ON THE ANALYTIC PROPERTIES OF THE STANDARD L-FUNCTION ATTACHED TO SIEGEL-JACOBI MODULAR FORMS

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In this work we study the analytic properties of the standard L-function attached to Siegel-Jacobi modular forms of higher index, generalizing previous results of Arakawa and Murase. Moreover, we obtain results on the analytic properties of Klingen-type Eisenstein series attached to Jacobi groups.

1. Introduction

Siegel-Jacobi modular forms - called here after [13] - are higher dimensional generalizations of classical Jacobi forms. As in the one-dimensional case they are very closely related to Siegel modular forms. Indeed, many examples may be naturally obtained from Fourier-Jacobi expansion of Siegel modular forms, but it is known (see for example [28]) that not all of them can be obtained as Fourier-Jacobi coefficients of Siegel modular forms.

The standard L-function attached to a Siegel modular form is perhaps one of the most well-studied automorphic L-functions. Indeed, its analytic properties have been extensively studied by many authors such as Andrianov and Kalinin [1], Böcherer [4, 5, 6], Garrett [11], Piatetski-Shapiro and Rallis [18], and Shimura [21, 22]. Moreover, if one assumes that the Siegel modular form is algebraic, in the sense that the Fourier coefficients at infinity are algebraic, then the values of the L-function at specific points (usually called special L-values), after dividing by appropriate powers of π and the Petersson self inner product, are algebraic. Results of this kind have been obtained first by Sturm [27], then extended by Böcherer and Schmidt [7] and Shimura [24].

The central object of study of this paper and its continuation [9] is a standard L-function attached to Siegel-Jacobi form. In particular, we investigate whether similar properties as in the previous paragraph (i.e. analytic continuation, algebraicity of special values) hold also for such an L-function. It is perhaps worth mentioning here that the underlying algebraic group, the Jacobi group, is not reductive, which means in particular that Siegel-Jacobi modular forms cannot be associated to Shimura varieties. However it is known (see [13, 14]) that they can be associated to mixed Shimura varieties.

We now introduce some notation in order to give a brief account of the main theorems proved in this paper. For simplicity we describe them here only for Siegel-Jacobi

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modular forms over the rational numbers, even though our results are more general and are proved over a totally real field.

Let $S \in M_{l,l}(\mathbb{Q})$ be a positive definite half-integral symmetric matrix, and f a Siegel-Jacobi modular form of weight k and index S for the congruence subgroup $\mathbf{\Gamma}_0(N)$. We give the detailed definition in section 3 but for the purposes of this introduction it is enough to say that f is a holomorphic function on the space $\mathcal{H}_{n,l} := \mathbb{H}_n \times M_{n,l}(\mathbb{C})$, where $\mathbb{H}_n$ is the Siegel upper half space, satisfying a particular modular property with respect to the group $\mathbf{\Gamma}_0(N) := H(\mathbb{Z}) \rtimes \Gamma_0(N)$, a congruence subgroup of the Jacobi group $\boldsymbol{G}^{n,l}(\mathbb{Q}) := H(\mathbb{Q}) \rtimes \mathrm{Sp}_n(\mathbb{Q})$. Here $H(\mathbb{Z})$ denotes the $\mathbb{Z}$-points of the Heisenberg group of degree n and index l, and $\Gamma_0(N)$ the classical congruence subgroup of level N in the theory of Siegel modular forms.

A study of Siegel-Jacobi modular forms of higher index and their L-functions was initiated by Shintani (unpublished), and then continued by Murase [15] and Murase and Sugano [17]. However, the only known results concern trivial level ($N = 1$). In this paper we generalize their work in various directions, one of them is that we consider a rather general congruence subgroup. Then, assuming that f is an eigenform for all Hecke operators $T(m)$ with eigenvalues $\lambda(m)$ and χ is a Dirichlet character of a conductor M, we considered a Dirichlet series $D(s, f, \chi) = \sum_{m=1}^{\infty} \lambda(m)\chi(m)m^{-s}$. This series is absolutely convergent for $\mathrm{Re}(s) > 2n + l + 1$ and - as we show in section 7 - after multiplying by an appropriate factor it possesses an Euler product representation. More precisely, we prove the following:

Theorem 1.1. *Assume that the matrix S satisfies the condition M_p^+ (see section 7 for a definition) for every prime ideal p with $(p, N) = 1$. Then*

$$\mathfrak{L}(\chi, s)D(s + n + l/2, f, \chi) = L(s, f, \chi) := \prod_p L_p(\chi(p)p^{-s})^{-1},$$

where for every prime number p

$$L_p(X) = \begin{cases} \prod_{i=1}^{n} \Big((1 - \mu_{p,i}X)(1 - \mu_{p,i}^{-1}X)\Big), \ \ \mu_{p,i} \in \mathbb{C}^\times, & \text{if } (p, N) = 1, \\ \prod_{i=1}^{n}(1 - \mu_{p,i}X), \ \ \mu_{p,i} \in \mathbb{C}, & \text{otherwise.} \end{cases}$$

Moreover, $\mathfrak{L}(\chi, s) = \prod_{(p,N)=1} \mathfrak{L}_p(\chi, s)$, where

$$\mathfrak{L}_p(\chi, s) := G_p(\chi, s) \begin{cases} \prod_{i=1}^{n} L_p(2s + 2n - 2i, \chi^2) & \text{if } l \in 2\mathbb{Z} \\ \prod_{i=1}^{n} L_p(2s + 2n - 2i + 1, \chi^2) & \text{if } l \notin 2\mathbb{Z}, \end{cases}$$

and $G_p(\chi, s)$ is a ratio of Euler factors which for almost all p is one.

The above theorem was originally shown by Murase and Sugano in the case of $N = 1$, $\chi = 1$ and $l = 1$. We extended it to any N, any character χ and any l. Together with generalization to any l certain new phenomena appear, such as for example the presence of the factor $G(\chi, s)$, which is equal to one in the case of $l = 1$. We defer a more detailed discussion to section 7.

