x}|}\int_V J_i(\mathbf{x}')\,d^3\mathbf{x}' + \frac{1}{|\mathbf{x}|^3}\int_V J_i(\mathbf{x}')\,x_j\,x'_j\,d^3\mathbf{x}' + \cdots\right). \tag{5.6.2}$$

At first sight, the leading term in (5.6.2) might look like a "magnetic monopole" contribution, in tension with Remark 5.1.1. In fact, for a steady current it vanishes. To see this, use equation (3.0.1) and note that,

$$\begin{aligned}\partial'_j\left(J_j(\mathbf{x}')\,x'_i\right) &= x'_i\,\partial'_j J_j(\mathbf{x}') + J_j(\mathbf{x}')\,\partial'_j x'_i = x'_i\,\partial'_j J_j(\mathbf{x}') + J_j(\mathbf{x}')\,\delta_{ji}\\ &= x'_i\,\partial'_j J_j(\mathbf{x}') + J_i(\mathbf{x}')\\ &= J_i(\mathbf{x}')\,,\end{aligned} \tag{5.6.3}$$

where ∂' denotes derivatives with respect to the primed coordinates. Integrating over V and applying Gauss' theorem yields,

$$\int_V J_i(\mathbf{x}')\,d^3\mathbf{x}' = \int_V \partial'_j\left(J_j(\mathbf{x}')\,x'_i\right)\,d^3\mathbf{x}' = \oint_{\partial V}\left(\mathbf{J}(\mathbf{x}')\cdot\hat{\mathbf{n}}\right)\,x'_i\,dS\,. \tag{5.6.4}$$

Since the current is localised, we may choose V so that $\mathbf{J} = 0$ on ∂V, and the surface term vanishes. Therefore the monopole-like contribution in (5.6.2) is absent.

The leading non-zero term is therefore the second term in (5.6.2). To simplify it, observe that,

$$\partial'_j\left(J_j(\mathbf{x}')\,x'_i\,x'_k\right) = x'_i\,J_k(\mathbf{x}') + x'_k\,J_i(\mathbf{x}')\,, \tag{5.6.5}$$

which again follows from (3.0.1). Integrating over V and using the same boundary conditions gives,

$$\int_V x'_i\,J_k(\mathbf{x}')\,d^3\mathbf{x}' = -\int_V x'_k\,J_i(\mathbf{x}')\,d^3\mathbf{x}'\,. \tag{5.6.6}$$

We can therefore exchange the indices at the cost of a minus sign. Using this identity, we rewrite,

$$\begin{aligned}\int_V J_i(\mathbf{x}')\, x_j\, x'_j\, d^3\mathbf{x}' &= x_j \int_V J_i(\mathbf{x}')\, x'_j\, d^3\mathbf{x}' \\ &= \frac{1}{2} x_j \int_V \left(J_i(\mathbf{x}')\, x'_j + J_i(\mathbf{x}')\, x'_j\right)\, d^3\mathbf{x}' \\ &= \frac{1}{2} x_j \int_V \left(J_i(\mathbf{x}')\, x'_j - J_j(\mathbf{x}')\, x'_i\right)\, d^3\mathbf{x}' \\ &= \frac{1}{2} \int_V \left((\mathbf{x}\cdot\mathbf{x}')\; J_i(\mathbf{x}') - (\mathbf{J}(\mathbf{x}')\cdot\mathbf{x})\; x'_i\right)\, d^3\mathbf{x}' \\ &= \frac{1}{2} \int_V \left[\mathbf{x}\times(\mathbf{J}(\mathbf{x}')\times\mathbf{x}')\right]_i\, d^3\mathbf{x}' \\ &= (\mathbf{m}\times\mathbf{x})_i\,, \end{aligned} \tag{5.6.7}$$

where we introduced the *magnetic dipole moment* of the current distribution,

$$\boxed{\mathbf{m} = \frac{1}{2}\int_V (\mathbf{x}'\times\mathbf{J}(\mathbf{x}'))\; d^3\mathbf{x}'\,.} \tag{5.6.8}$$

We can now read off the leading long-distance behaviour of the vector potential from (5.6.2),

$$\boxed{\mathbf{A}(\mathbf{x}) = \frac{\mu_0}{4\pi}\frac{\mathbf{m}\times\mathbf{x}}{|\mathbf{x}|^3} + \cdots\,.} \tag{5.6.9}$$

The corresponding magnetic field follows from (5.4.4). Using the vector identity (A.0.6), we obtain,

$$\mathbf{B}(\mathbf{x}) = \nabla\times\mathbf{A}(\mathbf{x}) = \frac{\mu_0}{4\pi}\left[\mathbf{m}\,\nabla\cdot\left(\frac{\mathbf{x}}{|\mathbf{x}|^3}\right) - (\mathbf{m}\cdot\nabla)\left(\frac{\mathbf{x}}{|\mathbf{x}|^3}\right)\right] + \cdots\,. \tag{5.6.10}$$

For $\mathbf{x}\neq 0$ we have $\nabla\cdot(\mathbf{x}/|\mathbf{x}|^3) = 0$, so the first term vanishes, and the second term gives the familiar dipole field,

$$\boxed{\mathbf{B}(\mathbf{x}) = \frac{\mu_0}{4\pi}\frac{1}{|\mathbf{x}|^3}\left(\frac{3\,(\mathbf{m}\cdot\mathbf{x})}{|\mathbf{x}|^2}\,\mathbf{x} - \mathbf{m}\right)\,,} \tag{5.6.11}$$

which has the same functional form as the electric dipole contribution in (4.8.4).

Remark 5.6.1. We have derived (5.6.11) by analysing the long-distance behaviour of classical current distributions. However, magnetic dipole moments also arise more fundamentally. In quantum mechanics, particles with spin typically carry an intrinsic magnetic dipole moment.

We now compute $\mathbf{m}$ for a current confined to a thin wire along a closed curve C. Using the curve current density (2.2.7) in (5.6.8) gives,

$$\boxed{\mathbf{m} = \frac{1}{2}\int_C I(s)\ (\mathbf{r}(s) \times \dot{\mathbf{r}}(s))\ ds\,.} \tag{5.6.12}$$

Example 5.6.2. *Magnetic moment of a circular loop*
We consider a circular loop of radius R carrying a constant current I. Place its centre at the origin and take it to lie in the $z = 0$ plane. If the current runs counterclockwise in the x–y plane, a convenient parametrisation is,

$$\mathbf{r}(s) = R\ (\cos(s)\ \hat{\mathbf{e}}_x + \sin(s)\ \hat{\mathbf{e}}_y)\ , \tag{5.6.13}$$

with $s \in [0, 2\pi)$. Then,

$$\mathbf{r}(s) \times \dot{\mathbf{r}}(s) = R^2\,\hat{\mathbf{e}}_z\,, \tag{5.6.14}$$

and (5.6.12) *yields,*

$$\mathbf{m} = \frac{R^2 I}{2}\int_0^{2\pi} ds\,\hat{\mathbf{e}}_z = I\,S\,\hat{\mathbf{e}}_z\,, \tag{5.6.15}$$

where $S = \pi R^2$ is the area enclosed by the loop.

This example shows that the dipole moment is determined by a simple geometric property of the loop. More generally, for any planar closed curve C carrying a constant current I, the magnetic dipole moment can be written as,

$$\boxed{\mathbf{m} = \frac{I}{2}\oint_C (\mathbf{r}(s) \times \dot{\mathbf{r}}(s))\ ds = I\,S_C\,\hat{\mathbf{n}}\,,} \tag{5.6.16}$$

where S_C is the area enclosed by C and $\hat{\mathbf{n}}$ is the unit normal to the plane, determined by the right-hand rule.

§5.7 Force and Potential for a Dipole

In this section we study how a spatially localised current distribution $\mathbf{J}$ interacts with an external magnetic field $\mathbf{B}$. Concretely, we analyse the Lorentz force,

$$\mathbf{F} = \int_V \mathbf{J}(\mathbf{x}) \times \mathbf{B}(\mathbf{x})\, d^3\mathbf{x} \tag{5.7.1}$$

exerted on the current distribution. We assume that we are sufficiently far from the sources of $\mathbf{B}$ so that, throughout the region V occupied by the current,

$$\nabla \times \mathbf{B}(\mathbf{x}) = 0\,, \tag{5.7.2}$$

for any $\mathbf{x} \in V$.

To make progress with the integral in (5.7.1), we Taylor expand the magnetic field about a reference point $\mathbf{x}_0$ associated with the distribution e.g. its centre of mass. For notational simplicity we choose coordinates so that $\mathbf{x}$ denotes the displacement from $\mathbf{x}_0$. The expansion reads,

$$\mathbf{B}(\mathbf{x}) = \mathbf{B}(\mathbf{x}_0) + ((\mathbf{x}-\mathbf{x}_0)\cdot\nabla)\,\mathbf{B}(\mathbf{x}_0) + \cdots\,. \tag{5.7.3}$$

Substituting this expansion into (5.7.1), and using (5.6.4) to drop the leading term proportional to $\mathbf{B}(\mathbf{x}_0)$, we obtain,

$$\mathbf{F} = \int_V \mathbf{J}(\mathbf{x}) \times ((\mathbf{x}\cdot\nabla)\,\mathbf{B}(\mathbf{x}_0))\; d^3\mathbf{x}\,. \tag{5.7.4}$$

Writing this expression in component notation gives,

$$\begin{aligned}
F_i &= \varepsilon_{ijk} \int_V J_j(\mathbf{x})\, x_l\, \partial_{x_{0l}} B_k(\mathbf{x}_0)\, d^3\mathbf{x} \overset{(5.7.2)}{=} \varepsilon_{ijk} \int_V J_j(\mathbf{x})\, x_l\, \partial_{x_{0k}} B_l(\mathbf{x}_0)\, d^3\mathbf{x} \\
&= \varepsilon_{ijk}\, \partial_{x_{0k}} \int_V J_j(\mathbf{x})\, x_l\, B_l(\mathbf{x}_0)\, d^3\mathbf{x} \overset{(5.6.6)}{=} \frac{1}{2}\varepsilon_{ijk}\, \partial_{x_{0k}} \int_V \left(J_j(\mathbf{x})\, x_l - J_l(\mathbf{x})\, x_j\right)\, B_l(\mathbf{x}_0)\, d^3\mathbf{x} \\
&= \frac{1}{2}\varepsilon_{ijk}\, \partial_{x_{0k}} \int_V \left((\mathbf{x}\cdot\mathbf{B}(\mathbf{x}_0))\; J_j(\mathbf{x}) - (\mathbf{J}(\mathbf{x})\cdot\mathbf{B}(\mathbf{x}_0))\; x_j\right)\; d^3\mathbf{x} \\
&= \frac{1}{2}\varepsilon_{ikj}\, \partial_{x_{0k}} \left[\mathbf{B}(\mathbf{x}_0) \times \int_V \mathbf{x}\times\mathbf{J}(\mathbf{x})\, d^3\mathbf{x}\right]_j \overset{(5.6.8)}{=} \varepsilon_{ikj}\, \partial_{x_{0k}} \left[\mathbf{B}(\mathbf{x}_0)\times\mathbf{m}\right]_j \\
&= \left[\nabla\times(\mathbf{B}\times\mathbf{m})\right]_i \overset{(A.0.6)}{=} \left[(\mathbf{m}\cdot\nabla)\,\mathbf{B} - (\nabla\cdot\mathbf{B})\,\mathbf{m}\right]_i \overset{(3.0.4)}{=} m_k\, \partial_{x_{0k}} B_i\,(\mathbf{x}_0) \\
&\overset{(5.7.2)}{=} m_k\, \partial_{x_{0i}} B_k\,(\mathbf{x}_0) = \partial_{x_{0i}}\,(\mathbf{m}\cdot\mathbf{B}(\mathbf{x}_0))\;.
\end{aligned} \tag{5.7.5}$$

Collecting the result, we can identify the force as the gradient of $\mathbf{m}\cdot\mathbf{B}$. Equivalently, we can write the force as $\mathbf{F} = -\nabla U$ for the potential energy,

$$\boxed{U = -\mathbf{m}\cdot\mathbf{B}\,.} \tag{5.7.6}$$

This expression is ubiquitous in physics.

Remark 5.7.1. As we mentioned in Remark 5.6.1, dipole moments can be intrinsic properties of fundamental particles. The potential (5.7.6) must then be included in addition to the Lorentz force for a moving charged particle. In particular, the Lagrangian (5.4.8) does not capture the full dynamics of a particle with spin.

Remark 5.7.2. The potential (5.7.6) shows that a magnetic dipole can reduce its potential energy in two ways. It can move to a region where the magnetic field is stronger, and it can rotate to increase the alignment of $\mathbf{m}$ with $\mathbf{B}$.

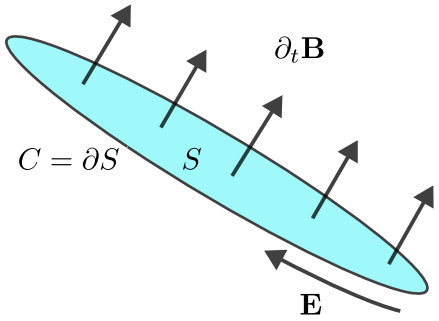


Figure 17: Illustration of Faraday's law of induction.

§6 Electrodynamics

In this section we study phenomena that rely on the time dependence of the electromagnetic field. We begin by revisiting (1.0.2) and seeing how a time-dependent magnetic field induces an electric field, even in the absence of electric charge. This is precisely the ingredient we were missing to understand how energy can be stored in a magnetic field, even in a static configuration. We then move on to the propagation of the electromagnetic field in vacuum where we will derive the wave equation and identify its solutions with light.

Finally, we return to the scalar and vector potentials in the time-dependent theory. Gauge symmetry is still present, and by choosing a gauge we can write the dynamics in terms of two unconstrained equations of motion. We will also derive these equations from an action principle.

§6.1 Faraday's Law of Induction

We now examine more closely the time-dependent Maxwell equation (1.0.2). It tells us that a time-varying magnetic field $\mathbf{B}$ induces an electric field $\mathbf{E}$, even in regions with no electric charge. Unlike the electrostatic field discussed in Section 4, this induced electric field generally has a non-zero curl, and therefore circulates around closed curves.

To make this statement precise, let C be a closed curve that bounds an oriented surface S, so that $C = \partial S$. Integrating both sides of (1.0.2) over S gives,

$$\int_S (\nabla \times \mathbf{E}) \cdot d\mathbf{S} = -\int_S \partial_t \mathbf{B} \cdot d\mathbf{S} \Rightarrow$$
$$\int_S (\nabla \times \mathbf{E}) \cdot d\mathbf{S} = -\partial_t \left(\int_S \mathbf{B} \cdot d\mathbf{S} \right) . \tag{6.1.1}$$

Applying Stokes' theorem to the left-hand side gives the circulation of $\mathbf{E}$ around C. Moreover, by Remark 5.4.2 the magnetic flux through S depends only on its boundary C.

Denoting this flux by $\Phi_C = \int_S \mathbf{B} \cdot d\mathbf{S}$, we obtain,

$$\boxed{\oint_C \mathbf{E} \cdot d\mathbf{r} = -\partial_t \Phi_C \,,} \tag{6.1.2}$$

known as *Faraday's law of induction*. A changing magnetic flux through a loop necessarily induces an electric field with non-trivial circulation around that loop. As we will discuss in the next section, the minus sign is physically crucial, it encodes *Lenz's law*.

To connect this to work and energy, take a small charge δq and move it once around the loop C. The work δW done by the electric force along this closed path is,

$$\delta W = \oint_C \mathbf{F} \cdot d\mathbf{r} = \delta q \oint_C \mathbf{E} \cdot d\mathbf{r} = \delta q \, \mathcal{E}_C \,, \tag{6.1.3}$$

where we defined the *electromotive force* $\mathcal{E}_C$ as the work done per unit charge in one circuit around C. With this definition, Faraday's law becomes,

$$\boxed{\mathcal{E}_C = -\partial_t \Phi_C \,.} \tag{6.1.4}$$

Note 6.1.1

The existence of an electric field with non-zero line integral (emf),

$$\mathcal{E}_C := \oint_C \mathbf{E} \cdot d\mathbf{r}$$

around a closed loop C is central to technological applications. Indeed, if a conducting loop of total resistance R follows the curve C, then a changing magnetic flux Φ_C produces an induced emf via Faraday's law,

$$\mathcal{E}_C = -\frac{d\Phi_C}{dt},$$

which drives an induced current. In a simple quasi-static model (neglecting self-inductance), Ohm's law gives

$$I_{\text{ind}} = \frac{\mathcal{E}_C}{R}.$$

This is closely analogous to a circuit containing a resistor and a battery. In electrostatics the electric field is conservative, $\mathbf{E} = -\nabla\varphi$, and the potential difference between two points A and B is,

$$\Delta\varphi = \varphi_B - \varphi_A = -\int_A^B \mathbf{E} \cdot d\mathbf{r}.$$

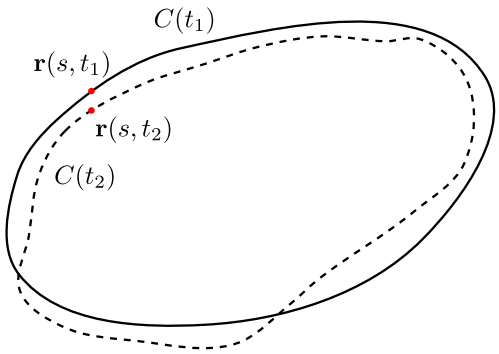


Figure 18: Illustration of a time dependent loop $C(t)$.

For an open wire of resistance R connected across a battery of voltage $\Delta\varphi$, the current is the familiar $I = \Delta\varphi/R$. In the induction case, by contrast, the emf $\mathcal{E}_C$ is generated by a non-conservative electric field, and it is the loop integral $\oint_C \mathbf{E}\cdot d\mathbf{r}$ (not a potential difference) that drives the current.

So far we have considered only time dependent magnetic field fluxes for stationary loops C. However, the magnetic field flux can also have time dependence coming from continuous deformations of C with time, as shown in Figure 18. In this case, we wouldn't be able to pull the time derivative outside the integral in equation (6.1.1), since the surface S would also depend on time. It would therefore seem that equation (6.1.4) does not hold in this case. As we will now see, Faraday's law (6.1.4) is still true, after a more general definition of the electromotive force $\mathcal{E}_C$.

To make the discussion more precise, we consider a family of closed loops $C(t)$ parametrised by $\mathbf{r}(s,t)$ with parameter $s \in [0,1)$ such that $\mathbf{r}(0,t) = \mathbf{r}(1,t)$ for all t. At a fixed time t, a charge δq moving with the wire has instantaneous velocity $\mathbf{v}(s,t) = \partial_t \mathbf{r}(s,t)$ and therefore the magnetic part of the Lorentz force $\delta\mathbf{F}_L = \delta q\,(\mathbf{E} + \mathbf{v}\times\mathbf{B})$ will be active. Dragging the charge around the loop would produce work the generalised expression,

$$\delta W = \int_{C(t)} \delta\mathbf{F}_L\cdot d\mathbf{r} = \int_{C(t)} \delta\mathbf{F}_L\cdot\partial_s\mathbf{r}(s,t)\,ds = \delta q\int_{C(t)} (\mathbf{E}+\mathbf{v}\times\mathbf{B})\cdot\partial_s\mathbf{r}(s,t)\,ds\,.$$

Since the electromotive force is the work done per unit charge, in this case it is appropriate to write,

$$\mathcal{E}_{C(t)} = \int_{C(t)} (\mathbf{E}+\mathbf{v}\times\mathbf{B})\cdot\partial_s\mathbf{r}(s,t)\,ds = \int_{C(t)} (\mathbf{E}+\mathbf{v}\times\mathbf{B})\cdot d\mathbf{r}\,. \tag{6.1.5}$$

The term $\oint \mathbf{E}\cdot d\mathbf{r}$ is often called the *transformer emf*, while $\oint(\mathbf{v}\times\mathbf{B})\cdot d\mathbf{r}$ is the *motional emf*.

We will now compute the time derivative of the flux $\Phi_{C(t)}$, taking into account the time

dependence of the loop $C(t)$. We start by writing the flux as,

$$\Phi_{C(t)} = \int_{S(t)} \mathbf{B}(t,\mathbf{x}) \cdot d\mathbf{S}\,,$$

where $S(t)$ is any surface with boundary $C(t) = \partial S(t)$. To make progress, we will use (5.4.9) to write the flux as the line integral of the vector potential $\mathbf{A}(t,\mathbf{x})$[4],

$$\Phi_{C(t)} = \oint_{C(t)} \mathbf{A} \cdot d\mathbf{r} = \int_0^1 A_i(t,\mathbf{r}(s,t))\,\partial_s r_i(s,t)\,ds\,.$$

The total time derivative gives three different contributions,

$$\begin{aligned}\frac{d}{dt}\Phi_{C(t)} &= \int_0^1 \partial_t A_i(t,\mathbf{r}(s,t))\,\partial_s r_i(s,t)\,ds + \int_0^1 \partial_j A_i(t,\mathbf{r}(s,t))\,\partial_t r_j(s,t)\,\partial_s r_i(s,t)\,ds \\ &\quad + \int_0^1 A_i(t,\mathbf{r}(s,t))\,\partial_s\partial_t r_i(s,t)\,ds\,. \end{aligned} \tag{6.1.6}$$

For the first term, we can write,

$$\begin{aligned}&\int_0^1 \partial_t \mathbf{A}(t,\mathbf{r}(s,t)) \cdot \partial_s \mathbf{r}(s,t)\,ds = \oint_{C(t)} \partial_t \mathbf{A} \cdot d\mathbf{r} \\ &= \int_{S(t)} (\nabla \times \partial_t \mathbf{A}) \cdot d\mathbf{S} = \int_{S(t)} \partial_t\,(\nabla \times \mathbf{A}) \cdot d\mathbf{S} = \int_{S(t)} \partial_t \mathbf{B} \cdot d\mathbf{S} \\ &\overset{(1.0.2)}{=} -\int_{S(t)} (\nabla \times \mathbf{E}) \cdot d\mathbf{S} = -\oint_{C(t)} \mathbf{E} \cdot d\mathbf{r}\,, \end{aligned} \tag{6.1.7}$$

where moving to the second line we used Stokes' theorem and we did the same in the last equality. After integrating by parts the third term in equation (6.1.6), its combination with the second term gives,

$$\begin{aligned}&\int_0^1 \partial_j A_i(t,\mathbf{r}(s,t))\,\partial_t r_j(s,t)\,\partial_s r_i(s,t)\,ds - \int_0^1 \partial_j A_i(t,\mathbf{r}(s,t))\,\partial_s r_j(s,t)\,\partial_t r_i(s,t)\,ds \\ &= \int_0^1 \left(\partial_j A_i(t,\mathbf{r}(s,t)) - \partial_i A_j(t,\mathbf{r}(s,t))\right)\,\partial_t r_j(s,t)\,\partial_s r_i(s,t)\,ds \\ &= \int_0^1 \varepsilon_{jik} B_k(t,\mathbf{r}(s,t))\,\partial_t r_j(s,t)\,\partial_s r_i(s,t)\,ds = \int_0^1 \varepsilon_{jik} B_k(t,\mathbf{r}(s,t))\,v_j(s,t)\,\partial_s r_i(s,t)\,ds \\ &= -\int_0^1 (\mathbf{v}(s,t) \times \mathbf{B}(t,\mathbf{r}(s,t))) \cdot \partial_s \mathbf{r}(s,t)\,ds = -\oint_{C(t)} (\mathbf{v} \times \mathbf{B}) \cdot d\mathbf{r}\,, \end{aligned} \tag{6.1.8}$$

where going from the second line to the third line we used that,

$$B_k = \varepsilon_{kij}\,\partial_i A_j \Rightarrow \partial_i A_j - \partial_j A_i = \varepsilon_{ijk}\,B_k.$$

[4]The derivation of equation (5.4.9) relied on being able to write $\mathbf{B} = \nabla \times \mathbf{A}$. For the time dependent case, we give the full argument in Section 6.7

Putting everything together, we see that the time derivative of the magnetic flux is,

$$\frac{d}{dt}\Phi_{C(t)} = -\oint_{C(t)} \mathbf{E}\cdot d\mathbf{r} - \oint_{C(t)} (\mathbf{v}\times\mathbf{B})\cdot d\mathbf{r} = -\mathcal{E}_{C(t)}\,, \tag{6.1.9}$$

after a simple comparison with the electromotive force in (6.1.5).