The theorem above establishes that the function $L(s, f, \chi)$ is absolutely convergent for $\mathrm{Re}(s) > n + \frac{l}{2} + 1$ and hence holomorphic. A suitable adjustment of the doubling method allows us to prove much more:

Theorem 1.2. *With notation as above, assuming that $\chi(-1) = (-1)^k$, the function $L(s, f, \chi)$ has a meromorphic continuation to the whole complex plane.*

Actually in the full version of the theorem (Theorem 9.3), after introducing an extra factor depending on the parity of l and some Gamma factors, we also provide information on the location of the poles of the function. Our theorem vastly extends previous work of Murase [15, 16]: we consider the case of totally real fields, non-trivial level and twisting by characters. However, perhaps the most important difference with the works [15, 16] is the method used.

The work of Murase has as its prototype the approach of Piatetski-Shapiro and Rallis [18] and their theory of zeta integrals. Murase uses an embedding of the form

$$\boldsymbol{G}^{n,l}(\mathbb{Q}) \times \boldsymbol{G}^{n,l}(\mathbb{Q}) \hookrightarrow \mathrm{Sp}_{2n+l}(\mathbb{Q}),$$

and computes an adelic zeta integral à la Piatetski-Shapiro and Rallis of a Siegel-type Eisenstein series of Sp_{2n+l} restricted to the image of the product $\boldsymbol{G}^{n,l}(\mathbb{A}_{\mathbb{Q}}) \times \boldsymbol{G}^{n,l}(\mathbb{A}_{\mathbb{Q}})$ against two copies of the adelic counterpart $\mathbf{f}$ of f.

Our approach is completely different. We use instead a map of Arakawa, [3],

$$\boldsymbol{G}^{n,l}(\mathbb{Q}) \times \boldsymbol{G}^{m,l}(\mathbb{Q}) \to \boldsymbol{G}^{m+n,l}(\mathbb{Q}),$$

which is not quite an embedding. This is a starting point in order to obtain a doubling method type identity: for a Dirichlet character χ with $\chi(-1) = (-1)^k$ and $m \geq n$,

$$< f(w), E^{n+m}(\mathrm{diag}[z, w], s; \chi, k, N) >= L(s, f, \chi, s) E^m(z, s; f, \chi, N), \quad (*)$$

where $E^{n+m}(\mathrm{diag}[z, w], s; \chi, k, N)$ is the restriction under the diagonal embedding $\mathcal{H}_{n,l} \times \mathcal{H}_{m,l} \hookrightarrow \mathcal{H}_{n+m,l}$ of a Siegel-type Jacobi Eisenstein series of degree $n+m$ associated to the character χ, and $E^m(z, s; f, \chi, N)$ is a Klingen-type Jacobi Eisenstein series of degree m associated to the cuspidal form f through parabolic induction.

It is important to note here that opposite to the first map used by Murase, we have the option to take $n \neq m$. And, indeed, we will make use of this in order to obtain results towards the analytic properties of Klingen-type Jacobi Eisenstein series (see Theorem 9.5).

The identity $(*)$ above was first obtained by Arakawa in [3] in the case of $N = 1$ and trivial χ (and hence k even), and in this paper is extended to the situation of totally real fields, arbitrary level as well as non-trivial characters χ. It should be stressed though that these generalizations are by all means not trivial and demand a different approach than the one taken by Arakawa. Indeed, Arakawa's approach is modeled on the work of Garrett in [11] who invented the doubling method and applied it to the case of Siegel modular forms over $\mathbb{Q}$ of trivial level and without twists by Dirichlet characters. Our approach follows techniques introduced by Shimura [22], where he massively extended Garrett's approach to the case of totally real field, arbitrary level as well as twisting by Hecke characters. However, as it will become clear in section 5 and especially Lemma 5.3, (see also the Remark 5.4) many new technical difficulties need to be addressed in the Jacobi setting.

It is worth to point out here that even though in some cases one can identify the standard L-function associated to a Siegel-Jacobi form with the standard L-function associated to a Siegel modular form (see for example the remark on page 252 in [16]), this is possible under some quite restrictive conditions on both index and level of the Siegel-Jacobi form. Actually, even in the situation of classical Jacobi forms this correspondence becomes quite complicated when one considers an index different than 1 and/or non-trivial level, which is very clear for example in the work of [26].

We would also like to emphasize that in this work not only we establish results for the standard L-function attached to a Siegel-Jacobi modular form, but also for the analytic properties of Klingen-type Eisenstein series of the Jacobi groups, something of interest on its own. Furthermore, the results presented in this paper are used in another work of ours ([9]) to establish algebraicity results for some critical values of the standard L-function attached to a Siegel-Jacobi modular form in the spirit of Deligne's period conjecture. Actually, an earlier version of this paper ([8]) included this application, but due to the considerable length of the paper we decided to separate the two. We have also shortened some computations, and therefore refer the interested reader to [8] for a more detailed account.

The reader will notice that in all the theorems we assume a particular parity condition between the sign of the twisting character χ and the weight k of the Siegel-Jacobi modular form. It is, of course, very important to be able to relax this condition and obtain the theorems for any finite character χ, independent of the weight k. This is the subject of a forthcoming work.

Brief description of each section: We finish this introduction by giving a short description of each section. In the second section we set most common notation used throughout this paper. In section three we introduce the notion of Siegel-Jacobi modular forms over a totally real field F, as well as the notion of adelic or automorphic Siegel-Jacobi forms. To the best of our knowledge their systematic study has not appeared before in the literature, notably Proposition 3.4 on the adelic Fourier expansion. In section four we develop the theory of Klingen-type Eisenstein series. We do this in greatest generality possible. Again, to the best of our knowledge, a systematic study of the adelized Klingen-type Jacobi Eisenstein series has not appeared before in the literature. In sections five and six we employ the doubling method in the way described above and compute the Petersson inner product of a restricted Siegel-type Jacobi Eisenstein series against a cuspidal Siegel-Jacobi form. In section seven we introduce the theory of Hecke operators in the Jacobi setting and extend previous results of Murase and Sugano. In the next section we turn our attention to the analytic properties of Siegel-type Jacobi Eisenstein series. We build on an idea going back to a work of Böcherer [4] and more recently of Heim [12]. After establishing the analytic properties of these Eisenstein series we use the results established in section 6 to obtain Theorem 9.3 on the analytic properties of the standard L-function. Moreover, we also establish Theorem 9.5 on the analytic continuation of Klinegn-type Jacobi Eisenstein series.