In the next section we will discuss the physical implications of (6.1.4), and in particular the meaning of the minus sign. Most importantly, it explains (and quantifies) why we must supply energy in order to build up a magnetic field. This is precisely why we postponed the discussion of magnetic-field energy in Section 5, where we ignored time-dependent effects.

§6.2 Energy Stored in a Magnetic Field

As we saw in the previous section, a time-varying magnetic flux through a loop induces an electric field that can do work on charges moving around the loop. We now consider a current $I(t)$ flowing around a fixed loop C. This current generates a magnetic field $\mathbf{B}(t,\mathbf{x})$ and hence a magnetic flux $\Phi_C(t)$ through any surface spanning C (with orientation fixed by the right-hand rule). Throughout this section we will work in the *quasi-static* regime, where $I(t)$ varies slowly in time. Physically, this means we can neglect energy carried away as electromagnetic radiation.

In the time-dependent theory we cannot use the magnetostatic equation (3.0.5). Instead, we have to use equation law (1.0.4). In the quasi-static regime, the displacement-current term $\mu_0\varepsilon_0\,\partial_t\mathbf{E}$ is suppressed. Roughly speaking, the induced electric field scales like $\mathbf{E}\propto\dot{I}(t)$, so $\partial_t\mathbf{E}\propto\ddot{I}(t)$, and by making the variation of $I(t)$ sufficiently slow we can make this contribution arbitrarily small compared to $\mu_0\,\mathbf{J}$. Dropping it to leading order, we approximate,

$$\nabla\times\mathbf{B} = \mu_0\,\mathbf{J} + \mu_0\,\varepsilon_0\,\partial_t\mathbf{E} \simeq \mu_0\,\mathbf{J}\,. \tag{6.2.1}$$

In words, we can determine the magnetic field by using the techniques we developed in magnetostatics by assuming that the time changes of $\mathbf{J}$ are propagated instantaneously everywhere in space.

Looking at the Biot-Savart law, we see that for a fixed geometry the magnetic field is proportional to the current, instantaneously, in the quasi-static regime. In particular, the magnetic flux through the loop is proportional to $I(t)$, so we can write,

$$\Phi_C(t) = L\,I(t)\,, \tag{6.2.2}$$

where the constant of proportionality L is the *inductance* of the loop, determined entirely by its geometry. For the purposes of this section we will not need an explicit expression for L.

We now attempt to increase the current from $I(t)$ to $I(t+\delta t)$ over a short time interval δt. This increases the flux to $\Phi_C(t+\delta t)$ and, by Faraday's law, induces an electric field $\mathbf{E}$

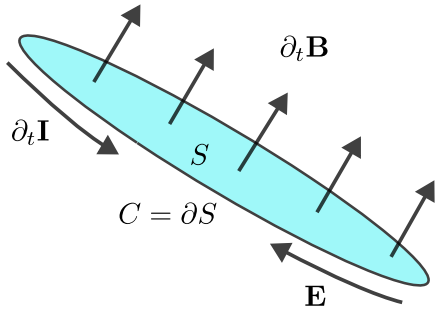


Figure 19: Electric field opposing the increase of current in a loop C.

around the loop C, as shown in Figure 19. The crucial point is that the induced field opposes the increase of flux (Lenz's law), encoded in the minus sign in (6.1.2). Consequently, we must do work to drive the current up.

During the interval δt, a charge,

$$\delta q = I(t)\,\delta t \tag{6.2.3}$$

is transported around the loop. The work done by the induced electric field on this charge is,

$$\begin{aligned}\delta W_{\text{ind}} &= \mathcal{E}_C\,\delta q = \mathcal{E}_C\,I(t)\,\delta t \overset{(6.1.4)}{=} -\partial_t \Phi_C(t)\,I(t)\,\delta t \overset{(6.2.2)}{=} -L\,\dot{I}(t)\,I(t)\,\delta t \\ &= -\frac{1}{2}\,L\,\frac{d}{dt} I(t)^2\,\delta t\,.\end{aligned} \tag{6.2.4}$$

For an increasing current, $\delta W_{\text{ind}} < 0$: the induced field removes energy from the charges. Therefore, the *external* work that must be supplied is,

$$\delta W_{\text{ext}} = -\delta W_{\text{ind}} = \frac{1}{2}\,L\,\frac{d}{dt} I(t)^2\,\delta t\,. \tag{6.2.5}$$

This shows that the energy U stored in the magnetic field grows at the rate,

$$\frac{d}{dt} U = \frac{1}{2}\,L\,\frac{d}{dt} I^2(t)\,, \tag{6.2.6}$$

and, taking $U = 0$ when $I = 0$, we obtain,

$$U = \frac{1}{2}\,L\,I^2(t) \overset{(6.2.2)}{=} \frac{1}{2}\,\Phi_C(t)\,I(t)\,. \tag{6.2.7}$$

This is already an important result, but we would like to express it directly in terms of

the magnetic field $\mathbf{B}$. Using (5.4.9) and parametrising the loop C by $\mathbf{r}(s)$, we can write,

$$\begin{aligned} U_{mag} &= \frac{1}{2} I(t) \oint_C \mathbf{A}(t,\mathbf{r}) \cdot d\mathbf{r} = \frac{1}{2} I(t) \oint_C \mathbf{A}(t,\mathbf{r}(s)) \cdot \dot{\mathbf{r}}\, ds \\ &= \frac{1}{2} \oint_C I(t) \left(\int_{\mathbb{R}^3} \mathbf{A}(t,\mathbf{x})\, \delta(\mathbf{x}-\mathbf{r}(s))\, d^3\mathbf{x} \right) \cdot \dot{\mathbf{r}}\, ds \\ &= \frac{1}{2} \int_{\mathbb{R}^3} \mathbf{A}(t,\mathbf{x}) \cdot \left(\oint_C I(t)\, \delta(\mathbf{x}-\mathbf{r}(s))\, \dot{\mathbf{r}}\, ds \right) d^3\mathbf{x} \\ &= \frac{1}{2} \int_{\mathbb{R}^3} \mathbf{A}(t,\mathbf{x}) \cdot \mathbf{J}(t,\mathbf{x})\, d^3\mathbf{x}\,, \end{aligned} \tag{6.2.8}$$

where in the last step we used (2.2.7) to recognise the loop integral as the three-dimensional current density of the wire, and from the first to the second line we inserted the Dirac delta.

To proceed, we would like to express $\mathbf{J}$ in terms of $\mathbf{B}$ through equation (6.2.1) allowing us to write,

$$\begin{aligned} U_{mag} &= \frac{1}{2\mu_0} \int_{\mathbb{R}^3} \mathbf{A}(t,\mathbf{x}) \cdot (\nabla \times \mathbf{B}(t,\mathbf{x}))\; d^3\mathbf{x} \\ &= \frac{1}{2\mu_0} \int_{\mathbb{R}^3} (\nabla \cdot (\mathbf{B}(t,\mathbf{x}) \times \mathbf{A}(t,\mathbf{x})) + \mathbf{B}(t,\mathbf{x}) \cdot (\nabla \times \mathbf{A}(t,\mathbf{x})))\; d^3\mathbf{x} \\ &= \frac{1}{2\mu_0} \int_{\mathbb{R}^3} (\nabla \cdot (\mathbf{B}(t,\mathbf{x}) \times \mathbf{A}(t,\mathbf{x})) + \mathbf{B}(t,\mathbf{x}) \cdot \mathbf{B}(t,\mathbf{x}))\; d^3\mathbf{x}\,, \end{aligned} \tag{6.2.9}$$

where we used (5.4.4). Assuming the fields fall off sufficiently fast at infinity, the integral of the divergence term reduces to a boundary term at infinity, which vanishes. We therefore obtain the elegant expression,

$$\boxed{U_{mag} = \frac{1}{2\mu_0} \int_{\mathbb{R}^3} |\mathbf{B}(t,\mathbf{x})|^2\; d^3\mathbf{x}\,.} \tag{6.2.10}$$

Note 6.2.1

For both the electric and the magnetic field, we have used a physical "work to assemble" argument to compute the stored energy. Later we will introduce an action principle for electromagnetism; within classical field theory one can also obtain the same result by constructing the energy–momentum tensor of the electromagnetic field.

§6.3 The Displacement Current

The last Maxwell equation we want to examine is (1.0.4). In the form we have written it, the time derivative of the electric field appears as a kind of "current", the *displacement*

current. The name is somewhat misleading as this term is not associated with the transport of conserved electric charge. Nevertheless, adding it was one of Maxwell's most significant contributions. As we will now see, without it Maxwell's equations are incompatible with the charge continuity equation (2.1.8).

Suppose that all we knew from experiment were equations (1.0.1), (1.0.2) and (1.0.3), and that we were trying to guess the remaining equation in the form,

$$\nabla \times \mathbf{B} = \mu_0 \, \mathbf{J} + \mathbf{X} \,, \tag{6.3.1}$$

for some vector field $\mathbf{X}$ that may depend on both space and time. Using this together with (1.0.1) we can write,

$$\rho = \varepsilon_0 \, \nabla \cdot \mathbf{E} \,, \tag{6.3.2}$$

$$\mathbf{J} = \frac{1}{\mu_0} \left(\nabla \times \mathbf{B} - \mathbf{X} \right) \,. \tag{6.3.3}$$

Substituting into the continuity equation (2.1.8) gives,

$$\begin{aligned} &\varepsilon_0 \, \partial_t \left(\nabla \cdot \mathbf{E} \right) + \frac{1}{\mu_0} \, \nabla \cdot \left(\nabla \times \mathbf{B} - \mathbf{X} \right) = 0 \Rightarrow \\ &\nabla \cdot \mathbf{X} = \varepsilon_0 \, \mu_0 \, \nabla \cdot \partial_t \mathbf{E} \,, \end{aligned} \tag{6.3.4}$$

where we used (A.0.2). This suggests that the most general form of $\mathbf{X}$ is,

$$\mathbf{X} = \varepsilon_0 \, \mu_0 \, \partial_t \mathbf{E} + \nabla \times \mathbf{Z} \,, \tag{6.3.5}$$

for some vector field $\mathbf{Z}$, since the last term is the most general divergence-free contribution. Plugging this back into our original guess yields,

$$\nabla \times \left(\mathbf{B} - \mathbf{Z} \right) = \mu_0 \, \mathbf{J} + \varepsilon_0 \, \mu_0 \, \partial_t \mathbf{E} \,. \tag{6.3.6}$$

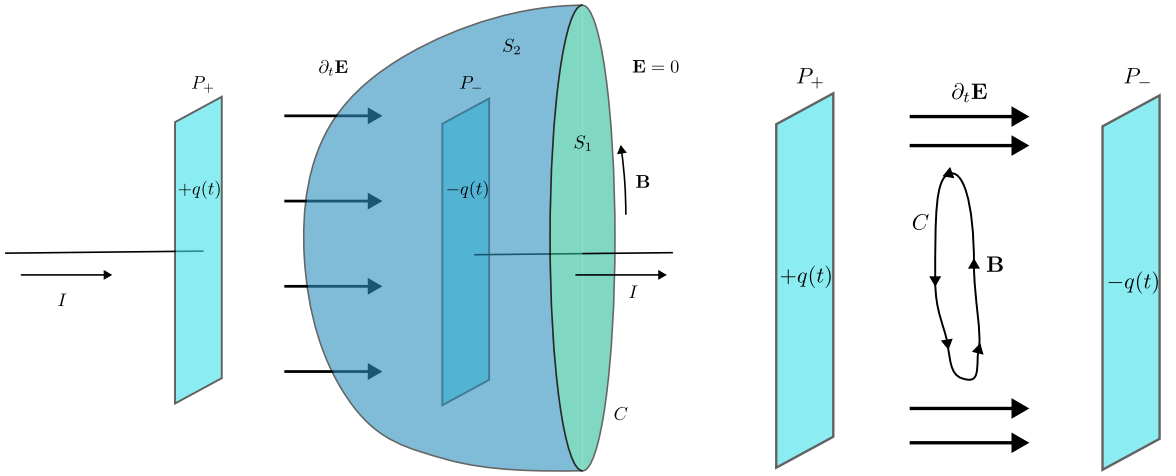
Equation (6.3.6) shows that, aside from the a priori undetermined term involving $\mathbf{Z}$, a contribution proportional to $\partial_t \mathbf{E}$ is required for consistency with charge conservation. The argument above used differential calculus. It is also instructive to see the same necessity through a thought experiment.

Consider the setup in Figure 20a. Two large conducting plates P_+ and P_- carry charges $+q(t)$ and $-q(t)$, respectively. A wire transfers charge from one plate to the other, so there is an electric current I in the wire. Charge conservation implies $I = \dot{q}(t)$.

Because the plates carry opposite charge, a time-dependent electric field $\mathbf{E}$ is established between them. Away from the plates the field falls off rapidly.[5] The current in the wire also produces a magnetic field circulating around the wire, as shown in Figure 20a.

Now choose a closed loop C that encloses the wire and lies far from the plates. For the moment, allow $\mathbf{Z}$ in (6.3.6) to be non-zero and consider the line integral of $\mathbf{B} - \mathbf{Z}$

[5]For two infinite parallel plates with opposite uniform surface charge densities, the electric field is non-zero only in the region between the plates.



(a) Necessity of the displacement current. (b) Experimental determination of $\mathbf{Z}$.

Figure 20: Maxwell's displacement current.

around C. By Stokes' theorem, this line integral equals the flux of $\nabla \times (\mathbf{B} - \mathbf{Z})$ through any open surface with boundary C. Choose two such surfaces. First, let S_1 intersect the wire while remaining far from the plates. Then the current flux through S_1 is I, while the electric field through S_1 is negligible. Second, let S_2 bulge out between the plates. Then the current flux through S_2 vanishes, while the electric field through S_2 is non-zero and time dependent. We can therefore write,

$$\begin{aligned} \oint_C (\mathbf{B} - \mathbf{Z}) \cdot d\mathbf{r} &= \int_{S_1} (\nabla \times (\mathbf{B} - \mathbf{Z})) \cdot d\mathbf{S} \overset{(6.3.6)}{=} \mu_0 \varepsilon_0 \int_{S_1} \partial_t \mathbf{E} \cdot d\mathbf{S} + \mu_0 \int_{S_1} \mathbf{J} \cdot d\mathbf{S} \\ &= \mu_0 \int_{S_1} \mathbf{J} \cdot d\mathbf{S} = \mu_0 \, I \,, \end{aligned} \tag{6.3.7}$$

and also,

$$\begin{aligned} \oint_C (\mathbf{B} - \mathbf{Z}) \cdot d\mathbf{r} &= \int_{S_2} (\nabla \times (\mathbf{B} - \mathbf{Z})) \cdot d\mathbf{S} \overset{(6.3.6)}{=} \mu_0 \varepsilon_0 \int_{S_2} \partial_t \mathbf{E} \cdot d\mathbf{S} + \mu_0 \int_{S_2} \mathbf{J} \cdot d\mathbf{S} \\ &= \varepsilon_0 \mu_0 \, \partial_t \int_{S_2} \mathbf{E} \cdot d\mathbf{S} \,. \end{aligned} \tag{6.3.8}$$

These two expressions refer to different physics, current flux through S_1 versus electric-flux change through S_2, but they must agree because they compute the same line integral around the same loop.

To relate them, consider the closed surface $S_1 \cup S_2$. The only net charge enclosed is the charge $-q(t)$ on the plate P_-. Applying Gauss' law and keeping track of orientations

gives,

$$\begin{aligned}
&\int_{S_1\cup S_2} \mathbf{E}\cdot d\mathbf{S}' = -\frac{q(t)}{\varepsilon_0} \Rightarrow \\
&\int_{S_1} \mathbf{E}\cdot d\mathbf{S}' + \int_{S_2} \mathbf{E}\cdot d\mathbf{S}' = -\frac{q(t)}{\varepsilon_0} \Rightarrow \\
&\int_{S_2} \mathbf{E}\cdot d\mathbf{S}' = -\frac{q(t)}{\varepsilon_0} \Rightarrow \\
&\int_{S_2} \mathbf{E}\cdot d\mathbf{S} = \frac{q(t)}{\varepsilon_0}\,. \qquad (6.3.9)
\end{aligned}$$

The extra minus sign between the third and the last line comes from the fact that, in Gauss' theorem, $d\mathbf{S}'$ must point outward from the enclosed volume, whereas in Stokes' theorem we orient $d\mathbf{S}$ using the right-hand rule relative to the chosen orientation of C. Substituting (6.3.9) into (6.3.8) and using $I = \dot{q}(t)$ then reproduces (6.3.7). This consistency is precisely what the displacement current term ensures.

The arguments above show that a term proportional to $\partial_t\mathbf{E}$ is necessary, but they do not determine $\mathbf{Z}$. To determine $\mathbf{Z}$ one must appeal to experiment. Consider a closed loop C in a region where there is no charge or current density. Integrating (6.3.6) over any surface S with boundary C gives,

$$\oint_C (\mathbf{B}-\mathbf{Z})\cdot d\mathbf{r} = \mu_0\varepsilon_0\,\partial_t \int_S \mathbf{E}\cdot d\mathbf{S}\,. \qquad (6.3.10)$$

Here we used Stokes' theorem for the $\mathbf{B}$ and $\mathbf{Z}$ terms, and we used that in a charge-free region $\nabla\cdot\mathbf{E}=0$, so the electric flux through S depends only on the loop C (by the same reasoning as in Remark 5.4.2 for $\mathbf{B}$). Measuring $\mathbf{E}$ and $\mathbf{B}$ for such loops shows that $\mathbf{Z}$ vanishes, so Maxwell-Ampère's law takes the form (1.0.4).

§6.4 Electromagnetic Waves

In the previous sections we saw that Maxwell's equations (1.0.2) and (1.0.4) imply a striking mutual interplay. A time-dependent magnetic field induces a time-dependent electric field, and a time-dependent electric field induces a time-dependent magnetic field. This suggests a mechanism that can sustain itself even in the absence of charges and currents. As we will now show, this is precisely what leads to the propagation of electromagnetic waves and, ultimately, to the light we see.

This also highlights why it is useful to study the electromagnetic field as a physical entity in its own right, rather than only as something determined instantaneously by its

sources. In vacuum, Maxwell's equations reduce to,

$$\boxed{\begin{aligned} \nabla \cdot \mathbf{E} &= 0\,, && (6.4.1)\\ \nabla \times \mathbf{E} &= -\partial_t \mathbf{B}\,, && (6.4.2)\\ \nabla \cdot \mathbf{B} &= 0\,, && (6.4.3)\\ \nabla \times \mathbf{B} &= \mu_0\, \varepsilon_0\, \partial_t \mathbf{E}\,. && (6.4.4) \end{aligned}}$$

These equations already contain the full story, but it is often convenient to derive equations for $\mathbf{E}$ and $\mathbf{B}$ separately, without explicit coupling between them. To do this for $\mathbf{E}$, take the curl of both sides of (6.4.2),

$$\begin{aligned} \nabla \times (\nabla \times \mathbf{E}) &= -\partial_t (\nabla \times \mathbf{B}) \overset{(6.4.4)}{\Rightarrow} \\ \nabla \times (\nabla \times \mathbf{E}) &= -\mu_0 \varepsilon_0\, \partial_t^2 \mathbf{E} \overset{(A.0.4)}{\Rightarrow} \\ \nabla (\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E} &= -\mu_0 \varepsilon_0\, \partial_t^2 \mathbf{E} \overset{(6.4.1)}{\Rightarrow} \\ \partial_t^2 \mathbf{E} - c^2\, \nabla^2 \mathbf{E} &= 0\,, \qquad (6.4.5) \end{aligned}$$

where we have defined,

$$\boxed{c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}}\,.} \qquad (6.4.6)$$

A completely analogous manipulation starting from (6.4.4) gives an identical equation for $\mathbf{B}$. Altogether, we obtain the decoupled system,

$$\boxed{\begin{aligned} \partial_t^2 \mathbf{E} - c^2\, \nabla^2 \mathbf{E} &= 0\,, && (6.4.7)\\ \partial_t^2 \mathbf{B} - c^2\, \nabla^2 \mathbf{B} &= 0\,. && (6.4.8) \end{aligned}}$$

Thus, in vacuum both the electric and magnetic fields satisfy the three-dimensional wave equation.[6] Using the measured values of ε_0 and μ_0, one finds $c \approx 3 \times 10^8\, \mathrm{m\,s^{-1}}$, the speed of light.

It is important to emphasise that the pair of wave equations (6.4.7)–(6.4.8) is weaker than the full set of vacuum Maxwell equations (6.4.1)–(6.4.4). Any vacuum solution of Maxwell's equations necessarily satisfies the wave equations, but the converse is not true. Maxwell's equations impose additional constraints, in particular on the allowed vector structure of the waves and on the relationship between $\mathbf{E}$ and $\mathbf{B}$.

Example 6.4.1. *Plane waves in the x-direction*
As an illustrative example, consider fields describing a plane wave propagating in the x-direction. We assume translational invariance in the transverse directions y and z, so that

[6] For a scalar field ψ, the three-dimensional wave equation takes the form $\partial_t^2 \psi - c^2\, \nabla^2 \psi = 0$, generalising the one-dimensional case you studied earlier in the term. The constant c is the speed of propagation.

the fields depend only on x and t. We can then write,

$$\mathbf{E}(x,t) = E_x(x,t)\,\hat{\mathbf{e}}_x + E_y(x,t)\,\hat{\mathbf{e}}_y + E_z(x,t)\,\hat{\mathbf{e}}_z\,, \tag{6.4.9}$$
$$\mathbf{B}(x,t) = B_x(x,t)\,\hat{\mathbf{e}}_x + B_y(x,t)\,\hat{\mathbf{e}}_y + B_z(x,t)\,\hat{\mathbf{e}}_z\,. \tag{6.4.10}$$

Substituting into (6.4.1) *and* (6.4.3) *gives,*

$$\begin{aligned} \partial_x E_x(x,t) = 0 \quad &\Rightarrow \quad E_x(x,t) = f_1(t)\,, \\ \partial_x B_x(x,t) = 0 \quad &\Rightarrow \quad B_x(x,t) = f_2(t)\,, \end{aligned}$$

so the components parallel to the direction of propagation cannot depend on x. Next, the x-components of (6.4.2) *and* (6.4.4) *imply,*

$$\begin{aligned} \partial_t E_x(x,t) = 0 \quad &\Rightarrow \quad \partial_t f_1(t) = 0\,, \\ \partial_t B_x(x,t) = 0 \quad &\Rightarrow \quad \partial_t f_2(t) = 0\,, \end{aligned}$$

so E_x and B_x are actually constant. Such constant background fields always solve Maxwell's equations, but they are not part of the propagating wave we want to study. Using linearity, we may therefore set these constant components to zero from now on.

With $E_x = B_x = 0$, the remaining components are constrained by (6.4.2) *and* (6.4.4). *The z-components give,*

$$\partial_x B_y(x,t) = \mu_0\varepsilon_0\,\partial_t E_z(x,t)\,, \tag{6.4.11}$$
$$\partial_x E_y(x,t) = -\partial_t B_z(x,t)\,, \tag{6.4.12}$$

while the y-components give,

$$\partial_x B_z(x,t) = -\mu_0\varepsilon_0\,\partial_t E_y(x,t)\,, \tag{6.4.13}$$
$$\partial_x E_z(x,t) = \partial_t B_y(x,t)\,. \tag{6.4.14}$$

We observe that (6.4.11) *and* (6.4.14) *form a closed subsystem, and* (6.4.12) *and* (6.4.13) *form another. These correspond to two independent polarisations. Without loss of generality, let us focus on the subsystem in which the electric field points in the y-direction,*

$$\mathbf{E}(x,t) = E_y(x,t)\,\hat{\mathbf{e}}_y\,, \tag{6.4.15}$$
$$\mathbf{B}(x,t) = B_z(x,t)\,\hat{\mathbf{e}}_z\,. \tag{6.4.16}$$

A key conclusion is already apparent. For a plane wave, $\mathbf{E}$ and $\mathbf{B}$ are orthogonal to each other, and both are orthogonal to the direction of propagation.

To obtain a wave equation for E_y, take an x-derivative of (6.4.12) *and a t-derivative of* (6.4.13), *and eliminate B_z,*

$$\begin{aligned} \partial_x^2 E_y(x,t) &= -\partial_x\partial_t B_z(x,t)\,, \\ \partial_t\partial_x B_z(x,t) &= -\mu_0\varepsilon_0\,\partial_t^2 E_y(x,t)\,. \end{aligned}$$

Combining these gives,

$$\partial_t^2 E_y(x,t) - c^2\,\partial_x^2 E_y(x,t) = 0\,, \tag{6.4.17}$$

which is the one-dimensional wave equation. Its most general solution is the d'Alembert form,

$$E_y(x,t) = f_R(x - c\,t) + f_L(x + c\,t)\,, \tag{6.4.18}$$

where f_R describes a right-moving wave and f_L a left-moving wave. Finally, the corresponding magnetic field follows from (6.4.12) *(or* (6.4.13)*) and is,*

$$B_z(x,t) = \frac{1}{c}\Big(f_R(x - c\,t) - f_L(x + c\,t)\Big)\,. \tag{6.4.19}$$

§6.5 General E/M Waves

In this section we describe the most general form of an electromagnetic wave propagating in vacuum. The building blocks will be solutions describing light of a single colour, known as *monochromatic plane waves*. For such a wave, the fields at any fixed point in space oscillate sinusoidally in time with a single angular frequency ω.

In Example 6.4.1 we found the general right/left-moving solution for a plane wave propagating in the x-direction. Choosing, for example, a purely right-moving monochromatic wave corresponds to taking,

$$f_R(x - c\,t) = E_0\,\sin\Big(\frac{\omega}{c}(x - c\,t)\Big) = E_0\,\sin\Big(\frac{\omega}{c}\,x - \omega t\Big) = E_0\,\sin(kx - \omega t)\,, \tag{6.5.1}$$

where we introduced the *wavenumber*,

$$k = \frac{\omega}{c}\,. \tag{6.5.2}$$

The time period is $T = 2\pi/\omega$, while the spatial period is the *wavelength* $\lambda = 2\pi/k$. The corresponding fields are,

$$\begin{aligned}\mathbf{E}(x,y,z,t) &= E_0\,\sin(kx - \omega t)\,\hat{\mathbf{e}}_y\,,\\ \mathbf{B}(x,y,z,t) &= \frac{E_0}{c}\,\sin(kx - \omega t)\,\hat{\mathbf{e}}_z\,.\end{aligned} \tag{6.5.3}$$

It will be convenient to use a complex representation. Different choices of phase (sine, cosine, or a phase-shifted sinusoid) describe the same physics, and complex exponentials keep the algebra simple. For instance, we may write,

$$\begin{aligned}\mathbf{E}(x,y,z,t) &= \mathbf{E}_0\,e^{i(kx-\omega t)}\,,\\ \mathbf{B}(x,y,z,t) &= \mathbf{B}_0\,e^{i(kx-\omega t)}\,,\end{aligned} \tag{6.5.4}$$

with constant (generally complex) vectors $\mathbf{E}_0$ and $\mathbf{B}_0$. The physical fields are then obtained by taking the real part.[7]

Motivated by this, we consider the general ansatz,

$$\boxed{\begin{aligned}\mathbf{E}(\mathbf{x},t) &= \mathbf{E}_0\, e^{i\,(\mathbf{k}\cdot\mathbf{x}-\omega\,t)}\,,\\ \mathbf{B}(\mathbf{x},t) &= \mathbf{B}_0\, e^{i\,(\mathbf{k}\cdot\mathbf{x}-\omega\,t)}\,,\end{aligned}} \tag{6.5.5}$$

where $\mathbf{E}_0$ and $\mathbf{B}_0$ are constant complex vectors and $\mathbf{k}$ is the *wavevector*. Its magnitude $|\mathbf{k}|$ equals the wavenumber, and its direction is the direction of propagation. (Indeed, $\mathbf{k}\cdot\mathbf{x}$ is constant when $\mathbf{x}$ is displaced orthogonally to $\mathbf{k}$.)

For the fields in (6.5.5) we have,

$$\begin{aligned}\nabla\cdot\mathbf{E} &= i\,\mathbf{k}\cdot\mathbf{E}_0\, e^{i\,(\mathbf{k}\cdot\mathbf{x}-\omega\,t)}\,,\\ \nabla\cdot\mathbf{B} &= i\,\mathbf{k}\cdot\mathbf{B}_0\, e^{i\,(\mathbf{k}\cdot\mathbf{x}-\omega\,t)}\,,\\ \nabla\times\mathbf{E} &= i\,\mathbf{k}\times\mathbf{E}_0\, e^{i\,(\mathbf{k}\cdot\mathbf{x}-\omega\,t)}\,,\\ \nabla\times\mathbf{B} &= i\,\mathbf{k}\times\mathbf{B}_0\, e^{i\,(\mathbf{k}\cdot\mathbf{x}-\omega\,t)}\,.\end{aligned}$$

Substituting into the vacuum Maxwell equations (6.4.1)–(6.4.4) gives the algebraic constraints,

$$\begin{aligned}\mathbf{k}\cdot\mathbf{E}_0 &= 0\,,\\ \mathbf{k}\cdot\mathbf{B}_0 &= 0\,,\\ \mathbf{k}\times\mathbf{B}_0 &= -\varepsilon_0\mu_0\,\omega\,\mathbf{E}_0\,,\\ \mathbf{k}\times\mathbf{E}_0 &= \omega\,\mathbf{B}_0\,.\end{aligned} \tag{6.5.6}$$

The first two conditions show that both $\mathbf{E}$ and $\mathbf{B}$ are transverse to the direction of propagation. Combining the last two equations yields,

$$\begin{aligned}&\mathbf{k}\times(\mathbf{k}\times\mathbf{E}_0) = \mathbf{k}\times(\omega\,\mathbf{B}_0) = \omega\,(\mathbf{k}\times\mathbf{B}_0) = -\varepsilon_0\mu_0\,\omega^2\,\mathbf{E}_0\\ \Rightarrow\qquad &(\mathbf{k}\cdot\mathbf{E}_0)\;\mathbf{k} - |\mathbf{k}|^2\,\mathbf{E}_0 = -\varepsilon_0\mu_0\,\omega^2\,\mathbf{E}_0\,,\end{aligned} \tag{6.5.7}$$

and using the transversality of $\mathbf{E}_0$, we obtain the vacuum dispersion relation,

$$\omega^2 = c^2\,|\mathbf{k}|^2\,. \tag{6.5.8}$$

To summarise, the monochromatic plane wave (6.5.5) solves the vacuum Maxwell equations provided,

$$\boxed{\mathbf{k}\cdot\mathbf{E}_0 = 0, \qquad \omega(\mathbf{k})\,\mathbf{B}_0 = \mathbf{k}\times\mathbf{E}_0, \qquad \omega^2(\mathbf{k}) = c^2\,|\mathbf{k}|^2\,.} \tag{6.5.9}$$

[7] This is justified because Maxwell's equations are linear: if the complex field solves the equations, then its real and imaginary parts solve them separately.

Note that the second condition also implies $\mathbf{k}\cdot\mathbf{B}_0 = 0$ automatically.

Finally, because Maxwell's equations are linear, we can superpose monochromatic plane waves. Allowing the amplitudes to depend on $\mathbf{k}$ and integrating over all wavevectors, we arrive at the Fourier representation of the most general vacuum solution,

$$\boxed{\begin{aligned}\mathbf{E}(\mathbf{x},t) &= \int_{\mathbb{R}^3} \mathbf{E}_0(\mathbf{k})\, e^{i\,(\mathbf{k}\cdot\mathbf{x}-\omega(\mathbf{k})\,t)}\, d^3\mathbf{k}\,,\\ \mathbf{B}(\mathbf{x},t) &= \int_{\mathbb{R}^3} \mathbf{B}_0(\mathbf{k})\, e^{i\,(\mathbf{k}\cdot\mathbf{x}-\omega(\mathbf{k})\,t)}\, d^3\mathbf{k}\,.\end{aligned}} \tag{6.5.10}$$

These fields satisfy the vacuum Maxwell equations provided the constraints (6.5.9) hold for each $\mathbf{k}$ appearing in the integrals.

§6.6 Energy Transport

In this section we study how electromagnetic energy is transported and how it is exchanged with matter. From the point of view of the electromagnetic field, the system is generally *not* isolated. In the presence of charges and currents the field can do work on the charges and, conversely, charges can supply energy to the field. Consequently, if we look at the electromagnetic energy stored in some spatial region V, its time variation should have two contributions,

- a flux of energy through the boundary ∂V, and
- a transfer of energy to or from the charges moving inside V.

To see the second effect explicitly, consider a small volume element δV carrying charge density ρ and moving with velocity $\mathbf{v}$ through the fields. Over a short time δt it is displaced by $\delta\mathbf{l} = \mathbf{v}\,\delta t$. The work done by the Lorentz force is,

$$\begin{aligned}\delta W &= \mathbf{F}_{\rm L}\cdot\delta\mathbf{l} = \delta q\,(\mathbf{E}+\mathbf{v}\times\mathbf{B})\cdot\delta\mathbf{l} = \delta q\,(\mathbf{E}+\mathbf{v}\times\mathbf{B})\cdot\mathbf{v}\,\delta t \\ &= \rho\,\delta V\,\mathbf{E}\cdot\mathbf{v}\,\delta t = \mathbf{J}\cdot\mathbf{E}\,\delta V\,\delta t\,,\end{aligned} \tag{6.6.1}$$

where in the last step we used $\mathbf{J} = \rho\,\mathbf{v}$, see Example 2.1.2. Thus, the rate at which the electromagnetic field transfers energy to matter (per unit volume) is simply $\mathbf{J}\cdot\mathbf{E}$.

We now derive the corresponding local conservation law for the electromagnetic field. Consider the electromagnetic energy density[8],

$$\boxed{\varepsilon(t,\mathbf{x}) = \frac{1}{2\mu_0}\,|\mathbf{B}(t,\mathbf{x})|^2 + \frac{\varepsilon_0}{2}\,|\mathbf{E}(t,\mathbf{x})|^2\,,} \tag{6.6.2}$$

[8]This move is very quick. We simply summed up the energy densities we computed for time independent configurations. One can argue that the true expression for energy density might contain time derivatives of fields. See Section 7.9 for a more rigorous treatment

which is the sum of the electric (4.9.8) and magnetic (6.2.10) contributions. Taking a time derivative gives,

$$\begin{aligned}\partial_t\varepsilon(t,\mathbf{x}) =& \frac{1}{\mu_0}\,\mathbf{B}(t,\mathbf{x})\cdot\partial_t\mathbf{B}(t,\mathbf{x}) + \varepsilon_0\,\mathbf{E}(t,\mathbf{x})\cdot\partial_t\mathbf{E}(t,\mathbf{x})\\ \overset{(1.0.2),(1.0.4)}{=}& -\frac{1}{\mu_0}\mathbf{B}(t,\mathbf{x})\cdot(\nabla\times\mathbf{E}(t,\mathbf{x})) + \frac{1}{\mu_0}\mathbf{E}(t,\mathbf{x})\cdot(\nabla\times\mathbf{B}(t,\mathbf{x}) - \mu_0\,\mathbf{J}(t,\mathbf{x}))\\ =& -\mathbf{E}(t,\mathbf{x})\cdot\mathbf{J}(t,\mathbf{x}) + \frac{1}{\mu_0}\left(\mathbf{E}(t,\mathbf{x})\cdot(\nabla\times\mathbf{B}(t,\mathbf{x})) - \mathbf{B}(t,\mathbf{x})\cdot(\nabla\times\mathbf{E}(t,\mathbf{x}))\right)\\ =& -\mathbf{E}(t,\mathbf{x})\cdot\mathbf{J}(t,\mathbf{x}) - \frac{1}{\mu_0}\,\nabla\cdot(\mathbf{E}(t,\mathbf{x})\times\mathbf{B}(t,\mathbf{x}))\;, \end{aligned} \tag{6.6.3}$$

where in the last step we used the vector identity $\nabla\cdot(\mathbf{E}\times\mathbf{B}) = \mathbf{B}\cdot(\nabla\times\mathbf{E}) - \mathbf{E}\cdot(\nabla\times\mathbf{B})$.

This motivates the definition of the *Poynting vector*,

$$\boxed{\mathcal{S} = \frac{1}{\mu_0}\mathbf{E}\times\mathbf{B}\,,} \tag{6.6.4}$$

in terms of which (6.6.3) becomes,

$$\boxed{\partial_t\varepsilon + \nabla\cdot\mathcal{S} = -\mathbf{E}\cdot\mathbf{J}\,.} \tag{6.6.5}$$

This has the same structure as the charge continuity equation (2.1.8), but now the right-hand side is generally non-zero. It represents the local power density transferred from the electromagnetic field to matter. In particular, when $\mathbf{E}\cdot\mathbf{J} > 0$ the electromagnetic field is doing work on the charges and its energy decreases accordingly. The Poynting vector is the analogue of the current density in this context as it describes how electromagnetic energy flows through space. This suggests that given an oriented surface S, the rate of energy P_S flowing through S is given by,

$$\boxed{P_S = \int_S \mathcal{S}\cdot d\mathbf{S}\,.} \tag{6.6.6}$$

Example 6.6.1. *Energy transport in monochromatic plane waves*
Let us check these results for a monochromatic plane wave in vacuum, as discussed in Section 6.5. Using real fields, we write,

$$\begin{aligned}\mathbf{E}(t,\mathbf{x}) &= \mathbf{E}_0\,\cos(\mathbf{k}\cdot\mathbf{x} - \omega\,t)\,,\\ \mathbf{B}(t,\mathbf{x}) &= \mathbf{B}_0\,\cos(\mathbf{k}\cdot\mathbf{x} - \omega\,t)\,,\end{aligned} \tag{6.6.7}$$

with the constraints,

$$\mathbf{k}\cdot\mathbf{E}_0 = 0, \qquad \omega^2 = c^2\,\mathbf{k}^2, \qquad \mathbf{k}\times\mathbf{E}_0 = \omega\,\mathbf{B}_0\,. \tag{6.6.8}$$

From (6.6.8) *we also obtain,*

$$|\mathbf{B}_0|^2 = \frac{1}{c^2}\,|\mathbf{E}_0|^2 \qquad \Rightarrow \qquad |\mathbf{B}_0|^2 = \varepsilon_0\mu_0\,|\mathbf{E}_0|^2\,. \tag{6.6.9}$$

Substituting into (6.6.2) *gives,*

$$\varepsilon(t,\mathbf{x}) = \varepsilon_0\,|\mathbf{E}_0|^2\,\cos^2(\mathbf{k}\cdot\mathbf{x}-\omega\,t)\,. \tag{6.6.10}$$

On the other hand, the Poynting vector (6.6.4) *becomes,*

$$\begin{aligned}\mathcal{S}(t,\mathbf{x}) &\overset{(6.6.7)}{=} \frac{1}{\mu_0}\,\mathbf{E}_0\times\mathbf{B}_0\,\cos^2(\mathbf{k}\cdot\mathbf{x}-\omega\,t) \overset{(6.6.8)}{=} \frac{1}{\omega\,\mu_0}\,\mathbf{E}_0\times(\mathbf{k}\times\mathbf{E}_0)\,\cos^2(\mathbf{k}\cdot\mathbf{x}-\omega\,t)\\ &= \frac{1}{\omega\,\mu_0}\,\big(|\mathbf{E}_0|^2\,\mathbf{k}-(\mathbf{k}\cdot\mathbf{E}_0)\,\mathbf{E}_0\big)\,\cos^2(\mathbf{k}\cdot\mathbf{x}-\omega\,t)\\ &\overset{(6.6.8)}{=} \frac{1}{c\mu_0}\,|\mathbf{E}_0|^2\,\cos^2(\mathbf{k}\cdot\mathbf{x}-\omega\,t)\,\hat{\mathbf{k}}\,.\end{aligned} \tag{6.6.11}$$

Since a plane wave in vacuum has $\mathbf{J}=0$, (6.6.5) *reduces to* $\partial_t\varepsilon+\nabla\cdot\mathcal{S}=0$, *which one can verify directly from* (6.6.10) *and* (6.6.11). *It is also instructive to note the relation,*

$$\mathcal{S}(t,\mathbf{x}) = c\,\varepsilon(t,\mathbf{x})\,\hat{\mathbf{k}}\,, \tag{6.6.12}$$

showing that, for a monochromatic plane wave, the energy flux points along the direction of propagation and has magnitude equal to the energy density times the propagation speed c.

§6.7 The Potentials – Gauge Invariance

In earlier sections we introduced the scalar potential φ and the vector potential $\mathbf{A}$ through (4.6.3) and (5.4.4). In each static setting, the point was the same. Two of Maxwell's equations simplify to constraint equations, and these can be solved automatically by writing the fields in terms of potentials. This reduces the problem to solving a single equation for φ or for $\mathbf{A}$, namely (4.6.4) and (5.4.6).

It is natural to ask whether the same strategy works in the time-dependent case. Introducing the vector potential via (5.4.4) still solves (1.0.3), since $\nabla\cdot\mathbf{B}=0$ is unchanged in the dynamical theory. However, writing $\mathbf{E}=-\nabla\varphi$ no longer solves (1.0.2), because Faraday's law involves a time derivative of $\mathbf{B}$. The correct time-dependent definitions are,

$$\boxed{\begin{aligned}\mathbf{E} &= -\nabla\varphi-\partial_t\mathbf{A}\,, &(6.7.1)\\ \mathbf{B} &= \nabla\times\mathbf{A}\,. &(6.7.2)\end{aligned}}$$

These solve (1.0.2) and (1.0.3) identically. For example,

$$\nabla\times\mathbf{E} \overset{(6.7.1)}{=} -\nabla\times(\nabla\varphi+\partial_t\mathbf{A}) = -\partial_t(\nabla\times\mathbf{A}) \overset{(6.7.2)}{=} -\partial_t\mathbf{B}\,. \tag{6.7.3}$$

A second important feature of the potential formulation is that it is *not unique*. One can check that the time-dependent *gauge transformations*,

$$\boxed{\begin{aligned}\mathbf{A} &\to \mathbf{A} + \nabla\chi\,,\\ \varphi &\to \varphi - \partial_t\chi\,,\end{aligned}} \tag{6.7.4}$$

with gauge parameter the scalar field $\chi(t,\mathbf{x})$, leave $\mathbf{E}$ and $\mathbf{B}$ unchanged as defined in (6.7.1)–(6.7.2).

Remark 6.7.1. What does this symmetry mean? The electromagnetic field $(\mathbf{E},\mathbf{B})$ contains six functions, while the potentials $(\varphi,\mathbf{A})$ involve four. The gauge symmetry tells us that even these four functions contain redundancy. Different choices of $(\varphi,\mathbf{A})$ related by (6.7.4) describe the same physical electric and magnetic fields.

Finally, let us translate the remaining two Maxwell equations, (1.0.1) and (1.0.4), into equations for the potentials. Substituting (6.7.1) and (6.7.2) gives,

$$\nabla\cdot(\nabla\varphi + \partial_t\mathbf{A}) + \frac{\rho}{\varepsilon_0} = 0\,, \tag{6.7.5}$$

$$c^2\,\nabla\times(\nabla\times\mathbf{A}) + \partial_t(\nabla\varphi + \partial_t\mathbf{A}) - \frac{1}{\varepsilon_0}\mathbf{J} = 0\,. \tag{6.7.6}$$

These are the dynamical equations satisfied by the potentials. To obtain a unique representation, one must supplement them with a choice of gauge.

§6.8 The Maxwell Action

In the previous section we derived the equations of motion (6.7.5) and (6.7.6) for the scalar and vector potentials. A natural next question is whether these equations can be obtained from an *action principle*. That is, can we find a functional $S[\varphi,\mathbf{A}]$ whose extrema (under arbitrary variations of φ and $\mathbf{A}$) reproduce the correct equations of motion?

As we will now show, the answer is yes. The appropriate functional is the *Maxwell action*,

$$\boxed{S[\varphi,\mathbf{A}] = \int_{\mathbb{R}^4}\left(\frac{\varepsilon_0}{2}\,|\mathbf{E}|^2 - \frac{1}{2\mu_0}\,|\mathbf{B}|^2 - \varphi\,\rho + \mathbf{A}\cdot\mathbf{J}\right)\,dt\,d^3\mathbf{x}\,,} \tag{6.8.1}$$

where $\mathbf{E}$ and $\mathbf{B}$ are expressed in terms of the potentials by (6.7.1) and (6.7.2). Writing the action explicitly in terms of φ and $\mathbf{A}$ gives,

$$S[\varphi,\mathbf{A}] = \int_{\mathbb{R}^4}\left(\frac{\varepsilon_0}{2}\,\left|\nabla\varphi + \partial_t\mathbf{A}\right|^2 - \frac{1}{2\mu_0}\,\left|\nabla\times\mathbf{A}\right|^2 - \varphi\,\rho + \mathbf{A}\cdot\mathbf{J}\right)\,dt\,d^3\mathbf{x}\,. \tag{6.8.2}$$

We extremise the action by requiring $\delta S = 0$ under arbitrary variations of φ and $\mathbf{A}$. Varying with respect to φ yields,

$$\begin{aligned}\delta S &= \int_{\mathbb{R}^4} (\varepsilon_0\ (\nabla\varphi + \partial_t\mathbf{A})\cdot\nabla\delta\varphi - \rho\,\delta\varphi)\ dt\,d^3\mathbf{x} \\ &= -\int_{\mathbb{R}^4} (\varepsilon_0\,\nabla\cdot(\nabla\varphi + \partial_t\mathbf{A}) + \rho)\ \delta\varphi\,dt\,d^3\mathbf{x} + \varepsilon_0\int_{\mathbb{R}^4}\nabla\cdot(\delta\varphi\ (\nabla\varphi + \partial_t\mathbf{A}))\ dt\,d^3\mathbf{x} \\ &= -\int_{\mathbb{R}^4} (\varepsilon_0\,\nabla\cdot(\nabla\varphi + \partial_t\mathbf{A}) + \rho)\ \delta\varphi\,dt\,d^3\mathbf{x}\,. \end{aligned} \tag{6.8.3}$$

In the last step we dropped the total divergence term, assuming suitable boundary conditions. For example that the variation $\delta\varphi$ vanishes sufficiently fast at spatial infinity. Since $\delta\varphi$ is arbitrary, the extremisation condition $\delta S = 0$ implies,

$$\nabla\cdot(\nabla\varphi + \partial_t\mathbf{A}) + \frac{\rho}{\varepsilon_0} = 0\,, \tag{6.8.4}$$

which is precisely (6.7.5).