2. Notation

Throughout the paper we use the following notation:

- F denotes a totally real algebraic number field of degree d, $\mathfrak{d}$ the different of F, and $\mathfrak{o}$ its ring of integers;

- $\mathbb{A}$ stands for the adeles of F; we write $\mathbf{a}$ and $\mathbf{h}$ for the sets of archimedean and non-archimedan places of F respectively, so that e.g. $\mathbb{A}_{\mathbf{h}} := \prod'_{v\in\mathbf{h}} F_v$ (restricted product) and $\mathbb{A}_{\mathbf{a}} := \prod_{v\in\mathbf{a}} F_v$ denote the finite and infinite adeles of F; for $x \in \mathbb{A}$ we will write $x_{\mathbf{h}}, x_{\mathbf{a}}$ meaning the finite and infinite part of x, correspondingly; for a ring R we use the superscript $R^\times$ to denote the invertible elements in R;
- A finite adele $a \in \mathbb{A}_{\mathbf{h}}$ corresponds to a fractional ideal $\mathfrak{a}$ of F via $\mathfrak{a} := \prod_{v\in\mathbf{h}} \mathfrak{p}_v^{n_v}$, where $a_v = \pi_v^{n_v}\mathfrak{o}_v^\times$, $n_v \in \mathbb{Z}$, π_v a uniformiser at v and $\mathfrak{p}_v$ the corresponding prime ideal at the finite place v. We will call $\mathfrak{a}$ the ideal corresponding to a.
- We define $\mathbb{Z}^{\mathbf{a}} := \mathbb{Z}^d$, and a typical element $k \in \mathbb{Z}^{\mathbf{a}}$ is of the form $k = (k_v)_{v\in\mathbf{a}}$ with $k_v \in \mathbb{Z}$. Moreover for an integer $\mu \in \mathbb{Z}$ we write $\mu\mathbf{a} := (\mu, \mu, \dots, \mu) \in \mathbb{Z}^{\mathbf{a}}$.
- For an adelic Hecke character $\chi : \mathbb{A}^\times/F^\times \to \mathbb{C}^\times$, we will write χ^* for the corresponding ideal Hecke character obtained by class field theory. Furthermore, if χ is finite, then its infinite part is of the form $\chi_{\mathbf{a}}(x_{\mathbf{a}}) = \prod_{v\in\mathbf{a}} \left(\frac{x_v}{|x_v|}\right)^{k_v}$, for $k_v \in \mathbb{Z}$. We then write $\mathrm{sgn}_{\mathbf{a}}(x_{\mathbf{a}})^k$ for $\chi_{\mathbf{a}}(x_{\mathbf{a}})$ where $k := (k_v) \in \mathbb{Z}^{\mathbf{a}}$.
- $M_{l,n}$ denotes the set of $l \times n$ matrices, and we set $M_n := M_{n,n}$. We write $Sym_n \subset M_n$ for the subset of symmetric matrices; if $A \in M_{l,n}$ and $B \in M_{l,m}$, then $(A\,B) \in M_{l,n+m}$ denotes concatenation of the matrices A, B; if $S \in Sym_l, x \in M_{l,n}$, we set $S[x] := {}^t x S x$;
- For an invertible matrix x we define $\tilde{x} := {}^t x^{-1}$;
- For two matrices $a \in M_n$ and $b \in M_m$ we define $\mathrm{diag}[a, b] := \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \in M_{n+m}$;
- We set $\mathbf{e}_{\mathbf{a}}(x) := \prod_{v\in\mathbf{a}} e(x_v) := \prod_{v\in\mathbf{a}} e^{2\pi i x_v}$ for $x = \prod_{v\in\mathbf{a}} x_v \in \mathbb{C}^{\mathbf{a}}$.
- G^n stands for the algebraic group Sp_n whose F-points are defined as follows:
$$\mathrm{Sp}_n(F) := \left\{ g \in \mathrm{SL}_{2n}(F) \colon {}^t g \left(\begin{smallmatrix} & -1_n \\ 1_n & \end{smallmatrix}\right) g = \left(\begin{smallmatrix} & -1_n \\ 1_n & \end{smallmatrix}\right) \right\};$$
For an element $g \in \mathrm{Sp}_n$ we write $g = \begin{pmatrix} a_g & b_g \\ c_g & d_g \end{pmatrix}$, where $a_g, b_g, c_g, d_g \in M_n$;
- For a fixed positive integer l, $\boldsymbol{G}^{n,l} := H^{n,l} \rtimes \mathrm{Sp}_n$ denotes the Jacobi group with $H^{n,l}$ denoting the Heisenberg subgroup, whose global points are defined as
$$\boldsymbol{G}^{n,l}(F) := \{\boldsymbol{g} = (\lambda, \mu, \kappa)g : \lambda, \mu \in M_{l,n}(F), \kappa \in Sym_l(F), g \in G^n(F)\},$$
$$H^{n,l}(F) := \{(\lambda, \mu, \kappa)1_{2n} \in \boldsymbol{G}^{n,l}(F)\};$$
the group law is given by
$$(\lambda, \mu, \kappa)g(\lambda', \mu', \kappa')g' := (\lambda + \tilde{\lambda}, \mu + \tilde{\mu}, \kappa + \kappa' + \lambda\,{}^t\tilde{\mu} + \tilde{\mu}\,{}^t\lambda + \tilde{\lambda}\,{}^t\tilde{\mu} - \lambda'\,{}^t\mu')gg',$$
where $(\tilde{\lambda}\,\tilde{\mu}) := (\lambda'\,\mu')g^{-1} = (\lambda'\,{}^t d - \mu'\,{}^t c \quad \mu'\,{}^t a - \lambda'\,{}^t b)$, the identity element of $\boldsymbol{G}^{n,l}(F)$ is $1_H 1_{2n}$, where $1_H := (0, 0, 0)$ denotes the identity element of $H^{n,l}(F)$, i.e. we always suppress the indices n, l in 1_H as its size will be clear from the context;
whenever it does not lead to any confusion, we omit superscripts and write $G, \boldsymbol{G}, \boldsymbol{G}^n$ or H;
following the convention described above, $G(\mathbb{A}) = \prod'_{v\in\mathbf{h}\cup\mathbf{a}} G(F_v) = G_{\mathbf{h}}G_{\mathbf{a}}$, where $G_{\mathbf{h}} = \prod'_{v\in\mathbf{h}} G(F_v)$, $G_{\mathbf{a}} = \prod_{v\in\mathbf{a}} G(F_v)$;
- $\mathcal{H}_{n,l} := (\mathbb{H}_n \times M_{l,n}(\mathbb{C}))^{\mathbf{a}}$, where $\mathbb{H}_n := \{\tau \in Sym_n(\mathbb{C}) : \mathrm{Im}(\tau) \text{ positive definite}\}$; an element $z \in \mathcal{H}_{n,l}$ will be written as $z = (z_v)_{v\in\mathbf{a}} = (\tau, w)$, where $\tau = (\tau_v)_{v\in\mathbf{a}} \in \mathbb{H}_n^{\mathbf{a}}$, $w = (w_v)_{v\in\mathbf{a}} \in M_{l,n}(\mathbb{C})^{\mathbf{a}}$; we distinguish an element $\boldsymbol{i}_0 := (\boldsymbol{i}, 0) \in \mathcal{H}_{n,l}$,