Next, vary with respect to $\mathbf{A}$ (keeping φ fixed). We find

$$\begin{aligned}\delta S &= \int_{\mathbb{R}^4}\left(\varepsilon_0\ (\nabla\varphi + \partial_t\mathbf{A})\cdot\partial_t\delta\mathbf{A} - \frac{1}{\mu_0}(\nabla\times\mathbf{A})\cdot(\nabla\times\delta\mathbf{A}) + \mathbf{J}\cdot\delta\mathbf{A}\right)\ dt\,d^3\mathbf{x} \\ &= \varepsilon_0\int_{\mathbb{R}^4}\partial_t((\nabla\varphi + \partial_t\mathbf{A})\cdot\delta\mathbf{A})\ dt\,d^3\mathbf{x} + \frac{1}{\mu_0}\int_{\mathbb{R}^4}\nabla\cdot((\nabla\times\mathbf{A})\times\delta\mathbf{A})\ dt\,d^3\mathbf{x} \\ &\quad + \int_{\mathbb{R}^4}\left(-\varepsilon_0\,\partial_t(\nabla\varphi + \partial_t\mathbf{A}) - \frac{1}{\mu_0}\nabla\times(\nabla\times\mathbf{A}) + \mathbf{J}\right)\cdot\delta\mathbf{A}\,dt\,d^3\mathbf{x} \\ &= \int_{\mathbb{R}^4}\left(-\varepsilon_0\,\partial_t(\nabla\varphi + \partial_t\mathbf{A}) - \frac{1}{\mu_0}\nabla\times(\nabla\times\mathbf{A}) + \mathbf{J}\right)\cdot\delta\mathbf{A}\,dt\,d^3\mathbf{x}\,. \end{aligned} \tag{6.8.5}$$

In going to the last line we again discarded total derivative terms, assuming $\delta\mathbf{A}$ vanishes suitably on the boundary. Since $\delta\mathbf{A}$ is arbitrary, $\delta S = 0$ gives

$$\frac{1}{\mu_0\varepsilon_0}\nabla\times(\nabla\times\mathbf{A}) + \partial_t(\nabla\varphi + \partial_t\mathbf{A}) - \frac{1}{\varepsilon_0}\mathbf{J} = 0\,, \tag{6.8.6}$$

which is equivalent to (6.7.6), after using $c^2 = 1/(\mu_0\varepsilon_0)$.

In Section 6.7 we saw that the potentials are not unique. The gauge transformation (6.7.4) leaves $\mathbf{E}$ and $\mathbf{B}$ unchanged. Since the field-strength terms in the action depend only on $\mathbf{E}$ and $\mathbf{B}$, they are automatically gauge invariant. The only part that can change under a gauge transformation is the coupling to sources,

$$\begin{aligned}S[\varphi - \partial_t\chi, \mathbf{A} + \nabla\chi] &= S[\varphi,\mathbf{A}] + \int_{\mathbb{R}^4}(\partial_t\chi\,\rho + \mathbf{J}\cdot\nabla\chi)\ dt\,d^3\mathbf{x} \\ &= S[\varphi,\mathbf{A}] + \int_{\mathbb{R}^4}(\partial_t(\chi\,\rho) + \nabla\cdot(\chi\,\mathbf{J}))\ dt\,d^3\mathbf{x} - \int_{\mathbb{R}^4}(\partial_t\rho + \nabla\cdot\mathbf{J})\,\chi\,dt\,d^3\mathbf{x}\,. \end{aligned} \tag{6.8.7}$$

If we choose gauge transformations χ that are localised in space and time or impose boundary conditions such that the total-derivative terms vanish, then the action is gauge invariant if and only if the last integral vanishes for arbitrary χ. This implies,

$$\partial_t \rho + \nabla \cdot \mathbf{J} = 0\,, \tag{6.8.8}$$

which is the charge continuity equation (2.1.8). Conversely, if the sources satisfy the continuity equation, then the Maxwell action is gauge invariant.

§7 Special Relativity and Maxwell's Theory

One of the central themes of the previous sections was a special symmetry of the electromagnetic field, gauge invariance. We now turn to a different symmetry question. Which transformations of spacetime leave Maxwell's theory unchanged? Pursuing this question reveals a fundamental tension with Newtonian intuition, Galilean boosts are not symmetries of electromagnetism. It was precisely this clash that helped Einstein set the stage for special relativity.

We begin by introducing Lorentz transformations as the correct spacetime symmetries of Maxwell's equations. Taking these symmetries seriously leads to a new understanding of space and, in particular, time. To develop the subject systematically, we then introduce the tensor formulation of special relativity. This language will allow us to write Maxwell's theory in a manifestly Lorentz-invariant form.

§7.1 Galilean Boosts and Waves

Physical laws should not depend on our choice of inertial frame. In other words, if a law is valid for one freely moving observer, it should be valid for all freely moving observers. The transformations relating inertial frames are therefore precisely those that leave the fundamental laws invariant. A first and familiar example is Newton's second law,

$$m\,\ddot{\mathbf{x}}(t) = \mathbf{F}\,. \tag{7.1.1}$$

Passing from one inertial frame to another should preserve the form of this equation. This requirement leads to the following transformations:

- **Translations:**

  A rigid translation is a constant shift of the spatial origin,

  $$\mathbf{y}(t) = \mathbf{x}(t) + \mathbf{a}\,, \tag{7.1.2}$$

  so that $\mathbf{x}(t) = \mathbf{y}(t) - \mathbf{a}$ and hence $\ddot{\mathbf{x}}(t) = \ddot{\mathbf{y}}(t)$. Newton's law then retains its form in the translated frame, provided the force is unchanged, $\mathbf{F}' = \mathbf{F}$.

- **Rotations:**

  A rigid rotation is given by,

  $$\mathbf{y}(t) = \mathbf{R}\,\mathbf{x}(t)\,, \tag{7.1.3}$$

  where $\mathbf{R}$ is a constant rotation matrix. Differentiating twice gives $\ddot{\mathbf{y}}(t) = \mathbf{R}\,\ddot{\mathbf{x}}(t)$, so Newton's law holds in the rotated frame as,

  $$m\,\ddot{\mathbf{y}}(t) = \mathbf{F}'\,, \qquad \mathbf{F}' = \mathbf{R}\,\mathbf{F}\,. \tag{7.1.4}$$

- **Galilean boosts:**

  Two inertial observers moving with constant relative velocity $\mathbf{v}$ are related by,

  $$\mathbf{y}(t) = \mathbf{x}(t) - \mathbf{v}\,t\,. \tag{7.1.5}$$

  Taking two time derivatives again gives $\ddot{\mathbf{y}}(t) = \ddot{\mathbf{x}}(t)$, so Newton's law is invariant under boosts as well (with $\mathbf{F}' = \mathbf{F}$).

A key point is that in all of these transformations we have not changed time. In Newtonian mechanics, t is a universal parameter shared by all observers. It is now natural to ask whether this picture is compatible with Maxwell's theory. The electric and magnetic fields may transform between observers, but if all inertial frames are equivalent then Maxwell's equations should keep the same form in every inertial frame.

Before analysing Maxwell's equations directly, it is helpful to look at a simpler model. As we saw in Section 6.4, the electromagnetic field in vacuum satisfies wave equations. To test the compatibility of wave propagation with Galilean boosts, we therefore consider a scalar field $\psi(t,x)$ in one spatial dimension satisfying,

$$\partial_t^2\psi(t,x) - c^2\,\partial_x^2\psi(t,x) = 0\,, \tag{7.1.6}$$

which describes waves propagating with speed c.

What happens to such a wave under a Galilean boost?[9] It is illuminating to first look at the solutions. The most general solution is the d'Alembert form,

$$\psi(t,x) = f_R(x - c\,t) + f_L(x + c\,t)\,, \tag{7.1.7}$$

describing right-moving and left-moving waves. Notice that both propagate with the same speed c, in opposite directions.

Now consider an observer moving to the right with constant velocity v. The Galilean transformation is,

$$x' = x - vt\,, \qquad t' = t\,, \tag{7.1.8}$$

so that $x = x' + vt'$. Since ψ is a scalar function, it transforms as a simple composition. The wave in the primed frame will be,

$$\psi'(t',x') = \psi(t(t',x'),x(t',x')) = \psi(t',x'+vt') = f_R\big(x' - (c-v)t'\big) + f_L\big(x' + (c+v)t'\big)\,. \tag{7.1.9}$$

As expected, the moving observer sees the right-moving part travelling at speed $c-v$, while the left-moving part travels at speed $c+v$.

[9] In one spatial dimension there are no non-trivial rotations, and spatial translations are straightforward.

Equivalently, we can see the mismatch at the level of the equation itself. Using the chain rule,

$$\partial_x = \frac{\partial x'}{\partial x}\,\partial_{x'} + \frac{\partial t'}{\partial x}\,\partial_{t'} = \partial_{x'}\,, \tag{7.1.10}$$

$$\partial_t = \frac{\partial x'}{\partial t}\,\partial_{x'} + \frac{\partial t'}{\partial t}\,\partial_{t'} = -v\,\partial_{x'} + \partial_{t'}\,. \tag{7.1.11}$$

Substituting into (7.1.6) gives,

$$\partial_{t'}^2\psi' - 2v\,\partial_{t'}\partial_{x'}\psi' - (c^2 - v^2)\,\partial_{x'}^2\psi' = 0\,, \tag{7.1.12}$$

which is manifestly different from (7.1.6).

This is exactly what happens for mechanical waves. For instance, if you drive parallel to ocean waves, you will measure a different wave speed from someone standing still on the shore. The reason is that mechanical waves propagate in a medium with a preferred rest frame. The simple wave equation (7.1.6) holds only in that rest frame; a moving observer must use a different equation, such as (7.1.12).

Electromagnetic waves are fundamentally different. They propagate in vacuum, without a material medium, and the same physical laws should apply in every inertial frame. If Maxwell's equations have the same form for all inertial observers, then the speed of light in vacuum must be the same for all inertial observers, and the same in all directions. This conclusion is incompatible with Galilean boosts. Resolving this tension led Einstein to replace Galilean invariance with Lorentz invariance, marking the birth of special relativity and, ultimately, the relativistic theory of gravity.

§7.2 Lorentz Boosts in 1+1 Dimensions

In the previous section we saw that Maxwell's theory, and the simple vacuum wave equation, cannot be compatible with Galilean boosts. If all inertial observers are to describe light propagating at the same speed c, then the transformations relating inertial frames must be different from those of Newtonian mechanics. In this section we determine the symmetry transformations of the one-dimensional wave equation and thereby arrive at Lorentz boosts in $1+1$ dimensions.

We consider the most general linear change of coordinates,

$$\begin{aligned} x' &= A\,x + B\,t + c_1\,, \\ t' &= E\,t + D\,x + c_2\,. \end{aligned} \tag{7.2.1}$$

Allowing t to mix with x is already a significant departure from the Newtonian picture. If $D \neq 0$ or $E \neq 1$, then time is no longer a universal parameter shared by all observers, but instead becomes a coordinate that can depend on the observer. The constants c_1 and c_2 represent constant shifts in space and time. Since translations trivially preserve the form of the wave equation, we set $c_1 = c_2 = 0$ from now on.

We demand that the transformation preserves the wave equation (7.1.6) with the same propagation speed c. Using the chain rule,

$$\begin{aligned}\partial_x &= \frac{\partial x'}{\partial x}\,\partial_{x'} + \frac{\partial t'}{\partial x}\,\partial_{t'} = A\,\partial_{x'} + D\,\partial_{t'}\,,\\ \partial_t &= \frac{\partial x'}{\partial t}\,\partial_{x'} + \frac{\partial t'}{\partial t}\,\partial_{t'} = B\,\partial_{x'} + E\,\partial_{t'}\,.\end{aligned} \tag{7.2.2}$$

Substituting these into (7.1.6) gives,

$$\left(E^2 - c^2 D^2\right)\partial_{t'}^2\psi - \left(c^2 A^2 - B^2\right)\partial_{x'}^2\psi + 2\left(BE - AD\,c^2\right)\partial_{t'}\partial_{x'}\psi = 0\,. \tag{7.2.3}$$

Requiring this to match (7.1.6) for *all* fields ψ leads to the three constraints,

$$E^2 - c^2\,D^2 = 1, \qquad c^2\,A^2 - B^2 = c^2, \qquad B\,E - A\,D\,c^2 = 0\,. \tag{7.2.4}$$

One might wonder whether we could allow an overall multiplicative factor on the left-hand side. However, such a factor would not be acceptable in the presence of sources, where the right-hand side would not transform in the same way.

The system (7.2.4) admits one continuous free parameter, which we identify with the relative velocity v between the two inertial frames. A convenient parametrisation is,

$$A = E = \gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}\,, \qquad B = -\gamma\,v\,, \qquad D = -\gamma\,\frac{v}{c^2}\,, \tag{7.2.5}$$

so that $B = c^2 D$.

Remark 7.2.1. One can also flip the sign of A or of E while still satisfying (7.2.4), corresponding to space reversal and time reversal respectively. Here we focus on the branch that is continuously connected to the identity transformation at $v = 0$.

The resulting transformation is the *Lorentz boost* in $1+1$ dimensions,

$$\boxed{\begin{aligned} x' &= \gamma\,(x - v\,t)\,,\\ t' &= \gamma\,\left(t - \frac{v}{c^2}\,x\right)\,,\\ \gamma &= \frac{1}{\sqrt{1 - (v/c)^2}}\,. \end{aligned}} \tag{7.2.6}$$

There are a few important remarks,

Remark 7.2.2. There is no notion of a global time shared by all observers. Time is a coordinate, and different inertial observers generally disagree about simultaneity. Concretely, if two events satisfy $\Delta t = t_2 - t_1 = 0$ in one frame while $\Delta x = x_2 - x_1 \neq 0$, then in a boosted frame,

$$\Delta t' = t'_2 - t'_1 = -\gamma\,\frac{v}{c^2}\,\Delta x \neq 0\,. \tag{7.2.7}$$

Remark 7.2.3. Galilean boosts are recovered when $v \ll c$. In that limit $\gamma \simeq 1$, and (7.2.6) reduces to $x' \simeq x - vt$ and $t' \simeq t$.

Remark 7.2.4. Galilean boosts preserve the spatial distance in one dimension, $\Delta x^2 = (x_2 - x_1)^2 = (x_2' - x_1')^2$. Lorentz boosts do *not* preserve Δx^2; instead they preserve the *spacetime distance*,

$$\boxed{\Delta l^2 = -c^2\,(\Delta t)^2 + (\Delta x)^2 = -c^2\,(\Delta t')^2 + (\Delta x')^2\,.} \tag{7.2.8}$$

Unlike a Euclidean distance, the spacetime interval can be,

- $\Delta l^2 < 0$ for events with *timelike* separation,
- $\Delta l^2 = 0$ for events with *lightlike* separation,
- $\Delta l^2 > 0$ for events with *spacelike* separation.

Recognising Δl^2 as the fundamental invariant distance between events marks the birth of *special relativity*.

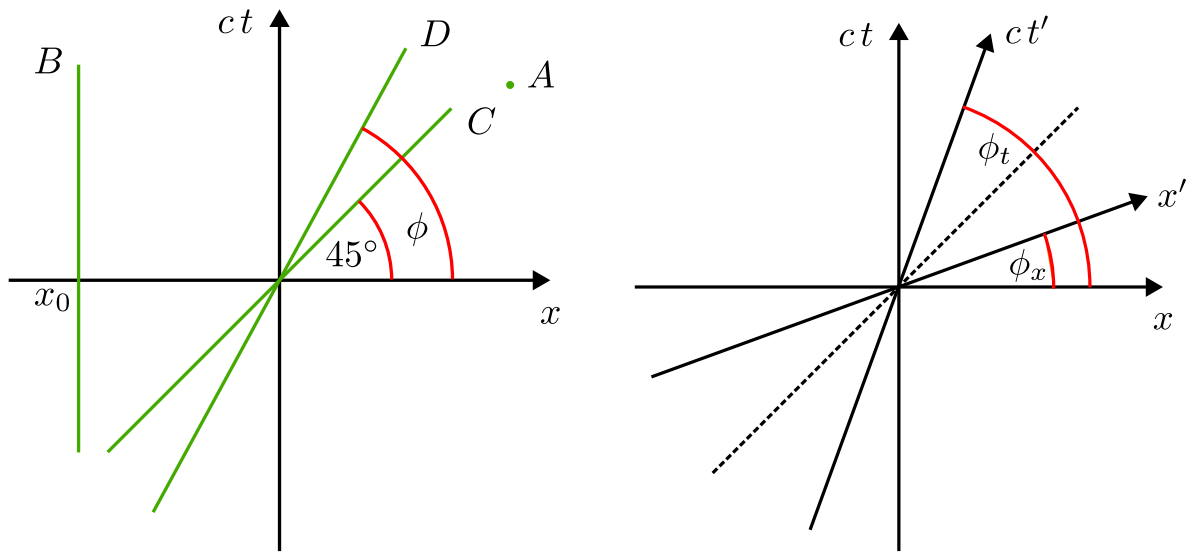
Remark 7.2.5. We began by asking for the linear transformations that preserve the wave equation (7.1.6), and we found the Lorentz boosts (7.2.6). These same transformations preserve the spacetime distance (7.2.8). One can also reverse the logic by taking (7.2.8) as the fundamental invariant, and then derive Lorentz transformations as the symmetries that preserve it. This is the viewpoint we will adopt when we move to higher dimensions.

§7.3 Measurement in Special Relativity

In this section we examine in more detail the physical implications of Lorentz boosts, now viewed as fundamental symmetries of spacetime. To streamline the discussion, it is useful to introduce some standard terminology. A *spacetime event* is represented by a point (t, x) in spacetime. The *worldline* of a point particle is its trajectory $(t(\lambda), x(\lambda))$ in spacetime, parametrised by a real parameter λ.

It is convenient to represent events on a *spacetime diagram*, with axes measured in units of length, x horizontally and ct vertically. In Figure 21a an event A is represented by a single point. A particle at rest has $x =$ constant and therefore a vertical worldline, such as line B. A photon travels at the speed of light, so its worldline satisfies $x = \pm ct$ and appears as a straight line at 45° to the x-axis, as shown by line C. More generally, a particle moving with constant speed $v < c$ has $x = vt$, hence a straight worldline making an angle ϕ with the horizontal such that,

$$\tan\phi = \frac{ct}{x} = \frac{c}{v}\,. \tag{7.3.1}$$



(a) Spacetime representation of worldlines. (b) Coordinate systems of different observers.

Figure 21: Spacetime diagrams.

In Figure 21a this is represented by the line D.

It is also useful to draw, on the same diagram, the coordinate axes of two inertial observers R and R' related by the Lorentz transformation (7.2.6), as shown in Figure 21b. The x' axis is the set of events with $t' = 0$. Using (7.2.6), the condition $t' = 0$ becomes,

$$t - \frac{v}{c^2}\, x = 0 \qquad \Rightarrow \qquad ct = \frac{v}{c}\, x\,, \tag{7.3.2}$$

so, from the point of view of R, the x' axis is a straight line through the origin at angle $\phi_x = \arctan(v/c)$ above the x-axis. Similarly, the ct' axis is the set of events with $x' = 0$, i.e.,

$$x - vt = 0 \qquad \Rightarrow \qquad ct = \frac{c}{v}\, x\,, \tag{7.3.3}$$

so it is a straight line at angle $\phi_t = \arctan(c/v)$ above the x-axis. Note that $\phi_x + \phi_t = \pi/2$, and that the light rays $x = \pm ct$ bisect the angle between the ct' and x' axes.

With these tools in place, we can discuss the implications of special relativity for measurements of space and time. We begin with simultaneity, a concept that is crucial for defining the length of an object.

Consider two events E_1 and E_2 which are simultaneous according to observer R and separated by a spatial distance l. If E_1 has coordinates (t, x) in R, then E_2 has coordinates $(t, x + l)$. Thus $\Delta t = 0$ and $\Delta x = l$. Now consider an observer R' moving with respect to R at velocity v. The Lorentz transformation (7.2.6) implies,

$$\Delta t' = \gamma \left(\Delta t - \frac{v}{c^2}\, \Delta x \right) = -\gamma\, \frac{v}{c^2}\, l \neq 0\,, \tag{7.3.4}$$

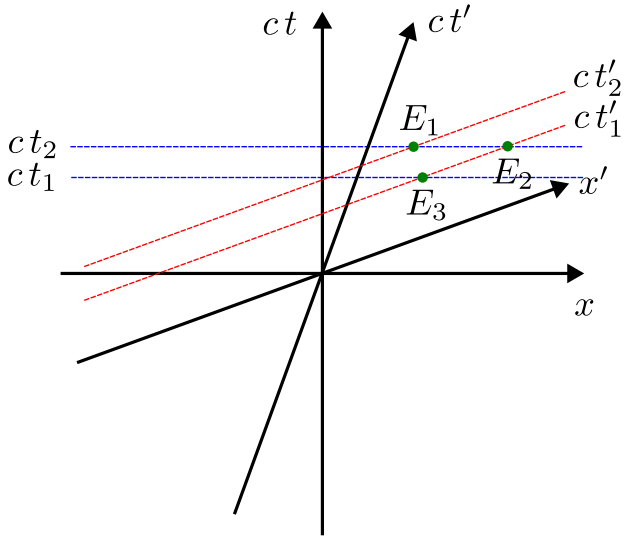
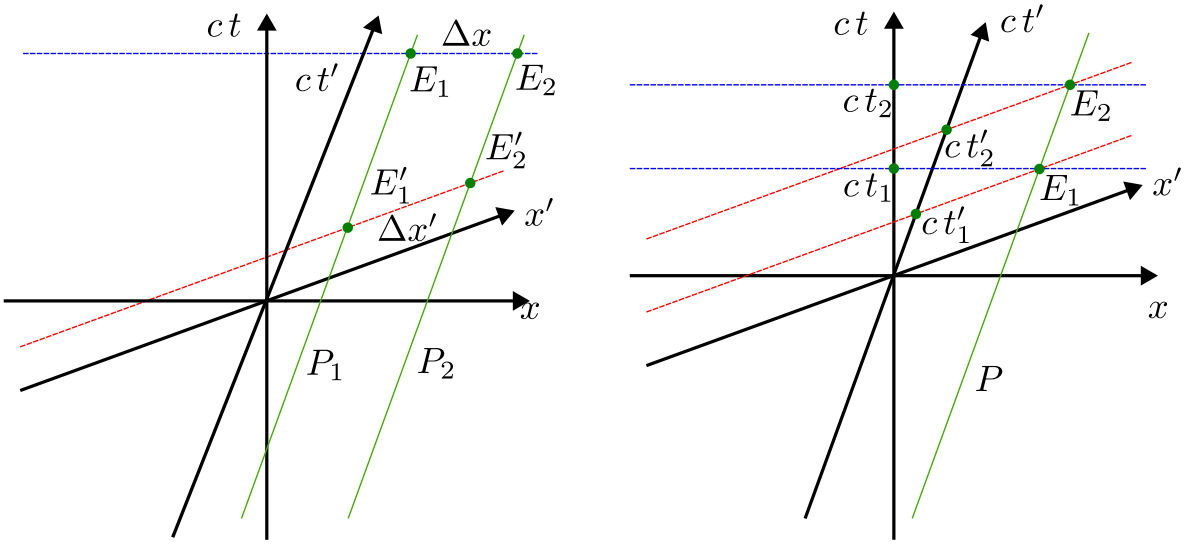


Figure 22: Spacetime diagram illustrating simultaneity in special relativity.



(a) Spacetime diagram for length contraction.

(b) Spacetime diagram for time dilation.

Figure 23: Measurement of length and time in special relativity.

so E_1 and E_2 are *not* simultaneous in R'. This is in sharp contrast with Newtonian physics, where simultaneity is absolute. The same idea is illustrated in Figure 22: E_1 and E_2 lie on a horizontal line (constant t) for observer R, but not for R'.

Simultaneity is essential when measuring the length of an object. Consider a rigid rod that is at rest in the frame R'. In Figure 23a, P_1 and P_2 are the worldlines of the two ends of the rod. The length measured in the rod's rest frame is obtained by comparing the positions of the two ends at the same time t', i.e. for events E_1' and E_2' with $\Delta t' = 0$.

Denote this *proper length* by,

$$\Delta x' = l \,. \tag{7.3.5}$$

An observer R who sees the rod moving must also compare the ends at the same time in their own frame, i.e. for events E_1 and E_2 with $\Delta t = 0$. Denote the length measured by R by,

$$\Delta x = l' \,. \tag{7.3.6}$$

Using the inverse Lorentz transformation (or equivalently rewriting (7.2.6) in terms of differences) gives,

$$\begin{aligned} \Delta x &= \gamma \left(\Delta x' + v \, \Delta t' \right) \,, \\ \Delta t &= \gamma \left(\Delta t' + \frac{v}{c^2} \, \Delta x' \right) \,. \end{aligned} \tag{7.3.7}$$

Imposing simultaneity in R means setting $\Delta t = 0$, which fixes,

$$\Delta t' = -\frac{v}{c^2} \, \Delta x' \,. \tag{7.3.8}$$

Substituting this into the expression for Δx yields,

$$\boxed{l' = \frac{l}{\gamma} \,.} \tag{7.3.9}$$

Thus a moving rod is measured to be shorter along the direction of motion. This effect is called *length contraction*. The rest-frame length l is the *proper length*.

Next, consider a clock that is at rest in R', with worldline P as in Figure 23b. Let E_1 and E_2 be two successive ticks of the clock. Since the clock is at rest in R', these events occur at the same position in R', so $\Delta x' = 0$. The time measured by the clock is therefore,

$$\Delta t' = \tau \,, \tag{7.3.10}$$

which is the *proper time* between the ticks. In frame R the same two events are separated by some coordinate time $\Delta t = \tau'$. Using (7.2.6) for differences and setting $\Delta x' = 0$ gives,

$$\boxed{\tau' = \gamma \, \tau \,.} \tag{7.3.11}$$

Thus, a moving clock is observed to tick more slowly. The time interval between ticks is longer in the frame in which the clock is moving. This is *time dilation*. The rest-frame interval τ is the *proper time*.

Note 7.3.1

Muons are elementary massive particles with an average lifetime of $\tau_0 = 2.2\,\mu$s when at rest. In a number of different experiments, it has been confirmed that their lifetime increases when they move at high speeds. This effect is most striking for cosmic-ray muons produced high in the atmosphere. Although τ_0 is only a few microseconds, relativistic time dilation allows many of them to reach the Earth's surface before decaying. The same phenomenon can be described from the muon's point of view. In the muon's rest frame, its lifetime is still τ_0, but the distance to the Earth's surface is length contracted by a factor of $1/\gamma$, so the trip appears shorter and can be completed before the muon decays.

In Section 7.1 we saw that the wave equation is preserved by Lorentz boosts. Here we examine how the frequency and wavenumber of a monochromatic wave change between inertial observers.

Consider a scalar field in $1+1$ dimensions with a superposition of left- and right-moving monochromatic waves in the frame R,

$$\psi(x,t) = A_R \sin\big(k(x - c\,t)\big) + A_L \sin\big(k(x + c\,t)\big)\,, \qquad \omega = c\,k\,, \tag{7.3.12}$$

where A_R and A_L are constant amplitudes. After a Lorentz boost with velocity v, so that R' moves to the right relative to R, the same field can be written in the form,

$$\psi(x',t') = A_R \sin\big(k'_R(x' - c\,t')\big) + A_L \sin\big(k'_L(x' + c\,t')\big)\,, \tag{7.3.13}$$

with transformed wavenumbers,

$$\begin{aligned} k'_R &= \gamma\left(1 - \frac{v}{c}\right) k = \sqrt{\frac{c-v}{c+v}}\,k\,, \\ k'_L &= \gamma\left(1 + \frac{v}{c}\right) k = \sqrt{\frac{c+v}{c-v}}\,k\,. \end{aligned} \tag{7.3.14}$$

Thus the moving observer generally sees different wavelengths (and hence different frequencies) for right- and left-moving waves, while the propagation speed remains c. This is the *relativistic Doppler effect*.

Note 7.3.2

In astrophysics, a *redshift* means that light is observed at a lower frequency, or equivalently longer wavelength, than it was emitted. One mechanism is precisely the relativistic Doppler effect. If a source is receding from the observer along the line of sight, the observed frequency is reduced, or *redshifted*, while an approaching source produces a *blueshift*. In observational cosmology one also encounters *cosmological redshift*, due to the expansion of space. At sufficiently small distances this agrees with a Doppler

interpretation, but it is conceptually distinct.

§7.4 *Lorentz Transformations in* $3+1$ *Dimensions*

So far we have worked in $1+1$ dimensions, where Lorentz boosts arise naturally as the transformations that preserve the wave equation and, equivalently, the spacetime distance. To apply these ideas to Maxwell's theory we now move to $3+1$ dimensions and introduce the notation that makes Lorentz symmetry transparent.

Following Remark 7.2.5, our starting point is the invariant spacetime interval in $3+1$ dimensions,

$$\boxed{\Delta s^2 = -c^2\,\Delta t^2 + \Delta x^2 + \Delta y^2 + \Delta z^2\,.} \tag{7.4.1}$$

As in the $1+1$ dimensional case, this is a *pseudo*-norm with indefinite signature; see Remark 7.2.4.

It is convenient to package spacetime coordinates into a single four-vector,

$$x^\mu = \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} = \begin{pmatrix} c\,t \\ x \\ y \\ z \end{pmatrix}\,, \tag{7.4.2}$$

where we multiplied time by c so that all components have units of length. Using Einstein's summation convention (also known as *contraction* over repeated indices), the interval can be written as,

$$\boxed{\begin{aligned}\Delta s^2 &= -\left(\Delta x^0\right)^2 + \left(\Delta x^1\right)^2 + \left(\Delta x^2\right)^2 + \left(\Delta x^3\right)^2 \\ &= \Delta x^\mu\,\eta_{\mu\nu}\,\Delta x^\nu\,,\end{aligned}} \tag{7.4.3}$$

where $\eta_{\mu\nu}$ is the *Minkowski metric*[10],

$$\boxed{\eta_{\mu\nu} = \mathrm{diag}(-1,1,1,1)\,.} \tag{7.4.4}$$

The "up–down" placement of repeated indices in (7.4.3) is deliberate and will become important later.

The inverse metric is denoted $\eta^{\mu\nu}$ and is defined by,

$$\boxed{\eta^{\alpha\mu}\eta_{\mu\beta} = \delta^\alpha_\beta\,,} \tag{7.4.5}$$

[10]Here we use the "mostly plus" conventino for the Minkowski metric. In some textbooks you will find the "mostly minus" definition $\eta_{\mu\nu} = \mathrm{diag}(1,-1,-1,-1)$. The only consequence is that a few signs in our equations will defer.

where δ^α_β is Kronecker's delta. For the Minkowski metric, the matrix $\eta^{\mu\nu}$ has the same diagonal form as $\eta_{\mu\nu}$.

A homogeneous Lorentz transformation acts linearly on spacetime vectors as,

$$\boxed{x^{\mu\prime} = \Lambda^\mu{}_\nu \, x^\nu \, .} \tag{7.4.6}$$

We will keep careful track of "up" and "down" indices. The free index on the right-hand side appears in the same position as the index on the left-hand side, as required by the contraction rule.

More generally, we define the Lorentz-invariant inner product of two four-vectors V^μ and W^μ by,

$$\boxed{V \cdot W \equiv V^\mu \eta_{\mu\nu} W^\nu \, .} \tag{7.4.7}$$

This satisfies the usual bilinearity and symmetry properties of an inner product, except that it is not positive definite.

Demanding that (7.4.6) preserves the interval (7.4.3) for all Δx^μ gives the defining condition for a Lorentz transformation,

$$\boxed{\Lambda^\rho{}_\mu \, \eta_{\rho\sigma} \, \Lambda^\sigma{}_\nu = \eta_{\mu\nu} \, .} \tag{7.4.8}$$

As an immediate consequence, the inner product (7.4.7) is preserved,

$$V^{\rho\prime} \eta_{\rho\sigma} W^{\sigma\prime} = V^\mu \eta_{\mu\nu} W^\nu \, , \tag{7.4.9}$$

provided that all vectors transform accordingly,

$$\boxed{W^{\mu\prime} = \Lambda^\mu{}_\nu \, W^\nu \, ,} \tag{7.4.10}$$

and similarly for V^μ.

In matrix notation (7.4.8) becomes,

$$\boxed{\Lambda^T \, \eta \, \Lambda = \eta \, ,} \tag{7.4.11}$$

where $(\Lambda^T)_\mu{}^\rho = \Lambda^\rho{}_\mu$. Multiplying by η^{-1}, one finds a useful expression for the inverse Lorentz transformation,

$$\boxed{\begin{aligned} \Lambda^{-1} &= \eta^{-1} \, \Lambda^T \, \eta \, , & (7.4.12) \\ \left(\Lambda^{-1}\right)^\nu{}_\mu &= \eta^{\nu\pi} \, \Lambda^\rho{}_\pi \, \eta_{\rho\mu} \, . & (7.4.13) \end{aligned}}$$

Equivalently, the inverse is obtained by taking a transpose and then raising/lowering indices with η.

Four-vectors carry an "up" index. Objects with a "down" index are called *covectors* or *dual vectors*. Given a vector W^ν, we define its associated covector after "lowering" the index by contractring with the Minkowski metric,

$$\boxed{W_\mu = \eta_{\mu\nu}\, W^\nu\,.} \tag{7.4.14}$$

Conversely, given a covector W_μ, we "raise" its index using the inverse metric to obtain a vector,

$$\boxed{W^\nu = \eta^{\nu\mu}\, W_\mu\,.} \tag{7.4.15}$$

With these conventions, the inner product (7.4.7) can be written in equivalent ways,

$$\boxed{V\cdot W = V_\mu W^\mu = V^\mu W_\mu\,.} \tag{7.4.16}$$

How do covectors transform? Starting from $W_\nu = \eta_{\nu\mu}W^\mu$ and using (7.4.10) gives,

$$W_{\nu\prime} = \eta_{\nu\mu}\, W^{\mu\prime} = \eta_{\nu\mu}\,\Lambda^\mu{}_\rho\, W^\rho = \eta_{\nu\mu}\,\Lambda^\mu{}_\rho\,\eta^{\rho\sigma}\, W_\sigma \equiv \Lambda_\nu{}^\sigma\, W_\sigma\,, \tag{7.4.17}$$

and therefore,

$$\boxed{W_{\nu\prime} = \Lambda_\nu{}^\sigma\, W_\sigma\,.} \tag{7.4.18}$$

This is one reason it is worth tracking index positions carefully, "up" and "down" indices correspond to objects with different Lorentz transformation rules. Note that we defined $\Lambda_\nu{}^\sigma$ after "raising" the "down" index and "lowering" the "up" index of $\Lambda^\mu{}_\rho$.

Example 7.4.1. *Partial derivatives as a covector.*
A key example of a covector is the gradient of a scalar field. Let ψ be a scalar function and define $V_\mu = \partial_\mu\psi$. Consider a Lorentz transformation (7.4.6)*. By the chain rule,*

$$\boxed{\partial_\mu = \frac{\partial x^{\alpha\prime}}{\partial x^\mu}\,\partial_{\alpha\prime} = \Lambda^\alpha{}_\mu\,\partial_{\alpha\prime}\,.} \tag{7.4.19}$$

Using (7.4.8) *one can show the identity,*

$$\boxed{\Lambda_\gamma{}^\mu\,\Lambda^\alpha{}_\mu = \delta^\alpha_\gamma\,,} \tag{7.4.20}$$

which allows us to invert (7.4.19)*,*

$$\boxed{\partial_{\gamma\prime} = \Lambda_\gamma{}^\mu\,\partial_\mu\,.} \tag{7.4.21}$$

Therefore,

$$V_{\alpha\prime} = \partial_{\alpha\prime}\psi = \Lambda_\alpha{}^\mu\,\partial_\mu\psi = \Lambda_\alpha{}^\mu\, V_\mu\,, \tag{7.4.22}$$

which is exactly the covector transformation rule (7.4.18)*.*

Remark 7.4.2. In Example 7.4.1 the object ∂_μ is a linear operator, but it transforms with the same components as a covector. This is a recurring and very useful theme in relativistic physics.

Equation (7.4.11) imposes ten independent conditions on the sixteen entries of a 4×4 matrix, leaving six independent parameters. Physically, these correspond to three boosts and three spatial rotations.

- **Boost in the x-direction.**
  Acting with the $1+1$ dimensional boost (7.2.6) in the (t, x) plane and leaving y and z unchanged gives,

$$\Lambda_x(v_x) = \begin{pmatrix} \gamma & -\frac{v_x}{c}\gamma & 0 & 0 \\ -\frac{v_x}{c}\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad \gamma = \frac{1}{\sqrt{1-(v_x/c)^2}}\,. \tag{7.4.23}$$

  It is a good exercise to verify directly that (7.4.11) is satisfied.

- **Boost in the y-direction.**
  Similarly,

$$\Lambda_y(v_y) = \begin{pmatrix} \gamma & 0 & -\frac{v_y}{c}\gamma & 0 \\ 0 & 1 & 0 & 0 \\ -\frac{v_y}{c}\gamma & 0 & \gamma & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad \gamma = \frac{1}{\sqrt{1-(v_y/c)^2}}\,. \tag{7.4.24}$$

- **Boost in the z-direction.**
  And for the z-direction we have,

$$\Lambda_z(v_z) = \begin{pmatrix} \gamma & 0 & 0 & -\frac{v_z}{c}\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\frac{v_z}{c}\gamma & 0 & 0 & \gamma \end{pmatrix}, \qquad \gamma = \frac{1}{\sqrt{1-(v_z/c)^2}}\,. \tag{7.4.25}$$

- **Spatial rotations.**
  Keeping the time coordinate fixed and rotating only the spatial coordinates corresponds to,

$$\Lambda_{\mathbf{R}} = \begin{pmatrix} 1 & 0 \\ 0 & \mathbf{R} \end{pmatrix}, \qquad \mathbf{R}^T\mathbf{R} = I_{3\times 3}\,, \tag{7.4.26}$$

  where $\mathbf{R}$ is a 3×3 rotation matrix with three independent parameters.

Lorentz transformations form a group under matrix multiplication the *Lorentz group*. The product of two Lorentz transformations is a Lorentz transformation, and every Lorentz transformation has an inverse. A convenient, though not unique, way to parameterise a general element is as a product of boosts and a spatial rotation, for example,

$$\Lambda(\mathbf{v}, \mathbf{R}) = \Lambda_x(v_x)\, \Lambda_y(v_y)\, \Lambda_z(v_z)\, \Lambda_{\mathbf{R}}\,. \tag{7.4.27}$$

Remark 7.4.3. The six-parameter family of Lorentz transformations described above satisfies the defining condition (7.4.8). However, as in Remark 7.2.1, there are also matrices that satisfy (7.4.8) but lie outside this connected family. To see this, take the determinant of both sides of (7.4.11),

$$\det(\Lambda^T \eta \Lambda) = \det(\eta) \Rightarrow (\det \Lambda)^2 = 1 \Rightarrow \det \Lambda = \pm 1\,.$$

All matrices in our family satisfy $\det \Lambda(\mathbf{v}, \mathbf{R}) = +1$. Therefore any Lorentz transformation with $\det \Lambda = -1$ is disconnected from the identity and cannot be reached by continuously varying the parameters $\mathbf{v}$ and $\mathbf{R}$. Simple examples are time reversal and a spatial reflection, respectively,

$$\Lambda_T = \mathrm{diag}(-1, 1, 1, 1)\,, \qquad \Lambda_P = \mathrm{diag}(1, -1, 1, 1)\,.$$

The group of matrices Λ with $\det \Lambda = 1$ is called the *proper Lorentz group* $SO(1, 3)$.

As a useful check of our promise, let us verify that the scalar wave equation in $3 + 1$ dimensions,

$$\partial_t^2 \psi - c^2\, \nabla^2 \psi = 0\,, \tag{7.4.28}$$

is Lorentz invariant. Using $x^0 = ct$ and $\partial_0 = \partial/\partial x^0$, this can be written as,

$$-\frac{\partial}{\partial x^0}\frac{\partial}{\partial x^0}\psi + \frac{\partial}{\partial x^1}\frac{\partial}{\partial x^1}\psi + \frac{\partial}{\partial x^2}\frac{\partial}{\partial x^2}\psi + \frac{\partial}{\partial x^3}\frac{\partial}{\partial x^3}\psi = 0\,, \tag{7.4.29}$$

or, more compactly,

$$\boxed{\eta^{\mu\nu}\, \partial_\mu \partial_\nu \psi = 0\,.} \tag{7.4.30}$$

Under a Lorentz transformation, the derivatives transform as in (7.4.19), leading to the primed frame equation,

$$\eta^{\mu\nu} \Lambda^\alpha{}_\mu \Lambda^\beta{}_\nu\, \partial_{\alpha'} \partial_{\beta'} \psi = 0 \overset{(7.4.8)}{\Rightarrow} \eta^{\alpha\beta}\, \partial_{\alpha'} \partial_{\beta'} \psi' = 0\,, \tag{7.4.31}$$

which is again the same wave equation.

§7.5 *Vectors, Covectors and Tensors*

In the previous section we introduced vectors and covectors, together with their transformation rules under Lorentz transformations. This is extremely useful when writing physical laws, because it allows us to build quantities that are independent of the inertial frame. The simplest example is the contraction of a vector V^μ with a covector W_μ,

$$V^\mu W_\mu \,, \tag{7.5.1}$$

which is essentially the Lorentz-invariant inner product between V^μ and the dual W^μ, see (7.4.7).

Another way to see the invariance of this contraction is to use the fact that vectors and covectors transform in "opposite" ways, according to (7.4.10) and (7.4.18). Identity (7.4.20) then guarantees,

$$V^{\pi'} W_{\pi'} = \Lambda^\pi{}_\alpha \, \Lambda_\pi{}^\beta \, V^\alpha \, W_\beta \overset{(7.4.20)}{=} \delta^\beta_\alpha \, V^\alpha \, W_\beta = V^\alpha \, W_\alpha \,. \tag{7.5.2}$$

This may look like a small observation, but it is the basic mechanism by which we construct many useful Lorentz-invariant objects. In particular, as we observed in Example 7.4.2, partial derivatives transform like covectors. Contracting a derivative with a vector therefore also produces an invariant quantity. Depending on what the derivative acts on, this leads to different examples. For instance, using (7.4.21) together with (7.4.20), one finds that the divergence of a vector is Lorentz invariant,

$$\boxed{\partial_{\alpha'} J^{\alpha'} = \partial_\mu J^\mu \,.} \tag{7.5.3}$$

Another example is the directional derivative of a scalar field,

$$V^{\alpha'} \, \partial_{\alpha'} \psi = V^\mu \, \partial_\mu \psi \,. \tag{7.5.4}$$

Remark 7.5.1. Since the last identity holds for *any* scalar field ψ, it is often convenient to view it as an identity of differential operators,

$$V^{\alpha'} \, \partial_{\alpha'} = V^\mu \, \partial_\mu \,. \tag{7.5.5}$$

This point of view becomes central in differential geometry and general relativity. Loosely speaking, the partial derivatives ∂_μ form a basis of a vector space, and a linear combination $V^\mu \partial_\mu$ is the geometric, coordinate-independent object.

More generally, we define *tensors* of type $\binom{m}{n}$ as objects with m vector indices and n covector indices. You can think of these as generalisations of matrices to more than two indices. What makes an object a tensor is its transformation law under the Lorentz group,

$$\boxed{T^{\mu'_1 \cdots \mu'_m}{}_{\nu'_1 \cdots \nu'_n} = \Lambda^{\mu'_1}{}_{\mu_1} \cdots \Lambda^{\mu'_m}{}_{\mu_m} \, \Lambda_{\nu'_1}{}^{\nu_1} \cdots \Lambda_{\nu'_n}{}^{\nu_n} \, T^{\mu_1 \cdots \mu_m}{}_{\nu_1 \cdots \nu_n} \,.} \tag{7.5.6}$$

With this definition, vectors are $\binom{1}{0}$ tensors, covectors are $\binom{0}{1}$ tensors, and scalars are $\binom{0}{0}$ tensors.

Note 7.5.2

From the definition one can infer a number of useful rules,

1. If a tensor is equal to zero in one frame, it is equal to zero in all frames.
2. The sum of two $\binom{m}{n}$ tensors is a tensor of the same type.
3. The product of a $\binom{m_1}{n_1}$ tensor and a $\binom{m_2}{n_2}$ tensor is a $\binom{m_1+m_2}{n_1+n_2}$ tensor.
4. Raising (lowering) an index of a $\binom{m}{n}$ tensor using the Minkowski metric produces a $\binom{m+1}{n-1}$ ($\binom{m-1}{n+1}$) tensor.
5. Contracting one vector index with one covector index of a $\binom{m}{n}$ tensor produces a $\binom{m-1}{n-1}$ tensor.
6. Taking a partial derivative of a $\binom{m}{n}$ tensor produces a $\binom{m}{n+1}$ tensor.

Remark 7.5.3. We have already seen the last property in action in Example 7.4.1. Starting from a scalar ψ (a $\binom{0}{0}$ tensor), taking one derivative produced $\partial_\mu \psi$ (a $\binom{0}{1}$ tensor). The general statement can be proven by combining the transformation rule (7.4.21) for derivatives with the tensor transformation rule (7.5.6).

Note 7.5.4

Be cautious with the *order* of indices. Starting from a $\binom{0}{3}$ tensor $T_{\alpha\beta\gamma}$, raising one index produces three distinct $\binom{1}{2}$ tensors,

$$T^{\alpha}{}_{\beta\gamma}\,, \qquad T_{\alpha}{}^{\beta}{}_{\gamma}\,, \qquad T_{\alpha\beta}{}^{\gamma}\,. \tag{7.5.7}$$

This notation makes it clear which index was raised. By contrast, a shorthand such as $T^{\alpha}_{\beta\gamma}$ is ambiguous. It does not uniquely indicate which of the original indices was raised.