where $\boldsymbol{i} := (i1_n)^{\mathbf{a}}$;
for $z = (\tau, w) \in \mathcal{H}_{n,l}$ we define $\delta(z) := \det(\mathrm{Im}(\tau)) := \prod_{v \in \mathbf{a}} \det(\mathrm{Im}(\tau_v)))$;

- For a fractional ideal $\mathfrak{b}$ and an integral ideal $\mathfrak{c}$ we define the following subgroups of $\boldsymbol{G}(\mathbb{A})$:
$$K[\mathfrak{b},\mathfrak{c}] := K^n[\mathfrak{b},\mathfrak{c}] := K_{\mathbf{h}}[\mathfrak{b},\mathfrak{c}]\boldsymbol{G}_{\mathbf{a}},$$
$$K_0[\mathfrak{b},\mathfrak{c}] := K_0^n[\mathfrak{b},\mathfrak{c}] := K_{\mathbf{h}}[\mathfrak{b},\mathfrak{c}] \times K_\infty \quad \text{and} \quad K := K^n := K_{\mathbf{h}}[\mathfrak{b},\mathfrak{c}](H_{\mathbf{a}}^{n,l} \rtimes D_\infty^{\mathbf{a}}),$$
where $K_\infty \simeq Sym_l(\mathbb{R})^{\mathbf{a}} \rtimes D_\infty^{\mathbf{a}} \subset H^{n,l}(\mathbb{R})^{\mathbf{a}} \rtimes \mathrm{Sp}_n(\mathbb{R})^{\mathbf{a}}$ is the stabilizer of the point $\mathbf{i}_0$, and D_∞ is the maximal compact subgroup of $\mathrm{Sp}_n(\mathbb{R})$,
$$K_{\mathbf{h}}[\mathfrak{b},\mathfrak{c}] := C_{\mathbf{h}}[\mathfrak{o},\mathfrak{b}^{-1},\mathfrak{b}^{-1}] \rtimes D_{\mathbf{h}}[\mathfrak{b}^{-1},\mathfrak{b}\mathfrak{c}] \subset \boldsymbol{G}_{\mathbf{h}},$$
$$C_{\mathbf{h}}[\mathfrak{o},\mathfrak{b}^{-1},\mathfrak{b}^{-1}] := \{(\lambda,\mu,\kappa) \in {\prod_{v\in\mathbf{h}}}' H(F_v) : \forall v \in \mathbf{h} \quad {\scriptstyle \begin{array}{c} \lambda_v \in M_{l,n}(\mathfrak{o}_v),\ \mu_v \in M_{l,n}(\mathfrak{b}_v^{-1}), \\ \kappa_v \in Sym_l(\mathfrak{b}_v^{-1}) \end{array}} \},$$
$$D_{\mathbf{h}}[\mathfrak{b}^{-1},\mathfrak{b}\mathfrak{c}] := \prod_{v\in\mathbf{h}} D_v[\mathfrak{b}^{-1},\mathfrak{b}\mathfrak{c}],$$
$$D_v[\mathfrak{b}^{-1},\mathfrak{b}\mathfrak{c}] := \left\{ x = \begin{pmatrix} a_x & b_x \\ c_x & d_x \end{pmatrix} \in G_v : {\scriptstyle \begin{array}{cc} a_x \in M_n(\mathfrak{o}_v), & b_x \in M_n(\mathfrak{b}_v^{-1}), \\ c_x \in M_n(\mathfrak{b}_v\mathfrak{c}_v), & d_x \in M_n(\mathfrak{o}_v) \end{array}} \right\};$$
- For $r \in \{0,1,\ldots,n\}$ we define parabolic subgroups of G^n and $\boldsymbol{G}^n$ as follows:
$$P^{n,r}(F) := \left\{ \begin{pmatrix} a_1 & 0 & b_1 & b_2 \\ a_3 & a_4 & b_3 & b_4 \\ c_1 & 0 & d_1 & d_2 \\ 0 & 0 & 0 & d_4 \end{pmatrix} \in G^n(F) : a_1, b_1, c_1, d_1 \in M_r(F) \right\},$$
$$\boldsymbol{P}^{n,r}(F) := \{((\lambda\, 0_{l,n-r}),\mu,\kappa)g : \lambda \in M_{l,r}(F), \mu \in M_{l,n}(F), \kappa \in Sym_l(F), g \in P^{n,r}(F)\};$$
additionally, we set $\boldsymbol{P}^n := \boldsymbol{P}^{n,0}$.

3. Siegel-Jacobi modular forms of higher index

In this section we introduce the notion of Siegel-Jacobi modular form, both from a classical and an adelic point of view, and then explain the relation between the two notions. The content of this section is well-known to researchers working on Jacobi forms, but to the best of our knowledge it has not been written elsewhere in such detail and generality. Our exposition follows mainly [15, 28].

3.1. **Siegel-Jacobi modular forms.** For two natural numbers l, n, we consider the Jacobi group $\boldsymbol{G} := \boldsymbol{G}^{n,l}$ of degree n and index l over a totally real algebraic number field F. Note that the global points $\boldsymbol{G}(F)$ may be viewed as a subgroup of $G^{l+n}(F) := \mathrm{Sp}_{l+n}(F)$ via the embedding

$$\boldsymbol{g} = (\lambda,\mu,\kappa)g \longmapsto \begin{pmatrix} 1_l & \lambda & \kappa - \mu\,{}^t\lambda & \mu \\ & 1_n & {}^t\mu & \\ & & 1_l & \\ & & -\,{}^t\lambda & 1_n \end{pmatrix} \begin{pmatrix} 1_l & & & \\ & a & & b \\ & & 1_l & \\ & c & & d \end{pmatrix}, \quad g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right). \tag{1}$$

We write $\{\sigma_v : F \hookrightarrow \mathbb{R},\ v \in \mathbf{a}\}$ for the set of real embeddings of F. Each σ_v induces an embedding $\boldsymbol{G}(F) \hookrightarrow \boldsymbol{G}(\mathbb{R})$; we will write $(\lambda_v,\mu_v,\kappa_v)g_v$ for $\sigma_v(\boldsymbol{g})$. The group $\boldsymbol{G}(\mathbb{R})^{\mathbf{a}}$ acts on $\mathcal{H}_{n,l} := (\mathbb{H}_n \times M_{l,n}(\mathbb{C}))^{\mathbf{a}}$ component wise via
$$\boldsymbol{g}z = \boldsymbol{g}(\tau,w) = (\lambda,\mu,\kappa)g(\tau,w) = \prod_{v\in\mathbf{a}} (g_v\tau_v, w_v\lambda(g_v,\tau_v)^{-1} + \lambda_v g_v \tau_v + \mu_v),$$

where $g_v\tau_v = (a_v\tau_v + b_v)(c_v\tau_v + d_v)^{-1}$ and $\lambda(g_v, \tau_v) := (c_v\tau_v + d_v)$ for $g_v = \left(\begin{smallmatrix} a_v & b_v \\ c_v & d_v \end{smallmatrix}\right)$.

For $k \in \mathbb{Z}^{\mathbf{a}}$ and a matrix $S \in Sym_l(\mathfrak{d}^{-1})$ we define the factor of automorphy of weight k and index S by

$$J_{k,S}\colon \boldsymbol{G}^{n,l}(F) \times \mathcal{H}_{n,l} \to \mathbb{C}$$

$$J_{k,S}(\boldsymbol{g}, z) = J_{k,S}(\boldsymbol{g}, (\tau, w)) := \prod_{v\in\mathbf{a}} j(g_v, \tau_v)^{k_v} \mathcal{J}_{S_v}(\boldsymbol{g_v}, \tau_v, w_v),$$

where $\boldsymbol{g} = (\lambda, \mu, \kappa)g$, $j(g_v, \tau_v) = \det(c_v\tau_v + d_v) = \det(\lambda(g_v, \tau_v))$ and

$$\begin{aligned}\mathcal{J}_{S_v}(\boldsymbol{g_v}, \tau_v, w_v) = e(&-\mathrm{tr}\,(S_v\kappa_v) + \mathrm{tr}\,(S_v[w_v]\lambda(g_v, \tau_v)^{-1}c_v) \\ &- 2\mathrm{tr}\,({}^t\lambda_v S_v w_v \lambda(g_v, \tau_v)^{-1}) - \mathrm{tr}\,(S_v[\lambda_v]g_v\tau_v))\end{aligned}$$

with $e(x) := e^{2\pi i x}$, and we recall that $S[x] = {}^t x S x$. A rather long but straightforward calculation shows that $J_{k,S}$ satisfies the usual cocycle relation:

(2) $$J_{k,S}(\boldsymbol{g}\boldsymbol{g'}, z) = J_{k,S}(\boldsymbol{g}, \boldsymbol{g'}\, z) J_{k,S}(\boldsymbol{g'}, z).$$

For a function $f\colon \mathcal{H}_{n,l} \to \mathbb{C}$ we define

(3) $$(f|_{k,S}\, \boldsymbol{g})(z) := J_{k,S}(\boldsymbol{g}, z)^{-1} f(\boldsymbol{g}\, z).$$

The property (2) implies that

$$(f|_{k,S}\, \boldsymbol{g}\boldsymbol{g'})(z) = (f|_{k,S}\, \boldsymbol{g}|_{k,S}\, \boldsymbol{g'})(z).$$

A subgroup $\boldsymbol{\Gamma}$ of $\boldsymbol{G}(F)$ will be called a congruence subgroup if there exist a fractional ideal $\mathfrak{b}$ and an integral ideal $\mathfrak{c}$ of F such that $\boldsymbol{\Gamma}$ is a subgroup of finite index of the group $G(F) \cap \boldsymbol{g}K[\mathfrak{b}, \mathfrak{c}]\boldsymbol{g}^{-1}$ for some $\boldsymbol{g} \in \boldsymbol{G}_{\mathbf{h}}$.