Example 7.5.5. *Contraction Let $T^{\alpha\beta}{}_{\gamma}$ be a $\binom{2}{1}$ tensor and define the contraction*[11]*,*

$$V^{\alpha} = T^{\alpha\beta}{}_{\beta}\,. \tag{7.5.8}$$

We claim that V^{α} transforms as a $\binom{1}{0}$ tensor. Indeed, using (7.5.6)*,*

$$\begin{aligned} V^{\alpha'} = T^{\alpha'\beta'}{}_{\beta'} &= \Lambda^{\alpha'}{}_{\alpha}\,\Lambda^{\beta'}{}_{\rho}\,\Lambda_{\beta'}{}^{\sigma}\,T^{\alpha\rho}{}_{\sigma} \overset{(7.4.20)}{=} \Lambda^{\alpha'}{}_{\alpha}\,\delta^{\sigma}{}_{\rho}\,T^{\alpha\rho}{}_{\sigma} \\ &= \Lambda^{\alpha'}{}_{\alpha}\,T^{\alpha\rho}{}_{\rho} = \Lambda^{\alpha'}{}_{\alpha}\,V^{\alpha}\,, \end{aligned}$$

[11] Contracting instead as $T^{\beta\alpha}{}_{\beta}$ would in general produce a different vector.

which is exactly the vector transformation rule.

If a physical law (for example, Maxwell's equations) can be written in tensorial form, then it automatically holds in every inertial frame and such equations are called *covariant*. In particular, while charge density, current density, and the electric and magnetic fields do transform between frames, the *tensorial* form of the equations they satisfy remains the same. In the following sections we will package Maxwell's equations into tensors with the result being remarkably compact.

Example 7.5.6. *As an illustration, let us revisit the Lorentz invariance of the scalar wave equation* (7.4.30)*. By the last property in Note 7.5.2, the object,*

$$C_{\mu\nu} = \partial_\mu \partial_\nu \psi \tag{7.5.9}$$

is a $\binom{0}{2}$ *tensor since* ψ *is a scalar and therefore a* $\binom{0}{0}$ *tensor. By the fourth property, raising one index gives,*

$$C^\mu{}_\nu = \eta^{\mu\lambda}\, \partial_\lambda \partial_\nu \psi \,, \tag{7.5.10}$$

which is a $\binom{1}{1}$ *tensor. The fifth property suggests that contracting results in,*

$$D = C^\mu{}_\mu = \eta^{\mu\lambda}\, \partial_\lambda \partial_\mu \psi \,, \tag{7.5.11}$$

a $\binom{0}{0}$ *tensor, hence Lorentz invariant. Therefore the equation* $D = 0$ *is invariant in all inertial frames by the first property, which is exactly the Lorentz invariance of* (7.4.30)*.*

§7.6 Charge Continuity in Special Relativity

One of the first fundamental statements we encountered in Maxwell's theory was the conservation of charge, (2.1.8). Since this is a basic physical principle, we expect it to hold in every inertial frame. In special relativity this means that the continuity equation should retain the same form under Lorentz transformations.

To make this manifest, we rewrite the continuity equation in spacetime notation. Starting from (2.1.8), we insert a factor of c and use $x^0 = ct$, so that $\partial/\partial x^0 = (1/c)\,\partial/\partial t$,

$$\frac{\partial}{\partial x^0}(c\rho) + \nabla \cdot \mathbf{J} = 0 \,.$$

This can be written compactly as,

$$\boxed{\partial_\mu J^\mu = 0 \,,} \tag{7.6.1}$$

where we introduced the *spacetime current*,

$$\boxed{J^\mu = \begin{pmatrix} c\,\rho \\ \mathbf{J} \end{pmatrix} .} \tag{7.6.2}$$

Using the Remark around (7.5.3), the equation (7.6.1) is Lorentz invariant provided J^μ transforms as a spacetime vector. In other words, charge density and current density are not independent relativistic objects. They are the time and space components of a single four-vector, and they mix under Lorentz boosts.

This observation has immediate physical consequences that are not visible in a purely Newtonian treatment. As a simple illustration, suppose that in some inertial frame we have a static charge distribution with no current,

$$J^\mu = \begin{pmatrix} c\,\rho \\ \mathbf{0} \end{pmatrix}. \tag{7.6.3}$$

What does a moving observer measure? Consider an observer boosted with velocity v_x along the x-axis. Under the Lorentz boost (7.4.23) we find,

$$J^{\mu'} = (\Lambda_x(v_x))^\mu{}_\nu\, J^\nu = \begin{pmatrix} \gamma\, c\,\rho \\ -\gamma\,\rho\, v_x \\ 0 \\ 0 \end{pmatrix}. \tag{7.6.4}$$

Thus the boosted observer measures,

$$\rho' = \gamma\,\rho, \qquad J'_x = -\gamma\,\rho\, v_x = -\rho'\, v_x\,. \tag{7.6.5}$$

There are two key messages,

- The charge density is *larger* in the moving frame by a factor of γ. This is exactly what we should expect from length contraction. The same total charge is squeezed into a shorter length along the direction of motion, so the density increases.

- The conclusion of example 2.1.2 seems to be wrong when we take into account relativistic effects as we need to include an extra factor of γ when we relate the fluid velocity to the current density. However, the relationship between the transformed charge density ρ' and the current J'_x is the same. Therefore our conclusion from 2.1.2 holds provided that we use the correct charge density that we see in our frame. The minus sign we found is due to the fact that the observer moves to right and sees the charge moving to their left.

In summary, (7.6.1) packages charge conservation into a manifestly Lorentz-invariant form and makes clear that ρ and $\mathbf{J}$ are frame-dependent components of the same relativistic current.

§7.7 Electromagnetic Field Tensor

In the previous sections we learned two lessons that, at first sight, seem unrelated,

- Maxwell's theory is gauge invariant, so it is naturally expressed in terms of potentials.
- Maxwell's theory is compatible with Lorentz transformations, so it should admit a covariant spacetime formulation.

In this section we combine these ideas and package the electric and magnetic fields into a single spacetime tensor.

Start from the scalar and vector potentials φ and $\mathbf{A}$ introduced in Section 6.7, and assemble them into the *four-potential*,

$$A^\mu = \begin{pmatrix} \varphi/c \\ \mathbf{A} \end{pmatrix} . \tag{7.7.1}$$

The natural assumption, which will be justified by what follows, is that A^μ transforms as a spacetime vector under Lorentz transformations (7.4.10). Lowering the index with the Minkowski metric then gives the associated covector,

$$\boxed{A_\mu = \eta_{\mu\nu} A^\nu = \begin{pmatrix} -\varphi/c \\ \mathbf{A} \end{pmatrix} .} \tag{7.7.2}$$

Recall that the scalar and vector potentials are not unique. They are defined only up to the gauge transformations (6.7.4). In spacetime notation this becomes the very compact statement,

$$\boxed{A_\mu \to A_\mu + \partial_\mu \chi \, ,} \tag{7.7.3}$$

where the gauge parameter $\chi(t, \mathbf{x})$ is a scalar field. For this reason one often calls A_μ a *gauge field*.

The gauge-invariant object built from A_μ and one derivative is the antisymmetric tensor,

$$\boxed{F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \, .} \tag{7.7.4}$$

Indeed, if we perform a gauge transformation (7.7.3), the new electromagnetic tensor is equal to the original one,

$$F'_{\mu\nu} = \partial_\mu A'_\nu - \partial_\nu A'_\mu = \partial_\mu \left(A_\nu + \partial_\nu \chi \right) - \partial_\nu \left(A_\mu + \partial_\mu \chi \right) = \partial_\mu A_\nu - \partial_\nu A_\nu = F_{\mu\nu} \, . \tag{7.7.5}$$

By construction it satisfies the antisymmetry relation,

$$\boxed{F_{\mu\nu} = -F_{\nu\mu} \, .} \tag{7.7.6}$$

Using $x^0 = ct$, $\partial_0 = (1/c)\partial_t$, together with $\mathbf{E} = -\nabla\varphi - \partial_t\mathbf{A}$ and $\mathbf{B} = \nabla \times \mathbf{A}$, one finds that $F_{\mu\nu}$ contains exactly the components of $\mathbf{E}$ and $\mathbf{B}$. In particular,

$$\boxed{\begin{aligned} F_{0i} &= \partial_0 A_i - \partial_i A_0 = -\frac{1}{c}\,E_i\,, \\ F_{ij} &= \partial_i A_j - \partial_j A_i = \varepsilon_{ijk}\,B_k\,. \end{aligned}} \tag{7.7.7}$$

In matrix form,

$$F_{\mu\nu} = \begin{pmatrix} 0 & -E_x/c & -E_y/c & -E_z/c \\ E_x/c & 0 & B_z & -B_y \\ E_y/c & -B_z & 0 & B_x \\ E_z/c & B_y & -B_x & 0 \end{pmatrix}. \tag{7.7.8}$$

Raising both indices gives,

$$F^{\mu\nu} = \eta^{\mu\alpha}\eta^{\nu\beta}F_{\alpha\beta} = \begin{pmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & B_z & -B_y \\ -E_y/c & -B_z & 0 & B_x \\ -E_z/c & B_y & -B_x & 0 \end{pmatrix}. \tag{7.7.9}$$

Thus, in special relativity the electric and magnetic fields are not separate objects, they are simply different components of a single antisymmetric rank-2 tensor.

Since A_μ is a covector and ∂_μ also transforms as a covector, the definition (7.7.4) implies that $F_{\mu\nu}$ transforms as a $\binom{0}{2}$ tensor,

$$F_{\mu'\nu'} = \Lambda_{\mu'}{}^{\alpha}\,\Lambda_{\nu'}{}^{\beta}\,F_{\alpha\beta}\,, \qquad F^{\mu'\nu'} = \Lambda^{\mu'}{}_{\alpha}\,\Lambda^{\nu'}{}_{\beta}\,F^{\alpha\beta}\,. \tag{7.7.10}$$

This immediately implies that, under boosts, the components of $\mathbf{E}$ and $\mathbf{B}$ must mix.

As an example, consider a boost in the x-direction with velocity v_x, the matrix $\Lambda_x(v_x)$ from (7.4.23). Applying the transformation to the matrix (7.7.9) gives,

$$F^{\mu'\nu'} = \begin{pmatrix} 0 & E_x/c & \gamma\,(E_y - v_x\,B_z)/c & \gamma\,(E_z + v_x\,B_y)/c \\ -E_x/c & 0 & \gamma\left(B_z - \frac{v_x}{c^2}\,E_y\right) & -\gamma\,\left(B_y + \frac{v_x}{c^2}\,E_z\right) \\ -\gamma\,(E_y - v_x\,B_z)/c & -\gamma\left(B_z - \frac{v_x}{c^2}\,E_y\right) & 0 & B_x \\ -\gamma\,(E_z + v_x\,B_y)/c & \gamma\,\left(B_y + \frac{v_x}{c^2}\,E_z\right) & -B_x & 0 \end{pmatrix}.$$

Comparing with the general form (7.7.9), we can read off the transformation laws for the fields,

$$\begin{aligned} E'_x &= E_x, & B'_x &= B_x\,, \\ E'_y &= \gamma\,(E_y - v_x\,B_z)\,, & B'_y &= \gamma\,\left(B_y + \frac{v_x}{c^2}\,E_z\right)\,, \\ E'_z &= \gamma\,(E_z + v_x\,B_y)\,, & B'_z &= \gamma\,\left(B_z - \frac{v_x}{c^2}\,E_y\right)\,. \end{aligned}$$

In particular, components parallel to the boost are unchanged, while transverse components mix. A purely electric field in one frame can appear partly magnetic in another and vice versa. This is exactly what we should expect if $\mathbf{E}$ and $\mathbf{B}$ are components of a single Lorentz tensor.

Our example was without loss of generality since we can apply a simple Euclidean rotation and write the transformation for a general velocity vector $\mathbf{v}$. Doing so yields,

$$\mathbf{E}'_{\parallel} = \mathbf{E}_{\parallel}, \qquad \mathbf{B}'_{\parallel} = \mathbf{B}_{\parallel}\,, \tag{7.7.11}$$

$$\mathbf{E}'_{\perp} = \gamma\,(\mathbf{E}_{\perp} + \mathbf{v}\times\mathbf{B})\,, \qquad \mathbf{B}'_{\perp} = \gamma\left(\mathbf{B}_{\perp} - \frac{\mathbf{v}}{c^2}\times\mathbf{E}\right)\,. \tag{7.7.12}$$

for the components parallel and perpendicular to the velocity $\mathbf{v}$.

§7.8 Covariant Form of Maxwell's Theory

Our next big question is the rewriting of Maxwells equations in terms of $F_{\mu\nu}$ and the spacetime current J^μ that we defined in (7.6.2). As we will show, in our tensorial variables, we have the correspondence,

$$\left.\begin{aligned}\nabla\cdot\mathbf{E} &= \tfrac{\rho}{\varepsilon_0}\\ \nabla\times\mathbf{B} &= \mu_0\,\mathbf{J} + \mu_0\,\varepsilon_0\,\partial_t\mathbf{E}\end{aligned}\right\} \quad\to\quad \partial_\mu F^{\mu\nu} = -\mu_0\,J^\nu\,, \tag{7.8.1}$$

$$\left.\begin{aligned}\nabla\cdot\mathbf{B} &= 0\\ \nabla\times\mathbf{E} &= -\partial_t\mathbf{B}\end{aligned}\right\} \quad\to\quad \partial_\mu F_{\kappa\lambda} + \partial_\kappa F_{\lambda\mu} + \partial_\lambda F_{\mu\kappa} = 0\,. \tag{7.8.2}$$

The first thing we want to make sure of is that we have the same number of equations. Equation (7.8.1) has four spacetime components, matching a scalar and a vector equation in three dimensions.

Counting independent components of (7.8.2) is slightly more involved, since it carries three free indices. The key observation is that its left-hand side is *totally antisymmetric* under permutations of these indices. To make this explicit, define the $\binom{0}{3}$ tensor,

$$W_{\kappa\lambda\mu} = \partial_\mu F_{\kappa\lambda} + \partial_\kappa F_{\lambda\mu} + \partial_\lambda F_{\mu\kappa}\,. \tag{7.8.3}$$

For instance, exchanging the first two indices yields,

$$W_{\lambda\kappa\mu} = \partial_\mu F_{\lambda\kappa} + \partial_\lambda F_{\kappa\mu} + \partial_\kappa F_{\mu\lambda} = -\,W_{\kappa\lambda\mu}\,, \tag{7.8.4}$$

where we used only the antisymmetry (7.7.6) of the field-strength tensor $F_{\mu\nu}$. More generally, for any permutation σ of three objects,

$$W_{\sigma(\kappa)\sigma(\lambda)\sigma(\mu)} = \epsilon(\sigma)\,W_{\kappa\lambda\mu}\,, \tag{7.8.5}$$

with $\epsilon(\sigma) = +1$ for even permutations and $\epsilon(\sigma) = -1$ for odd permutations. Total antisymmetry implies that $W_{\kappa\lambda\mu} = 0$ whenever any two indices coincide, so the only potentially non-vanishing components are those with three distinct indices. In four spacetime dimensions this leaves,

$$\binom{4}{3} = 4$$

independent components of W.

We will now check the validity of our claim. We will separately examine the time and spatial components of (7.8.1). Considering the time component gives us equation (1.0.1),

$$\partial_0 F^{00} + \partial_i F^{i0} = -\mu_0\, J^0 \overset{(7.7.9)}{\Rightarrow} \frac{1}{c}\partial_i E_i = \mu_0\, J^0 \overset{(7.6.2)}{\Rightarrow} \partial_i E_i = \frac{\rho}{\varepsilon_0}\,, \tag{7.8.6}$$

while a spatial component gives,

$$\begin{aligned} &\partial_0 F^{0i} + \partial_j F^{ji} = -\mu_0\, J^i \overset{(7.7.9)}{\Rightarrow} \frac{1}{c^2}\partial_t E_i + \varepsilon_{jik}\partial_j\, B_k = -\mu_0\, J_i \Rightarrow \\ &\varepsilon_{ijk}\partial_j B_k = \mu_0\varepsilon_0\, \partial_t E_i + \mu_0\, J_i\,, \end{aligned} \tag{7.8.7}$$

which is equation (1.0.4) written in component notation.

Since equation (7.8.2) is totally antisymmetric, we can only have one of its components to be along the time direction. Therefore, we have two cases, one in which one of the components is 0 or not. Examining the first case gives,

$$\begin{aligned} &\partial_0 F_{ij} + \partial_i F_{j0} + \partial_j F_{0i} = 0 \overset{(7.7.8)}{\Rightarrow} \varepsilon_{ijk}\, \partial_t B_k + \partial_i E_j - \partial_j E_i = 0 \Rightarrow \\ &\varepsilon_{lij}\, (\varepsilon_{ijk}\, \partial_t B_k + \partial_i E_j - \partial_j E_i) = 0 \overset{(A.0.1)}{\Rightarrow} \varepsilon_{lij}\, \partial_i E_j = -\partial_t B_l\,, \end{aligned} \tag{7.8.8}$$

which is the same with equation (1.0.2). Choosing all three indices to be spatial yields a scalar equation in three dimensions,

$$\begin{aligned} &\partial_i\, F_{jk} + \partial_j\, F_{ki} + \partial_k\, F_{ij} = 0 \Rightarrow \varepsilon_{jkl}\, \partial_i B_l + \varepsilon_{kil}\, \partial_j B_l + \varepsilon_{ijl}\, \partial_k B_l = 0 \Rightarrow \\ &\varepsilon_{ijk}\, (\varepsilon_{jkl}\, \partial_i B_l + \varepsilon_{kil}\, \partial_j B_l + \varepsilon_{ijl}\, \partial_k B_l) = 0 \overset{(A.0.1)}{\Rightarrow} \partial_k B_k = 0\,, \end{aligned} \tag{7.8.9}$$

which is our familiar divergence free condition (1.0.3) for the magnetic field. As we should expect, writing our electromagnetic tensor in terms of the gauge field as in (7.7.4), automatically solves equation (7.8.2). This is the reason this equation is also known as the *Bianchi identity*. This was the point of introducing the scalar and vector potentials in Section 6.7, to solve equations (1.0.2) and (1.0.3). These equations are now packaged in a single equation (7.8.2).

The final step is to rewrite Maxwell's action (6.8.1) in covariant form. We first notice that,

$$F_{\mu\nu}\, F^{\mu\nu} \overset{(7.7.8),\,(7.7.9)}{=} -2\,\mu_0 \left(\varepsilon_0\, |\mathbf{E}|^2 - \frac{1}{\mu_0}\, |\mathbf{B}|^2\right) = 2\, \left(|\mathbf{B}|^2 - \frac{1}{c^2}\, |\mathbf{E}|^2\right)\,, \tag{7.8.10}$$

and also,

$$A_\mu J^\mu \overset{(7.6.2),\,(7.7.2)}{=} -\varphi\,\rho + \mathbf{A}\cdot\mathbf{J}\,. \tag{7.8.11}$$

This allows us to write Maxwell's action in the elegant form,

$$\begin{aligned} S[A_\mu] &= \frac{1}{c}\int_{\mathbb{R}^4}\left(-\frac{1}{4\,\mu_0}\,F^{\mu\nu}\,F_{\mu\nu} + A_\mu J^\mu\right)\,d^4x \\ &= \int_{\mathbb{R}^4}\left(-\frac{1}{4\,\mu_0}\,F^{\mu\nu}\,F_{\mu\nu} + A_\mu J^\mu\right)\,dt\,d^3\mathbf{x}\,, \end{aligned} \tag{7.8.12}$$

The integrand is built out of tensor contractions and it is therefore invariant as a Lorentz scalar. It is worth pointing out that the whole integral is also invariant since the Jacobian of a proper Lorentz transformation has determinant equal to one (see Remark 7.4.3).

§7.9 The Energy-Momentum Tensor

This section is non-examinable.

In Sections 4.9 and 6.2 we computed the energy densities stored in static electric and magnetic fields. However, in Section 6.6, we took their sum to be the energy density of time dependent configurations. In this section, we will use the technology of classical field theory and justify our earlier results more rigorously.

The canonical energy-momentum tensor of the relativistic theory is given by[12],

$$T^\mu{}_\nu = -\frac{\partial L_{\max}}{\partial\,(\partial_\mu A_\lambda)}\,\partial_\nu A_\lambda + \delta^\mu_\nu\,L_{\max}\,, \tag{7.9.1}$$

where $L_{\max}$ is the Lagrangian density for the electromagnetic field withough the external sources,

$$L_{\max} = -\frac{1}{4\mu_0}F_{\mu\nu}F^{\mu\nu}\,. \tag{7.9.2}$$

After noting that,

$$\frac{\partial F_{\alpha\beta}}{\partial\,(\partial_\mu A_\lambda)} = \delta^\mu_\alpha\,\delta^\lambda_\beta - \delta^\mu_\beta\,\delta^\lambda_\alpha\,, \tag{7.9.3}$$

[12]You will notice a sign difference with the definition in the first half of the term. This is due to our convention for the Minkowski being "mostly plus".

we can easily compute,

$$\frac{\partial L_{\max}}{\partial\left(\partial_\mu A_\lambda\right)} = -\frac{1}{\mu_0}\, F^{\mu\lambda}\,, \tag{7.9.4}$$

to obtain the expression,

$$T^{\mu}{}_{\nu} = \frac{1}{\mu_0}\left(F^{\mu\lambda}\,\partial_\nu A_\lambda - \frac{1}{4}\delta^{\mu}_{\nu}\, F^{\alpha\beta}F_{\alpha\beta}\right)\,. \tag{7.9.5}$$

After raising both indices, we have,

$$T^{\mu\nu} = \frac{1}{\mu_0}\left(F^{\mu\lambda}\,\partial^\nu A_\lambda - \frac{1}{4}\eta^{\mu\nu}\, F^{\alpha\beta}F_{\alpha\beta}\right)\,. \tag{7.9.6}$$

In the vacuum, with $J^\mu = 0$, we can easily check that,

$$\begin{aligned}\partial_\mu T^{\mu\nu} &= \frac{1}{\mu_0}\left(\partial_\mu F^{\mu\lambda}\,\partial^\nu A_\lambda + F^{\mu\lambda}\,\partial^\nu\partial_\mu A_\lambda - \frac{1}{2}F^{\alpha\beta}\,\partial^\nu F_{\alpha\beta}\right)\\ &\overset{(7.8.1)}{=} \frac{1}{\mu_0}\left(F^{\mu\lambda}\,\partial^\nu\partial_\mu A_\lambda - \frac{1}{2}F^{\alpha\beta}\,\partial^\nu F_{\alpha\beta}\right)\\ &= \frac{1}{2\,\mu_0}\left(F^{\mu\lambda}\,\partial^\nu\left(\partial_\mu A_\lambda - \partial_\lambda A_\mu\right) - F^{\alpha\beta}\,\partial^\nu F_{\alpha\beta}\right) = 0\,.\end{aligned} \tag{7.9.7}$$

However, there is a couple of issues with this particular expression for a physically relevant energy-momentum tensor. The first one is that it is not a symmetric tensor. Most importantly, this expression is not gauge invariant. As you know, the components of the energy-momentum tensor are directly related to physically measurable quantities. For example, T_{00} is the energy density while T_{0i} is the momentum density of the system. As such, we would like to have an expression which is manifestly invariant under (7.7.3). Moreover,

Instead of the canonical energy-momentum tensor, we can always consider the improved energy-momentum tensor,

$$\tilde{T}^{\mu\nu} = T^{\mu\nu} + \partial_\lambda W^{\lambda\mu\nu}\,, \tag{7.9.8}$$

which satisfies,

$$\partial_\mu \tilde{T}^{\mu\nu} = 0\,, \tag{7.9.9}$$

as long as the $\binom{3}{0}$ tensor $W^{\lambda\mu\nu}$ is antisymmetric in its first two indices, $W^{\lambda\mu\nu} = -W^{\mu\lambda\nu}$. Given that the tensor $W^{\lambda\mu\nu}$ vanishes at infinity, it does not alter the total energy or momentum of the system since it only contributes a spatial divergence when integrating over a spatial slice of space-time.