Of particular interest will be the congruence subgroup,

$$\begin{aligned}\boldsymbol{\Gamma}_0(\mathfrak{b}, \mathfrak{c}) := \boldsymbol{\Gamma}_0^{n,l}(\mathfrak{b}, \mathfrak{c}) := \{(\lambda, \mu, \kappa)\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \boldsymbol{G}(F)\colon \lambda \in M_{l,n}(\mathfrak{o}), \mu \in M_{l,n}(\mathfrak{b}^{-1}), \kappa \in Sym_l(\mathfrak{b}^{-1}),\\ a, d \in M_n(\mathfrak{o}), b \in M_n(\mathfrak{b}^{-1}), c \in M_n(\mathfrak{b}\mathfrak{c})\}.\end{aligned}$$

Often we will be given a congruence subgroup $\boldsymbol{\Gamma}$ equipped with a homomorphism $\chi : \boldsymbol{\Gamma} \to \mathbb{C}^\times$. For example, given a Hecke character χ of F of conductor $\mathfrak{f}_\chi$ dividing $\mathfrak{c}$, we can extend it to a homomorphism

$$\chi : \boldsymbol{\Gamma}_0(\mathfrak{b}, \mathfrak{c}) \to \mathbb{C}^\times,\;\; \chi\left((\lambda, \mu, \kappa)\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) := \chi(\det d).$$

We now consider an $S \in \mathfrak{b}\mathfrak{d}^{-1}\mathcal{T}_l$ where

(4) $$\mathcal{T}_l := \{x \in Sym_l(F) : \mathrm{tr}\,(xy) \in \mathfrak{o} \text{ for all } y \in Sym_l(\mathfrak{o})\}.$$

Moreover we assume that S is positive definite in the sense that if we write $S_v := \sigma_v(S) \in Sym_l(\mathbb{R})$ for $v \in \mathbf{a}$, then all S_v are positive definite.

Definition 3.1. Let k and S be as above, and $\boldsymbol{\Gamma}$ a congruence subgroup equipped with a homorphism χ. A Siegel-Jacobi modular form of weight $k \in \mathbb{Z}^{\mathbf{a}}$, index S, level Γ and Nebentypus χ is a holomorphic function $f\colon \mathcal{H}_{n,l} \to \mathbb{C}$ such that

(1) $f|_{k,S}\, \boldsymbol{g} = \chi(\boldsymbol{g})f$ for every $\boldsymbol{g} \in \boldsymbol{\Gamma}$,

(2) for each $g \in G^n(F)$, $f|_{k,S}\, g$ admits a Fourier expansion of the form

$$f|_{k,S}\, g(\tau, w) = \sum_{\substack{t \in L \\ t \geq 0}} \sum_{r \in M} c(\boldsymbol{g}; t, r) \mathbf{e}_{\mathbf{a}}(\mathrm{tr}\,(t\tau)) \mathbf{e}_{\mathbf{a}}(\mathrm{tr}\,({}^t r w)) \quad (*)$$

for some appropriate lattices $L \subset Sym_n(F)$ and $M \subset M_{l,n}(F)$, where $t \geq 0$ means that t_v is semi-positive definite for each $v \in \mathbf{a}$.

We will denote the space of such functions by $M^n_{k,S}(\mathbf{\Gamma}, \chi)$.

The second property is really needed only in the case of $n = 1$ and $F = \mathbb{Q}$ thanks to the Köcher principle for Siegel-Jacobi forms, as it is explained for example in [28, Lemma 1.6].

We note that if $f \in M^n_{k,S}(\mathbf{\Gamma}_0(\mathfrak{b}, \mathfrak{c}), \chi)$, then

$$f(\tau, w) = \sum_{\substack{t \in \mathfrak{b}\mathfrak{d}^{-1}\mathcal{T}_n \\ t \geq 0}} \sum_{r \in \mathfrak{b}\mathfrak{d}^{-1}\mathcal{T}_{l,n}} c(t, r) \mathbf{e}_{\mathbf{a}}(\mathrm{tr}\,(t\tau)) \mathbf{e}_{\mathbf{a}}(\mathrm{tr}\,({}^t r w)),$$

where $\mathcal{T}_{l,n} := \{x \in M_{l,n}(F) : \mathrm{tr}\,({}^t x y) \in \mathfrak{o} \text{ for all } y \in M_{l,n}(\mathfrak{o})\}$.

We say that f is a cusp form if in the expansion $(*)$ above for every $g \in G^n(F)$, we have $c(\mathbf{g}; t, r) = 0$ unless $\begin{pmatrix} S_v & r_v \\ {}^t r_v & t_v \end{pmatrix}$ is positive definite for every $v \in \mathbf{a}$. The space of cusp forms will be denoted by $S^n_{k,S}(\mathbf{\Gamma}, \chi)$.

We now introduce the notion of Petersson inner product for Jacobi forms, following [28]. Let f and g be Jacobi forms of weight k, one of which is a cusp form. Moreover, assume that both f and g are of level $\mathbf{\Gamma}$. For $z = (\tau, w) \in \mathcal{H}_{n,l}$ we write $\tau = x + iy$ with $x, y \in Sym_n(F_{\mathbf{a}})$ and $w = u + iv$ with $u, v \in M_{l,n}(F_{\mathbf{a}})$. Let $dz := d(\tau, w) := \det(y)^{-(l+n+1)} dx dy du dv$ and set $\Delta_{S,k}(z) := \det(y)^k \mathbf{e}_{\mathbf{a}}(-4\pi \mathrm{tr}\,({}^t v S v y^{-1}))$. Then we define

$$< f, g >_{\Gamma} := \int_A f(z) \overline{g(z)} \Delta_{S,k}(z) dz, \quad A := \mathbf{\Gamma} \setminus \mathcal{H}_{n,l},$$

and

$$< f, g > := vol(A)^{-1} \int_A f(z) \overline{g(z)} \Delta_{S,k}(z) dz,$$

so that the latter is independent of the group $\mathbf{\Gamma}$ as long as both f and g are in $M^n_{k,S}(\mathbf{\Gamma}, \chi)$. As it is explained in [28], the volume differential dz is selected in such a way that $vol(A) = vol(\Gamma \setminus \mathbb{H}^{\mathbf{a}}_n)$ where Γ is the symplectic part of $\mathbf{\Gamma}$.

3.2. **Adelic Siegel-Jacobi modular forms.** We keep writing $\boldsymbol{G} := \boldsymbol{G}^{n,l}$ for the Jacobi group of degree n and index l. For two ideals $\mathfrak{b}$ and $\mathfrak{c}$ of F, of which $\mathfrak{c}$ is integral, we recall that we have defined the open subgroups $K_{\mathbf{h}}[\mathfrak{b}, \mathfrak{c}] \subset \boldsymbol{G}_{\mathbf{h}}$, $D_{\mathbf{h}}[\mathfrak{b}^{-1}, \mathfrak{b}\mathfrak{c}] \subset G^n_{\mathbf{h}}$ in Section 2.

Lemma 3.2. *The strong approximation theorem holds for the algebraic group $\boldsymbol{G}$. In particular,*

$$\boldsymbol{G}(\mathbb{A}) = \boldsymbol{G}(F) K_{\mathbf{h}}[\mathfrak{b}, \mathfrak{c}] \boldsymbol{G}_{\mathbf{a}}.$$

Proof. We give a sketch of the proof. We first observe that the strong approximation holds for the Heisenberg group. Indeed, its center Z is isomorphic to the group Sym_l of symmetric matrices, and we have $H^{n,l}/Z \cong M_{n,l} \times M_{n,l}$. Furthermore, the strong approximation holds for the symmetric matrices (as an additive group) and the same holds also for $M_{n,l} \times M_{n,l}$. From this it is easy to see that the strong approximation holds for $H^{n,l}$. Then, for the whole Jacobi group, it is enough to observe that the strong approximation holds for Sp_n with respect to the subgroup $D[\mathfrak{b}^{-1}, \mathfrak{b}\mathfrak{c}]$ (see [22]), and hence the statement follows by observing that the Heisenberg group is, by definition, a normal subgroup of $\boldsymbol{G}$. ☐

We now fix once and for all an additive character $\Psi : \mathbb{A}/F \to \mathbb{C}^\times$ as follows. We write $\Psi = \prod_{v \in \mathbf{h}} \Psi_v \prod_{v \in \mathbf{a}} \Psi_v$ and define

$$\Psi_v(x_v) := \begin{cases} e(-y_v), & v \in \mathbf{h} \\ e(x_v), & v \in \mathbf{a}, \end{cases}$$

where $y_v \in \mathbb{Q}$ is such that $\mathrm{Tr}_{F_v/\mathbb{Q}_p}(x_v) - y_v \in \mathbb{Z}_p$ for $p := v \cap \mathbb{Q}$. Given a symmetric matrix $S \in Sym_l(F)$ we define a character $\psi_S : Sym_l(\mathbb{A})/Sym_l(F) \to \mathbb{C}^\times$ by taking $\psi_S(\kappa) := \Psi(\mathrm{tr}\,(S\kappa)$.

We consider an adelic Hecke character $\chi : \mathbb{A}_F^\times/F^\times \to \mathbb{C}^\times$ of F of finite order such that $\chi_v(x) = 1$ for all $x \in \mathfrak{o}_v^\times$ with $x - 1 \in \mathfrak{c}_v$. We extend this character to a character of the group $K_0[\mathfrak{b}, \mathfrak{c}]$ by setting $\chi(w) := \prod_{v|\mathfrak{c}} \chi_v(\det(a_g))^{-1}$ for $w = hg \in K_0[\mathfrak{b}, \mathfrak{c}]$.

Now, let $k \in \mathbb{Z}^{\mathbf{a}}$ and $S \in Sym_l(F)$ be such that $S \in \mathfrak{b}\mathfrak{d}^{-1}\mathcal{T}_l$ with $\mathcal{T}_l$ as in (4). Moreover, let K be an open subgroup of $K[\mathfrak{b}, \mathfrak{c}]$ for some $\mathfrak{b}$ and $\mathfrak{c}$.

Definition 3.3. An adelic Siegel-Jacobi modular form of degree n, weight k, index S and character χ, with respect to the congruence subgroup K is a function $\mathbf{f} : \boldsymbol{G}(\mathbb{A}) \to \mathbb{C}$ such that

1. $\mathbf{f}\left((0, 0, \kappa)\gamma \boldsymbol{g} w\right) = \chi(w) J_{k,S}(w, \mathbf{i}_0)^{-1} \psi_S(\kappa) \mathbf{f}(\boldsymbol{g})$, for all $\kappa \in Sym_l(\mathbb{A})$, $\gamma \in \boldsymbol{G}(F)$, $\boldsymbol{g} \in \boldsymbol{G}(\mathbb{A})$ and $w \in K \cap K_0[\mathfrak{b}, \mathfrak{c}]$;
2. for every $\boldsymbol{g} \in \boldsymbol{G}_{\mathbf{h}}$ the function $f_{\boldsymbol{g}}$ on $\mathcal{H}_{n,l}$ defined by the relation
$$(f_{\boldsymbol{g}}|_{k,S}\boldsymbol{y})(\mathbf{i}_0) := \mathbf{f}(\boldsymbol{g}\boldsymbol{y}) \quad \text{for all } \boldsymbol{y} \in \boldsymbol{G}_{\mathbf{a}}$$
is a Siegel-Jacobi modular form for the congruence group $\boldsymbol{\Gamma}^{\boldsymbol{g}} := \boldsymbol{G}(F) \cap \boldsymbol{g} K \boldsymbol{g}^{-1}$.

Note that the relation (1) is well defined. Indeed, thanks to the strong approximation for Sym_l we may write $\kappa = \kappa_F \kappa_{\mathbf{h}} \kappa_{\mathbf{a}}$ with $\kappa_F \in Sym_l(F)$, $\kappa_{\mathbf{h}} \in \prod_{v \in \mathbf{h}} Sym_l(\mathfrak{b}_v^{-1})$ and $\kappa_{\mathbf{a}} \in \prod_{v \in \mathbf{a}} Sym_l(\mathbb{R})$. Furthermore, observe that $\psi_S(\kappa) = \prod_{v \in \mathbf{a}} \psi_{S,v}(\kappa_v) = J_{k,S}((0,0,\kappa), \mathbf{i}_0)^{-1}$ since $\psi_{S,\mathbf{h}}(\kappa_{\mathbf{h}}) = 1$ by our choice of the matrix S.