The goal is to choose a suitable $W^{\lambda\mu\nu}$ such that $\tilde{T}^{\mu\nu}$ is symmetric and gauge invariant. In particular, choosing $W^{\lambda\mu\nu} = \frac{1}{\mu_0} F^{\lambda\mu} A^{\nu}$ gives us,

$$\begin{aligned}
\tilde{T}^{\mu\nu} &= T^{\mu\nu} + \frac{1}{\mu_0} \partial_\lambda \left(F^{\lambda\mu} A^{\nu} \right) \\
&\overset{(7.8.1)}{=} T^{\mu\nu} - \frac{1}{\mu_0} F^{\mu\lambda} \partial_\lambda A^{\nu} \\
&= \frac{1}{\mu_0} \left(F^{\mu\lambda} \left(\partial^{\nu} A_\lambda - \partial_\lambda A^{\nu} \right) - \frac{1}{4} \eta^{\mu\nu} F^{\alpha\beta} F_{\alpha\beta} \right) \Rightarrow
\end{aligned}$$

$$\boxed{\tilde{T}^{\mu\nu} = \frac{1}{\mu_0} \left(F^{\mu\lambda} F^{\nu}{}_{\lambda} - \frac{1}{4} \eta^{\mu\nu} F^{\alpha\beta} F_{\alpha\beta} \right) .} \tag{7.9.10}$$

This energy-momentum tensor satisfies all our requirements and is known as the *Hilbert* energy-momentum tensor and it is guaranteed to satisfy,

$$\partial_\mu \tilde{T}^{\mu\nu} = 0 \tag{7.9.11}$$

So far, we have considered the electromagnetic field in the vacuum, with $J^\mu = 0$. From our discussion in Section 6.6, we expect the right hand side of (7.9.11) to be non-zero. This is due to the fact that the Maxwell field will exchange energy and momentum with the charged particles. Taking the divergence of (7.9.10) gives,

$$\begin{aligned}
\partial_\mu \tilde{T}^{\mu\nu} &= \frac{1}{\mu_0} \left(\partial_\mu F^{\mu\lambda} F^{\nu}{}_{\lambda} + F^{\mu\lambda} \partial_\mu F^{\nu}{}_{\lambda} - \frac{1}{2} F^{\alpha\beta} \partial^{\nu} F_{\alpha\beta} \right) \\
&\overset{(7.8.1)}{=} -F^{\nu}{}_{\lambda} J^{\lambda} + \frac{1}{\mu_0} \left(F^{\mu\lambda} \partial_\mu F^{\nu}{}_{\lambda} - \frac{1}{2} F^{\alpha\beta} \partial^{\nu} F_{\alpha\beta} \right) \\
&= -F^{\nu}{}_{\lambda} J^{\lambda} + \frac{1}{\mu_0} \eta^{\nu\rho} \left(F^{\mu\lambda} \partial_\mu F_{\rho\lambda} - \frac{1}{2} F^{\alpha\beta} \partial_\rho F_{\alpha\beta} \right) \\
&= -F^{\nu}{}_{\lambda} J^{\lambda} + \frac{1}{\mu_0} \eta^{\nu\rho} \left(F^{\alpha\beta} \partial_\alpha F_{\rho\beta} - \frac{1}{2} F^{\alpha\beta} \partial_\rho F_{\alpha\beta} \right) \\
&\overset{(7.8.2)}{=} -F^{\nu}{}_{\lambda} J^{\lambda} + \frac{1}{\mu_0} \eta^{\nu\rho} \left(F^{\alpha\beta} \partial_\alpha F_{\rho\beta} + \frac{1}{2} F^{\alpha\beta} \left(\partial_\alpha F_{\beta\rho} + \partial_\beta F_{\rho\alpha} \right) \right) \\
&= -F^{\nu}{}_{\lambda} J^{\lambda} + \frac{1}{\mu_0} \eta^{\nu\rho} \left(F^{\alpha\beta} \partial_\alpha F_{\rho\beta} - \frac{1}{2} F^{\alpha\beta} \left(\partial_\alpha F_{\rho\beta} + \partial_\alpha F_{\rho\beta} \right) \right) ,
\end{aligned}$$

where in the last line we simply used the antisymmetry of the field strength tensor. In particular, for the second term inside the nested bracket we wrote $F^{\alpha\beta}\partial_\beta F_{\rho\alpha} = F^{\beta\alpha}\partial_\alpha F_{\rho\beta} = -F^{\alpha\beta}\partial_\alpha F_{\rho\beta}$. After noticing the terms in the brackets cancel out, we have the elegant result,

$$\boxed{\partial_\mu \tilde{T}^{\mu\nu} = -F^{\nu}{}_{\lambda} J^{\lambda} .} \tag{7.9.12}$$

It is instructive to compute the different components of the energy-momentum tensor as these are related to energy density $\mathcal{E}$, momentum density $\mathcal{P}^i$, and the stress tensor σ^{ij} of the system. We have,

$$\begin{aligned}
\mathcal{E} = T^{00} &= \frac{1}{\mu_0}\left(F^{0i}\,F^0{}_i + \frac{1}{4}F^{\alpha\beta}\,F_{\alpha\beta}\right) \overset{(7.7.9),(7.8.10)}{=} \frac{1}{2\,\mu_0}\left(|\mathbf{B}|^2 + \frac{1}{c^2}|\mathbf{E}|^2\right)\,,\\
\mathcal{P}^i = \frac{1}{c}T^{0i} &= \frac{1}{\mu_0 c}F^{0j}\,F^i{}_j = \frac{1}{\mu_0 c^2}\varepsilon_{ijk}\,E_j\,B_k = \frac{1}{\mu_0 c^2}\,(\mathbf{E}\times\mathbf{B})_i \overset{(6.6.4)}{=} \frac{1}{c^2}\,\mathcal{S}_i\,,\\
\sigma_{ij} = T_{ij} &= \frac{1}{\mu_0}\left(F_i{}^k\,F^{jk} + F_{i0}\,F_j{}^0 - \frac{1}{4}\delta_{ij}F^{\alpha\beta}\,F_{\alpha\beta}\right)\\
&= \frac{1}{\mu_0}\left(\varepsilon_{ikl}\,\varepsilon_{jkr}\,B_l\,B_r + \frac{1}{c^2}\,E_i\,E_j - \frac{1}{2}\delta_{ij}\left(|\mathbf{B}|^2 - \frac{1}{c^2}|E|^2\right)\right)\\
&\overset{(A.0.1)}{=} \frac{1}{\mu_0}\left(\frac{1}{2}\delta_{ij}|\mathbf{B}|^2 - B_i\,B_j\right) + \frac{1}{\mu_0 c^2}\left(\frac{1}{2}\delta_{ij}|\mathbf{E}|^2 - E_i\,E_j\right)\,. \qquad (7.9.13)
\end{aligned}$$

The above results justify the identification for the energy density of equation (6.6.2). It is also reassuring that we recovered the Poynting vector of equation (6.6.4) which plays the role of current in the energy continuity equation (6.6.3). Our final step is to write out the time and spatial components of the right hand side of (7.9.12),

$$\begin{aligned}
F^0{}_\lambda J^\lambda = F^{0}{}_i\,J^i &\overset{(7.7.9),(7.6.2)}{=} \frac{1}{c}\,\mathbf{E}\cdot\mathbf{J}\,,\\
F_{i\lambda}J^\lambda = F_{i0}J^0 + F_{ik}J^k &\overset{(7.7.8),(7.6.2)}{=} \rho\,E_i + \varepsilon_{ikl}\,J_k\,B_l = (\rho\,\mathbf{E} + \mathbf{J}\times\mathbf{B})_i\,. \qquad (7.9.14)
\end{aligned}$$

It turns out that the time component is precisely the right hand side of equation (6.6.3) that we identified in section 6.6 as the energy exchanged between the electromagnetic field and the moving charges. We can write,

$$\boxed{\begin{aligned}
\partial_t\mathcal{E} + \partial_i\mathcal{S}_i &= -\mathbf{E}\cdot\mathbf{J},\\
\partial_t\mathcal{P}_i + \partial_j\sigma_{ji} &= -\left(\rho\,\mathbf{E} + \mathbf{J}\times\mathbf{B}\right)_i\,.
\end{aligned}} \qquad (7.9.15)$$

The term that appears on the right for the equation governing momentum density is nothing but the Lorentz force on charge. It is precisely the rate at which the two systems exchange momentum.

§8 Beyond Maxwell

This section is non-examinable.

This final section is *non-examinable*. Its purpose is to give a glimpse of two directions in which the potential-based and action-based formulation of electromagnetism becomes more than a convenient reformulation,

- In quantum mechanics the potentials can have observable effects even in regions where the classical fields vanish.
- Keeping gauge and Lorentz invariance one can consistently deform Maxwell theory into non-linear electrodynamics.

§8.1 The Aharonov–Bohm effect

In classical electromagnetism, the electric and magnetic fields $(\mathbf{E}, \mathbf{B})$ are the directly observable objects. The potentials $(\varphi, \mathbf{A})$ are extremely practical, but they are not unique. Different potentials related by gauge transformations describe the same observable fields. As we have already seen in Remarks 4.6.2 and 5.4.4 for classical particles, the potentials appear in their Lagrangian. However, in the equations of motion all that appears is the Lorentz force in terms of the electric and magnetic fields. In Quantum Mechanics, the important thing is the Lagrangian itself. This adds a crucial twist, the wavefunction of a charged particle is sensitive to the potentials through its *phase*. This can lead to measurable consequences even in regions where $\mathbf{E} = \mathbf{B} = 0$.

Consider a very long, thin solenoid along the z-axis carrying a steady current such that the magnetic field is confined to its interior. Idealising the situation, we assume,

$$\mathbf{B} = \mathbf{0}\,, \tag{8.1.1}$$

outside the solenoid, while the total magnetic flux through a cross-section of the solenoid is,

$$\Phi_B = \int \mathbf{B} \cdot d\mathbf{S}\,. \tag{8.1.2}$$

Using Stokes' theorem, for any closed loop C winding once around the cylinder, we can write,

$$\oint_C \mathbf{A} \cdot d\mathbf{r} = \int_S (\nabla \times A) \cdot d\mathbf{S} = \int_S \mathbf{B} \cdot d\mathbf{S} = \Phi_B\,, \tag{8.1.3}$$

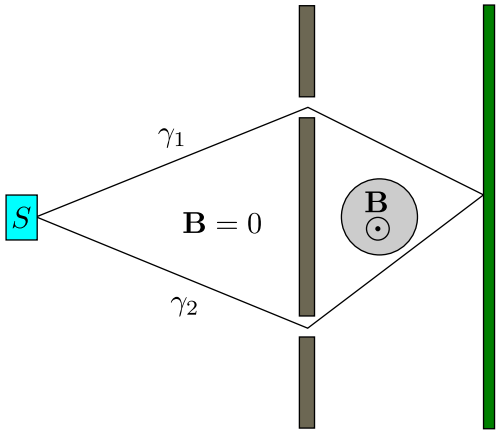


Figure 24: Aharonov–Bohm set-up. A long solenoid confines magnetic field $\mathbf{B}$ to its interior , producing a net magnetic flux Φ_B through its cross-section. An electron beam is split into two paths γ_1 and γ_2 that travel entirely in the field-free region and then recombine on a detector.

where S is a surface bounded by C. The point is that any such S must intersect the solinoid if C encircles it and in that region $\mathbf{B} \neq 0$. This is how the flux enters even though $\mathbf{B} = \mathbf{0}$ along the particle path C.

In non-relativistic quantum mechanics, a particle of charge q couples to the electromagnetic potentials by the *minimal coupling* replacement,

$$-i\hbar\nabla \;\longrightarrow\; -i\hbar\nabla - q\,\mathbf{A}\,, \qquad i\hbar\,\partial_t \;\longrightarrow\; i\hbar\,\partial_t - q\,\varphi\,. \tag{8.1.4}$$

A robust consequence is that along a path γ the wavefunction picks up an extra phase proportional to the line integral of the potential,

$$\boxed{\psi \;\sim\; \exp\!\left(\frac{iq}{\hbar}\int_\gamma \mathbf{A}\cdot d\mathbf{r}\right)\,.} \tag{8.1.5}$$

Now imagine a double-slit arrangement in which a beam can follow either path γ_1 or γ_2 and travel in the field-free region outside the solenoid before it is detected by a suitable apparatus, as shown in Figure 24. Both of these paths contribute to the wavefunction of the electron and the relative phase shift between the two is determined by the *closed* loop obtained by following γ_1 forward and γ_2 backward,

$$\boxed{\Delta\theta_{\mathrm{AB}} = \frac{q}{\hbar}\oint \mathbf{A}\cdot d\mathbf{r} = \frac{q}{\hbar}\,\Phi_B\,.} \tag{8.1.6}$$

Thus the interference fringes shift as we vary the enclosed flux Φ_B, even though the particles never pass through a region with nonzero $\mathbf{B}$.

Two comments are worth stressing,

- This does not contradict classical electromagnetism. Classically, a point particle responds only to **E** and **B**, so there is no force in the field-free region. This effect is a statement about quantum phases and interference.

- The quantity $\oint \mathbf{A} \cdot d\mathbf{r}$ is gauge-invariant for closed loops. So the effect is physical even though **A** itself is gauge-dependent.

- This is a good point to stress the importance of something that we didn't quite stress in the lectures. Everywhere outside the cylinder we have

$$\mathbf{B} = 0 \Rightarrow \nabla \times A = 0\,.$$

  The fact that the curl of **A** vanishes outside the cylinder might tempt us to write $\mathbf{A} = \nabla g$ for some function g. However, since the curl of **A** doesn't vanish *everywhere*, including the region inside the cylinder, prevents g from being a well defined function. To see this, compute the flux once again,

$$\oint_C \mathbf{A} \cdot d\mathbf{r} = \oint_C \nabla g \cdot d\mathbf{r}\,,$$

  which should be zero for a well defined function g.

§8.2 Nonlinear electrodynamics

In the Maxwell theory, one of the most efficient ways to organise the subject is via the action principle. Starting from a gauge-invariant and Lorentz-invariant action for the potential A_μ, the Euler–Lagrange equations reproduce Maxwell's equations, while gauge symmetry explains the redundancy $A_\mu \to A_\mu + \partial_\mu \chi$. This viewpoint also makes it clear what it means to go "beyond Maxwell" in a controlled way. We keep the gauge invariance and Lorentz invariance but allow a more general Lagrangian than the quadratic Maxwell form. Maxwell's linearity and the superposition principle are tied to the fact that $\mathcal{L}$ is quadratic in the field strength. Once $\mathcal{L}$ becomes nonlinear in $F_{\mu\nu}$, electromagnetic fields can interact with themselves and superposition generally fails.

But why would we want to do this? A good example is provided by quantum electrodynamics where the vacuum behaves as a polarizable medium at extremely strong fields. At low energies this is captured by an *effective* Lagrangian with non-linear corrections, leading to effects such as vacuum birefringence and light-by-light scattering. The key point is that "vacuum" need not remain perfectly linear once quantum fluctuations are included.

Gauge invariance implies that A_μ can enter the action only through $F_{\mu\nu}$ and through the coupling $A_\mu J^\mu$. Lorentz invariance then implies that the Lagrangian density depends on $F_{\mu\nu}$ only through Lorentz scalars.

To proceed, we introduce the four-dimensional Levi-Civita tensor $\varepsilon_{\kappa\lambda\mu\nu}$ which is totally antisymmetric and in our conventions $\varepsilon_{0123} = -1$ and therefore $\varepsilon^{0123} = 1$. We can easily

check that this quantity transforms as a tensor. Indeed, we can check,

$$\varepsilon^{0'1'2'3'} = \Lambda^{0'}{}_{\kappa}\,\Lambda^{1'}{}_{\lambda}\,\Lambda^{2'}{}_{\mu}\,\Lambda^{3'}{}_{\nu}\,\varepsilon^{\kappa\lambda\mu\nu} = \det\Lambda = 1\,, \tag{8.2.1}$$

and similarly for the other components. After this definition, we can see that two standard invariants are,

$$\mathcal{I}_1 := F_{\mu\nu}F^{\mu\nu}\,, \qquad \mathcal{I}_2 := F_{\mu\nu}\tilde{F}^{\mu\nu}\,, \tag{8.2.2}$$

where the dual tensor is defined by,

$$\tilde{F}^{\mu\nu} := \frac{1}{2}\,\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}\,. \tag{8.2.3}$$

With our metric signature $\eta_{\mu\nu} = \mathrm{diag}(-1,1,1,1)$, one finds (up to the usual factors of c)

$$\mathcal{I}_1 = 2\left(|\mathbf{B}|^2 - \frac{|\mathbf{E}|^2}{c^2}\right), \qquad \mathcal{I}_2 = -\frac{4}{c}\,\mathbf{E}\cdot\mathbf{B}\,. \tag{8.2.4}$$

A general nonlinear electrodynamics can then be described by an action of the form,

$$S[A] = \int_{\mathbb{R}^4}\Big(\mathcal{L}(\mathcal{I}_1,\mathcal{I}_2) + A_\mu J^\mu\Big)\,d^4x\,, \tag{8.2.5}$$

where $\mathcal{L}$ reduces to the Maxwell Lagrangian density in the weak-field limit.

To derive the equations of motion, vary $A_\mu \to A_\mu + \delta A_\mu$. Since $\delta F_{\mu\nu} = \partial_\mu\delta A_\nu - \partial_\nu\delta A_\mu$, integration by parts yields an expression of the form,

$$\delta S = \int_{\mathbb{R}^4}\left[-2\,\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial F_{\mu\nu}}\right) + J^\nu\right]\delta A_\nu\,d^4x\,, \tag{8.2.6}$$

up to boundary terms. It is therefore natural to introduce the antisymmetric tensor,

$$\boxed{H^{\mu\nu} := -\,2\,\frac{\partial\mathcal{L}}{\partial F_{\mu\nu}}\,.} \tag{8.2.7}$$

We note that for Maxwell's theory, $\mathcal{L} = -(4\mu_0)^{-1}F_{\mu\nu}F^{\mu\nu}$ and one gets $H^{\mu\nu} = (1/\mu_0)\,F^{\mu\nu}$. In general, the equations of motion take the compact form,

$$\boxed{\partial_\mu H^{\mu\nu} = -\,J^\nu\,, \qquad \partial_\mu F_{\kappa\lambda} + \partial_\kappa F_{\lambda\mu} + \partial_\lambda F_{\mu\kappa} = 0\,,} \tag{8.2.8}$$

where the second equation is still the Bianchi identity.

A famous nonlinear, gauge-invariant and Lorentz-invariant modification of Maxwell theory is *Born-Infeld* electrodynamics. It is characterised by a new scale b with dimensions of field strength and Lagrangian density,

$$\boxed{\mathcal{L}_{\mathrm{BI}} = \frac{b^2}{\mu_0}\left(1 - \sqrt{1 + \frac{1}{2b^2}\,\mathcal{I}_1 - \frac{1}{16b^4}\,\mathcal{I}_2^2}\right).} \tag{8.2.9}$$

For weak fields ($|\mathbf{E}|/c \ll b$ and $|\mathbf{B}| \ll b$) one can expand the square root and recover Maxwell theory plus higher-order corrections, schematically of the form

$$\mathcal{L}_{\rm BI} = -\frac{1}{4\mu_0}\,\mathcal{I}_1 + \mathcal{O}\!\left(\frac{\mathcal{I}_1^2}{b^2}, \frac{\mathcal{I}_2^2}{b^2}\right) . \tag{8.2.10}$$

The important qualitative message is that the field equations become *nonlinear*, so superposition no longer holds. This allows genuinely new phenomena (modified wave propagation, field self-interactions, etc.), while still respecting gauge symmetry and special relativity.

Additional material

§A Facts and Identities of Vector Calculus

This appendix contains several useful identities and properties from vector calculus that are heavily used in the main text. Unless otherwise stated, all functions and components of vector fields are assumed to be at least twice differentiable, with continuous partial derivatives. In terms of notation, if our coordinates are labelled by x^i, the corresponding partial derivatives are $\partial_i \equiv \frac{\partial}{\partial x^i}$. In this way, we will assume that $\partial_i \partial_j = \partial_j \partial_i$ as an operator acting on all our functions and vector components.

In these notes, we will often need to deal with the Levi-Civita symbol ϵ_{ijk}, which is totally antisymmetric in its indices and $\epsilon_{123} = 1$. The identity,

$$\epsilon_{ijk}\epsilon_{pqk} = \delta_{ip}\delta_{jq} - \delta_{iq}\delta_{jp}\,. \tag{A.0.1}$$

will allow us to show a couple of very useful identities regarding the curl operator.

For any smooth vectors $\mathbf{A}$, $\mathbf{C}$ and function f we can show that the following identities hold.

$$\nabla \cdot (\nabla \times \mathbf{A}) = 0 \tag{A.0.2}$$

Proof. We can use index notation to write,

$$\begin{aligned}
\nabla \cdot (\nabla \times \mathbf{A}) = \partial_i \left(\nabla \times \mathbf{A}\right)_i &= \epsilon_{ijk}\, \partial_i \partial_j A_k \\
&= \frac{1}{2}\left(\epsilon_{ijk}\, \partial_i \partial_j A_k + \epsilon_{ijk}\, \partial_i \partial_j A_k\right) \\
&= \frac{1}{2}\left(\epsilon_{ijk}\, \partial_i \partial_j A_k - \epsilon_{jik}\, \partial_i \partial_j A_k\right) \\
&= \frac{1}{2}\left(\epsilon_{ijk}\, \partial_i \partial_j A_k - \epsilon_{ijk}\, \partial_j \partial_i A_k\right) \\
&= \frac{1}{2}\epsilon_{ijk}\, \left(\partial_i \partial_j A_k - \partial_j \partial_i A_k\right) = 0\,.
\end{aligned}$$

In the third line we used the antisymmetry of the Levi-Civita symbol $\epsilon_{ijk} = -\epsilon_{jik}$ and in the fourth line we simply relabelled the dummy indices which are summed over. In the last line we used the fact that we can commute the order of the partial derivatives. □

$$\nabla \times \nabla f = 0 \tag{A.0.3}$$

Proof. Using similar ingredients,

$$(\nabla \times \nabla f)_i = \epsilon_{ijk} \partial_j \partial_k f = \frac{1}{2} \epsilon_{ijk} \left(\partial_j \partial_k f - \partial_k \partial_j f \right) = 0 \,.$$

□

$$\boxed{\nabla \times (\nabla \times \mathbf{A}) = \nabla \left(\nabla \cdot \mathbf{A} \right) - \nabla^2 \mathbf{A}} \tag{A.0.4}$$

Proof. In index notation we have,

$$\begin{aligned} \left(\nabla \times (\nabla \times \mathbf{A}) \right)_i &= \epsilon_{ijk} \partial_j \left(\nabla \times \mathbf{A} \right)_k = \epsilon_{ijk} \, \epsilon_{kpq} \, \partial_j \partial_p A_q \\ &= \epsilon_{ijk} \, \epsilon_{pqk} \, \partial_j \partial_p A_q \overset{(A.0.1)}{=} \left(\delta_{ip} \, \delta_{jq} - \delta_{iq} \, \delta_{jp} \right) \partial_j \partial_p A_q \\ &= \partial_i \left(\partial_j A_j \right) - \partial_j \partial_j A_i \,, \end{aligned}$$

which proves the identity after using equation (A.0.1). □

$$\boxed{\nabla \times (f \, \mathbf{A}) = \nabla f \times \mathbf{A} + f \, \nabla \times \mathbf{A}} \tag{A.0.5}$$

Proof.

$$\begin{aligned} \left(\nabla \times (f \, \mathbf{A}) \right)_i &= \epsilon_{ijk} \, \partial_j \left(f \, A_k \right) = \epsilon_{ijk} \, \left(\partial_j f \, A_k + f \, \partial_j A_k \right) \\ &= \epsilon_{ijk} \, \partial_j f \, A_k + f \, \epsilon_{ijk} \, \partial_j A_k \\ &= \left(\nabla f \times \mathbf{A} \right)_i + f \, \left(\nabla \times \mathbf{A} \right)_i \end{aligned}$$

□

$$\boxed{\nabla \times (\mathbf{A} \times \mathbf{C}) = (\nabla \cdot \mathbf{C}) \, \mathbf{A} + (\mathbf{C} \cdot \nabla) \, \mathbf{A} - (\nabla \cdot \mathbf{A}) \, \mathbf{C} - (\mathbf{A} \cdot \nabla) \, \mathbf{C}} \tag{A.0.6}$$

Proof.

$$\begin{aligned} \left(\nabla \times (\mathbf{A} \times \mathbf{C}) \right)_i &= \epsilon_{ijk} \, \partial_j \left(\mathbf{A} \times \mathbf{C} \right)_k = \epsilon_{ijk} \, \epsilon_{kpq} \, \partial_j \left(A_p \, C_q \right) \\ &\overset{(A.0.1)}{=} \left(\delta_{ip} \, \delta_{jq} - \delta_{iq} \, \delta_{jp} \right) \, \partial_j \left(A_p \, C_q \right) \\ &= \partial_j \left(A_i \, C_j \right) - \partial_j \left(A_j \, C_i \right) \\ &= \left(\partial_j C_j \right) \, A_i + C_j \, \partial_j A_i - \left(\partial_j A_j \right) \, C_i - A_j \, \partial_j C_i \,, \end{aligned}$$

where once again we used property (A.0.1) in the second line. □

$$\nabla \cdot (\mathbf{A} \times \mathbf{C}) = \mathbf{C} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{C}) \tag{A.0.7}$$

Proof.

$$\begin{aligned} \partial_i \, (\mathbf{A} \times \mathbf{C})_i &= \partial_i \, (\varepsilon_{ijk} \, A_j \, C_k) = C_k \, (\varepsilon_{ijk} \, \partial_i A_j) + A_j \, (\varepsilon_{ijk} \, \partial_i C_k) \\ &= C_k \, (\nabla \times \mathbf{A})_k - A_j \, (\nabla \times \mathbf{C})_j \end{aligned}$$

□

A couple of theorems that we heavily use in the main text are Gauss' and Stokes' theorems. To state Gauss' theorem we consider a volume region $V \subset \mathbb{R}^3$ with boundary $S = \partial V$. The theorem then relates the volume integral of the divergence of a vector $\mathbf{A}$ to its surface integral,

$$\int_V \nabla \cdot \mathbf{A} \, d^3\mathbf{x} = \int_S \mathbf{A} \cdot d\mathbf{S} \,, \tag{A.0.8}$$

with the surface S oriented in a way such that its normal vector points away from the volume V.

Stokes' theorem applies to surfaces $S \subset \mathbb{R}^3$ with a closed curved boundary $C = \partial S$, oriented according to the right hand rule. In particular it relates the surface integral of the curl of a vector field $\mathbf{A}$ to its circulation around C,

$$\int_S (\nabla \times \mathbf{A}) \cdot d\mathbf{S} = \oint_C \mathbf{A} \cdot d\mathbf{r} \,. \tag{A.0.9}$$

§B Dirac's Delta Function

This Appendix aims to introduce Dirac's delta function $\delta(\mathbf{x})$ along with some of its properties which will be useful for the purposes of this course. Strictly speaking, $\delta(\mathbf{x})$ is not a function but rather a distribution which we are supposed to convolve with test functions $f(x)$. In one dimension in particular, the defining property of Dirac's delta function is,

$$\int_a^b \delta(x - x_0) \, f(x) \, dx = \begin{cases} f(x_0), & \text{if } x_0 \in (a, b) \\ 0, & \text{if } x_0 \notin [a, b] \end{cases} .$$

Based on this property, attempting to visualise Dirac's delta results in a picture where $\delta(x)$ is zero everywhere except for the points where its argument vanishes. Moreover, in order

for the integral to be finite, such a function would have to blow up at that point. To see this, we can choose $f(x) = 1$, $x_0 = 0$ and $b = -a = \varepsilon$ in the definition to conclude that,

$$\lim_{\varepsilon \to 0^+} \int_{-\varepsilon}^{\varepsilon} \delta(x) = 1 \,,$$

which would not be possible for a function that remains finite at $x = 0$.

Even though such a quantity does not exist as a strict function, we can understand it as a $\varepsilon \to 0$ limit of a family of functions $\delta_\varepsilon(x)$. This is what we call a *representation* of Dirac's delta and satisfies the definition only after taking a limit,

$$\lim_{\varepsilon \to 0^+} \int_a^b \delta_\varepsilon(x - x_0)\, f(x)\, dx = \begin{cases} f(x_0), & \text{if } x_0 \in [a, b] \\ 0, & \text{otherwise} \end{cases} .$$

Two such examples are the *rectangular distribution*,

$$\delta_\varepsilon(x) = \begin{cases} 1/\varepsilon & \text{if } |x| \leq \varepsilon/2 \\ 0 & \text{if } |x| > \varepsilon/2 \end{cases} ,$$

and the *heat kernel*,

$$\delta_\varepsilon(x) = \frac{1}{\sqrt{2\pi\varepsilon}}\, e^{-\frac{x^2}{2\varepsilon}} \,.$$

For both cases we see that,

$$\lim_{\varepsilon \to 0^+} \delta_\varepsilon(x) = \begin{cases} \infty & \text{if } x = 0 \\ 0 & \text{if } x \neq 0 \end{cases} , \tag{B.0.1}$$

as one would expect.

To conclude the one-dimensional case, we mention that we can also make sense of derivatives of Dirac's delta as distributions. The way to understand this is through a combination of repeated integration by parts and the definition,

$$\int_a^b \delta^{(n)}(x - x_0)\, f(x)\, dx = (-1)^n \int_a^b \delta(x - x_0)\, f^{(n)}(x)\, dx = \begin{cases} (-1)^n\, f^{(n)}(x_0), & \text{if } x_0 \in [a, b] \\ 0, & \text{otherwise} \end{cases} .$$

In the above argument we dropped all boundary terms since Dirac's delta vanishes in an open set containing them. In this situation, one can argue that all their derivatives will have to vanish as well at those points.

The generalisation of Dirac's delta definition to n dimensions $\delta(\mathbf{x})$ is now straightforward. Given a volume $V \subset \mathbb{R}^n$, for any real valued function $f(\mathbf{x})$ Dirac's delta satisfies,

$$\int_V \delta(\mathbf{x} - \mathbf{x}_0)\, f(\mathbf{x})\, d^n\mathbf{x} = \begin{cases} f(\mathbf{x}_0) & \text{if } \mathbf{x}_0 \in V \\ 0 & \text{if } \mathbf{x}_0 \notin V \end{cases} .$$

Note B.0.1

We point out that since the delta function is non-zero only at the points for which its argument vanishes, we can always write,

$$f(\mathbf{x})\,\delta(\mathbf{x}-\mathbf{x}_0) = f(\mathbf{x}_0)\,\delta(\mathbf{x}-\mathbf{x}_0)\,.$$

Finally, we will mention an identity relating the derivative of the Heaviside step function $\Theta(x)$ to Dirac's delta. In particular, the definition of the delta function can be written as,

$$\Theta(x) = \int_{-\infty}^{x} \delta(y)\,dy\,.$$

Another way to phrase this is the important identity,

$$\boxed{\frac{d}{dx}\Theta(x) = \delta(x)\,.} \tag{B.0.2}$$

§C Fluxes of Lower Dimensional Current Densities

In this Appendix we will show that the current densities (2.2.7) and (2.2.9) give the expressions we would expect when integrated over a surface S_p. For the case of current flowing along a curve, we consider the situation captured in Figure 3 with the surface S_p parametrised by $\mathbf{x}(s_1, s_2)$. For the flux we can write,

$$\begin{aligned} I_{S_p}(t) &= \int_{S_p} \mathbf{J}(t, \mathbf{x}(s_1,s_2)) \cdot d\mathbf{S}_p \\ &\overset{(2.2.7)}{=} \int_{S_p}\int_{C} I(t,s)\,\delta(\mathbf{x}(s_1,s_2)-\mathbf{r}(s))\,\dot{\mathbf{r}}(s)\cdot d\mathbf{S}_p\,ds \\ &= \int_{S_p}\int_{C} I(t,s)\,\delta(\mathbf{x}(s_1,s_2)-\mathbf{r}(s))\,\dot{\mathbf{r}}(s)\cdot\left(\partial_{s_1}\mathbf{x}(s_1,s_2)\times\partial_{s_2}\mathbf{x}(s_1,s_2)\right)\,ds\,ds_1\,ds_2\,. \end{aligned} \tag{C.0.1}$$

The above integral is over a three-dimensional space and contains a delta function in three dimensions. We expect that in the end we should not be left with any integration at all. From the properties of Dirac's delta we know that all the non-zero contribution will come from the points at which its argument vanishes. It therefore makes sense to perform the change of variables,

$$\mathbf{w} = \mathbf{x}(s_1,s_2) - \mathbf{r}(s)\,, \tag{C.0.2}$$

with inverse Jacobian and determinant,

$$\mathbf{J}^{-1} = \begin{pmatrix} \partial_s \mathbf{w} & \partial_{s_1}\mathbf{w} & \partial_{s_2}\mathbf{w} \end{pmatrix} = \begin{pmatrix} -\dot{\mathbf{r}} & \partial_{s_1}\mathbf{x} & \partial_{s_2}\mathbf{x} \end{pmatrix} , \tag{C.0.3}$$

$$\det\left(\mathbf{J}^{-1}\right) = -\dot{\mathbf{r}} \cdot (\partial_{s_1}\mathbf{x} \times \partial_{s_2}\mathbf{x}) \ . \tag{C.0.4}$$

In these variables, the integral becomes,

$$I_{S_p}(t) = \int_V \delta(\mathbf{w}) \, \frac{\dot{\mathbf{r}} \cdot (\partial_{s_1}\mathbf{x} \times \partial_{s_2}\mathbf{x})}{|\dot{\mathbf{r}} \cdot (\partial_{s_1}\mathbf{x} \times \partial_{s_2}\mathbf{x})|} \, I(t, s(\mathbf{w})) \, d^3\mathbf{w} = \varepsilon \, I(t, s_0) \, , \tag{C.0.5}$$

where we have set $s(\mathbf{w} = 0) = s_0$ and $\varepsilon^2 = 1$. We have therefore shown the result we quoted in the main text for the flux through the intersection point at $\mathbf{w} = 0$.

We now move on to the case of current flow on a two-dimensional surface and we imagine the situation we described in the main text and illustrated in Figure 3. The one-dimensional intersection $C = S \cap S_p$ is parametrised according to the condition,

$$\mathbf{z}(\tau) = \mathbf{x}(s_1(\tau), s_2(\tau)) = \mathbf{r}(\lambda_1(\tau), \lambda_2(\tau)) \, , \tag{C.0.6}$$

and with tangent vector,

$$\begin{aligned} \mathbf{z}(\tau) &= \dot{s_1}(\tau) \, \partial_{s_1}\mathbf{x}(s_1(\tau), s_2(\tau)) + \dot{s_2}(\tau) \, \partial_{s_2}\mathbf{x}(s_1(\tau), s_2(\tau)) \\ &= \dot{\lambda_1}(\tau) \, \partial_{\lambda_1}\mathbf{r}(\lambda_1(\tau), \lambda_2(\tau)) + \dot{\lambda_2}(\tau) \, \partial_{\lambda_2}\mathbf{r}(\lambda_1(\tau), \lambda_2(\tau)) \, . \end{aligned} \tag{C.0.7}$$

The above shows that $\dot{\mathbf{z}} \perp \mathbf{N}_S$ and $\dot{\mathbf{z}} \perp \mathbf{N}_{S_p}$ on the intersection $S \cap S_p$. For the flux through S_p we have,

$$\begin{aligned} I_{S_p}(t) &= \int_{S_p} \mathbf{J}(t, \mathbf{x}(s_1, s_2)) \cdot d\mathbf{S}_p \\ &\overset{(2.2.9)}{=} \int_{S_p} \int_S \delta(\mathbf{x}(s_1, s_2) - \mathbf{r}(\lambda_1, \lambda_2)) \, \mathbf{K}(\lambda_1, \lambda_2) \cdot d\mathbf{S}_p \, dS \\ &= \int_{S_p} \int_S \delta(\mathbf{x}(s_1, s_2) - \mathbf{r}(\lambda_1, \lambda_2)) \, |\mathbf{N}_S(\lambda_1, \lambda_2)| \, \left(\mathbf{K}(\lambda_1, \lambda_2) \cdot \mathbf{N}_{S_p}(s_1, s_2)\right) \, ds_1 \, ds_2 \, d\lambda_1 \, d\lambda_2 \, . \end{aligned} \tag{C.0.8}$$

The above expression shows that we need to carry out an integral in a four-dimensional volume V_4 and over an integrant with a delta function in three dimensions. This suggests that we should try to reduce it to a one. More specifically it should be the intersection C, at which the argument of the delta function vanishes. To simplify the integral, we perform the change of variables,

$$\begin{aligned} \mathbf{w} &= \mathbf{x}(s_1, s_2) - \mathbf{r}(\lambda_1, \lambda_2) \, , \\ f &= g(s_1, s_2, \lambda_1, \lambda_2) \, , \end{aligned} \tag{C.0.9}$$

with g a smooth function. Inverting the above transformation would give λ_i and s_i as a function $\mathbf{w}$ and f. The inverse Jacobian and its determinant are then,

$$\mathbf{J}^{-1} = \begin{pmatrix} \partial_{s_1} f & \partial_{s_2} f & \partial_{\lambda_1} f & \partial_{\lambda_2} f \\ \partial_{s_1}\mathbf{w} & \partial_{s_2}\mathbf{w} & \partial_{\lambda_1}\mathbf{w} & \partial_{\lambda_2}\mathbf{w} \end{pmatrix} = \begin{pmatrix} \partial_{s_1} g & \partial_{s_2} g & \partial_{\lambda_1} g & \partial_{\lambda_2} g \\ \partial_{s_1}\mathbf{x} & \partial_{s_2}\mathbf{x} & -\partial_{\lambda_1}\mathbf{r} & -\partial_{\lambda_2}\mathbf{r} \end{pmatrix},$$
$$\det\left(\mathbf{J}^{-1}\right) = \partial_{s_1} g\ (\partial_{s_2}\mathbf{x}\cdot\mathbf{N}_S) - \partial_{s_2} g\ (\partial_{s_1}\mathbf{x}\cdot\mathbf{N}_S) - \partial_{\lambda_1} g\ \left(\partial_{\lambda_2}\mathbf{r}\cdot\mathbf{N}_{S_p}\right) + \partial_{\lambda_2} g\ \left(\partial_{\lambda_1}\mathbf{r}\cdot\mathbf{N}_{S_p}\right). \tag{C.0.10}$$

After performing the change of variables, the flux takes the form,

$$I_{S_p}(t) = \int_{V_4} \delta(\mathbf{w})\, |\mathbf{N}_S(\lambda_1,\lambda_2)|\ \left(\mathbf{K}(\lambda_1,\lambda_2)\cdot\mathbf{N}_{S_p}(s_1,s_2)\right)\ |\det(\mathbf{J})|\ d^3\mathbf{w}\, df\,, \tag{C.0.11}$$

where we understand that λ_i and s_i are functions $\mathbf{w}$ and f. As anticipated, the delta function restricts the integrant at $\mathbf{w}=0$, which is presicely the curve of intersection C. We can therefore write,

$$I_{S_p}(t) = \int_C |\mathbf{N}_S(\lambda_1(\tau),\lambda_2(\tau))|\ \left(\mathbf{K}(\lambda_1(\tau),\lambda_2(\tau))\cdot\mathbf{N}_{S_p}(s_1(\tau),s_2(\tau))\right)\ |\det(\mathbf{J})|\ \dot{f}(\tau)\, d\tau\,, \tag{C.0.12}$$

where we once again changed variable of integration for the remaining one-dimensional integral according to,

$$f = g(s_1(\tau), s_2(\tau), \lambda_1(\tau), \lambda_2(\tau))\,, \tag{C.0.13}$$

and all quantities in the integrand are now restricted on the intersection. Using the cyclic property of the mixed product we can write,

$$\begin{aligned} \mathbf{K}\cdot\mathbf{N}_{S_p} &= \mathbf{K}\cdot(\partial_{s_1}\mathbf{x}\times\partial_{s_2}\mathbf{x}) = \partial_{s_2}\mathbf{x}\cdot(\partial_{s_1}\mathbf{x}\times\mathbf{K}) \\ &= \partial_{s_2}\mathbf{x}\cdot\left(\left(\partial_{s_1}\mathbf{x} + \frac{\dot{s}_2}{\dot{s}_1}\,\partial_{s_2}\mathbf{x}\right)\times\mathbf{K}\right) = \frac{1}{\dot{s}_1}\,\partial_{s_2}\mathbf{x}\cdot((\dot{s}_1\,\partial_{s_1}\mathbf{x} + \dot{s}_2\,\partial_{s_2}\mathbf{x})\times\mathbf{K}) \\ &\overset{(C.0.7)}{=} \frac{1}{\dot{s}_1}\partial_{s_2}\mathbf{x}\cdot(\dot{\mathbf{z}}\times\mathbf{K})\,. \end{aligned} \tag{C.0.14}$$

We now simplify the Jacobian expression (C.0.10) when evaluated on C. For the second term we can write,

$$\partial_{s_1}\mathbf{x}\cdot\mathbf{N}_S \overset{(C.0.7)}{=} \left(\dot{\mathbf{z}} - \frac{\dot{s}_2}{\dot{s}_1}\partial_{s_2}\mathbf{x}\right)\cdot\mathbf{N}_S = -\frac{\dot{s}_2}{\dot{s}_1}\partial_{s_2}\,\mathbf{x}\cdot\mathbf{N}_S\,, \tag{C.0.15}$$

where in the last equality we used that $\dot{\mathbf{z}}$ is orthogonal to the normal vector $\mathbf{N}_S$ of S everywhere on the intersection C. For the third term we can write,

$$\begin{aligned} \partial_{\lambda_2}\mathbf{r}\cdot\mathbf{N}_{S_p} &= \partial_{\lambda_2}\mathbf{r}\cdot(\partial_{s_1}\mathbf{x}\times\partial_{s_2}\mathbf{x}) \overset{(C.0.7)}{=} \frac{1}{\dot{s}_1}\partial_{\lambda_2}\mathbf{r}\cdot\left(\left(\dot{\lambda}_1\,\partial_{\lambda_1}\mathbf{r} + \dot{\lambda}_2\,\partial_{\lambda_2}\mathbf{r} - \dot{s}_2\,\partial_{s_2}\mathbf{x}\right)\times\partial_{s_2}\mathbf{x}\right) \\ &= \frac{\dot{\lambda}_1}{\dot{s}_1}\,\partial_{\lambda_2}\mathbf{r}\cdot(\partial_{\lambda_1}\mathbf{r}\times\partial_{s_2}\mathbf{x}) = -\frac{\dot{\lambda}_1}{\dot{s}_1}\partial_{s_2}\mathbf{x}\cdot(\partial_{\lambda_1}\mathbf{r}\times\partial_{\lambda_2}\mathbf{r}) \\ &= -\frac{\dot{\lambda}_1}{\dot{s}_1}\partial_{s_2}\mathbf{x}\cdot\mathbf{N}_S\,, \end{aligned} \tag{C.0.16}$$

where in the second line we used the cyclic property of the mixed product[13]. Similar manipulations for the last term in the Jacobian determinant yield,

$$\partial_{\lambda_1}\mathbf{r}\cdot\mathbf{N}_{S_p} = \frac{\dot{\lambda}_2}{\dot{s}_1}\partial_{s_2}\mathbf{x}\cdot\mathbf{N}_S\,. \tag{C.0.17}$$

Putting everything together we obtain the expression,

$$\det\left(\mathbf{J}^{-1}\right) = \frac{1}{\dot{s}_1}\left(\dot{s}_1\,\partial_{s_1}g + \dot{s}_2\,\partial_{s_2}g + \dot{\lambda}_1\,\partial_{\lambda_1}g + \dot{\lambda}_2\,\partial_{\lambda_2}g\right)\,\partial_{s_2}\mathbf{x}\cdot\mathbf{N}_S = \frac{\dot{f}}{\dot{s}_1}\,\partial_{s_2}\mathbf{x}\cdot\mathbf{N}_S\,. \tag{C.0.18}$$

Using the expressions (C.0.14) and (C.0.18) in the flux integral (C.0.12), we obtain,

$$I_{S_p}(t) = \int_C \frac{\dot{s}_1\,\dot{f}}{\left|\dot{s}_1\,\dot{f}\right|}|\mathbf{N}_S|\,\frac{\partial_{s_2}\mathbf{x}\cdot(\dot{\mathbf{z}}\times\mathbf{K})}{|\partial_{s_2}\mathbf{x}\cdot\mathbf{N}_S|}\,d\tau\,. \tag{C.0.19}$$

It is now important to remember that $\mathbf{N}_S$ is orthogonal to both $\dot{\mathbf{z}}$ and $\mathbf{K}$ since these are always tangent to the surface S. This suggests that there is a scalar function u which relates $\mathbf{N}_S = u\,(\dot{\mathbf{z}}\times\mathbf{K})$. We therefore see that the flux takes the final form,

$$I_{S_p}(t) = \int_{S\cap S_p} \varepsilon\;|\dot{\mathbf{z}}\times\mathbf{K}|\;d\tau\,, \tag{C.0.20}$$

with $\varepsilon^2 = 1$.

Going back to the definition (2.2.9) of flux through a curve C and using the cyclic property of the inner product we can write,

$$I_C(t) = -\int_C \hat{\mathbf{n}}\cdot(\mathbf{K}\times\dot{\mathbf{y}})\;ds\,. \tag{C.0.21}$$

Once again since the unit norm vector $\hat{\mathbf{n}}$ is simultaneously orthogonal to $\mathbf{K}$ and $\dot{\mathbf{y}}$, we can write $\mathbf{K}\times\dot{\mathbf{y}} = -\sigma\;|\mathbf{K}\times\dot{\mathbf{y}}|\;\hat{\mathbf{n}}$ with $\sigma^2 = 1$ yielding,

$$I_C(t) = \int_C \sigma\;|\mathbf{K}\times\dot{\mathbf{y}}|\;ds\,, \tag{C.0.22}$$

which has the same form with the flux we computed from the intersection of S with S_p.

[13]For any three vectors we can always write $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = \mathbf{c}\cdot(\mathbf{a}\times\mathbf{b}) = \mathbf{b}\cdot(\mathbf{c}\times\mathbf{a})$