We denote the space of adelic Siegel-Jacobi modular forms by $\mathcal{M}_{k,S}^n(K, \chi)$. As in the case of Siegel modular forms (see for example [23, Lemma 10.8]) we can use Lemma 3.2 to establish a bijection between adelic Siegel-Jacobi forms and Siegel-Jacobi modular forms. Indeed, for any given $\boldsymbol{g} \in \boldsymbol{G}_{\mathbf{h}}$ we have the bijective map

$$\mathcal{M}_{k,S}^n(K, \chi) \to M_{k,S}^n(\boldsymbol{\Gamma}^{\boldsymbol{g}}, \chi_{\boldsymbol{g}}) \tag{5}$$

given by $\mathbf{f} \mapsto f_{\boldsymbol{g}}$, with notation as in the Definition 3.3 and $\chi_{\boldsymbol{g}}$ the character on $\boldsymbol{\Gamma}^{\boldsymbol{g}}$ defined as $\chi(\gamma) := \chi(\boldsymbol{g}^{-1}\gamma\boldsymbol{g})$. Furthermore, we say that $\mathbf{f}$ is a cusp form, and we denote this space by $\mathcal{S}^n_{k,l}(K,\chi)$ if in the above notation $f_{\boldsymbol{g}}$ is a cusp form for all $\boldsymbol{g} \in \boldsymbol{G}_{\mathbf{h}}$. We will often use the bijection above with $\boldsymbol{g}=1$. In this case, if we start with an adelic Siegel-Jacobi form $\mathbf{f}$, we will write f for the Siegel-Jacobi modular form corresponding to $\mathbf{f}$.

We finish this section with a formula for Fourier expansion of adelic Siegel-Jacobi forms.

Proposition 3.4. *Every Siegel-Jacobi form $\mathbf{f} \in \mathcal{M}^n_{k,S}(K[\mathfrak{b},\mathfrak{c}],\chi)$ admits Fourier expansion of the form*

$$\mathbf{f}\left((\lambda,\mu,0)\begin{pmatrix} q & \sigma\tilde{q} \\ & \tilde{q}\end{pmatrix}\right) = \sum_{\substack{t\in L\\ t\geq 0}}\sum_{r\in M} c(t,r;q,\lambda)\mathbf{e}_{\mathbb{A}}(\mathrm{tr}\,(t\sigma))\mathbf{e}_{\mathbb{A}}(\mathrm{tr}\,({}^t r\lambda\sigma + {}^t r\mu)), \tag{6}$$

where $\sigma \in Sym_n(\mathbb{A}), q\in \mathrm{GL}_n(\mathbb{A}), \lambda,\mu \in M_{l,n}(\mathbb{A})$ are such that $\lambda_v q_v \in M_{l,n}(\mathfrak{b}_v^{-1})$ for all $v\in\mathbf{h}$. Moreover, the coefficients $c(t,r;q,\lambda)$ satisfy the following properties:

(1) $c(t,r;q,\lambda) = \Psi_{\mathbf{a}}(\mathrm{tr}\,(S[\lambda]\sigma))\mathbf{e}_{\mathbf{a}}(\mathrm{tr}\,(S[\lambda](iq\,{}^tq)))(\det q)^k_{\mathbf{a}}\mathbf{e}_{\mathbf{a}}(i\mathrm{tr}\,({}^tqtq + {}^tq\,{}^tr\lambda q))c_0(t,r;q,\lambda)$, where $c_0(t,r;q,\lambda)$ is a complex number that depends only on $\mathbf{f},t,r,q_{\mathbf{h}}$ and $\lambda_{\mathbf{h}}$.
(2) $c(t,r;aq,\lambda a^{-1}) = \chi(\det a)c({}^tata, ra;q,\lambda)$ for every $a\in\mathrm{GL}_n(F)$.
(3) $c(t,r;q,\lambda)\neq 0$ only if $({}^tqtq)_v \in (\mathfrak{b}\mathfrak{d}^{-1}\mathcal{T}_n)_v$ and $e_v(\mathrm{tr}\,({}^tq_v\,{}^tr_v(M_{l,n}(\mathfrak{b}_v^{-1})))=1$ for every $v\in\mathbf{h}$.

Proof. First of all, note that it is enough to provide a formula for $\mathbf{f}$ at $(\lambda,\mu,\kappa)g$ with $\kappa=0$ (thanks to the relation (1)) and g of the form as in the hypothesis.

Let $X_{l,n} := \{\nu\in M_{l,n}(\mathbb{A}) : \nu_v \in M_{l,n}(\mathfrak{b}_v^{-1})$ for all $v\in\mathbf{h}\}$ and $X := \{x\in X_{n,n} \colon x = {}^tx\}$. As it was observed in [23, Lemma 9.6], we can write $\sigma = s + qx\,{}^tq$ and $\lambda s+\mu = m+\nu\,{}^tq$ with $s\in Sym_n(F), x\in X, m\in M_{l,n}(F)$ and $\nu\in X_{l,n}$. Then:

$$\begin{aligned}\mathbf{f}((\lambda,\mu,0)\begin{pmatrix} q & \sigma\tilde{q}\\ & \tilde{q}\end{pmatrix}) &= \mathbf{f}(\begin{pmatrix}1 & s\\ & 1\end{pmatrix}(\lambda,\lambda s+\mu,\lambda s^t\lambda)\begin{pmatrix} q & qx\\ & \tilde{q}\end{pmatrix})\\ &= \mathbf{f}((0,m,0)(\lambda,\nu\,{}^tq,\lambda s^t\lambda)_{\mathbf{a}}(\lambda,0,0)_{\mathbf{h}}(0,\nu\,{}^tq,\kappa)_{\mathbf{h}}\begin{pmatrix} q & qx\\ & \tilde{q}\end{pmatrix})\\ &= \mathbf{f}((\lambda,\nu\,{}^tq,\lambda s^t\lambda)_{\mathbf{a}}(\lambda,0,0)_{\mathbf{h}}\mathrm{diag}[q,\tilde{q}](0,\nu,\kappa)_{\mathbf{h}}\begin{pmatrix} 1_n & x\\ & 1_n\end{pmatrix}_{\mathbf{a}})\\ &= \psi_S(\kappa_{\mathbf{h}})\left(f_{\boldsymbol{p}}|_{k,S}(\lambda,\nu\,{}^tq,\lambda s^t\lambda)_{\mathbf{a}}\begin{pmatrix} q & qx\\ & \tilde{q}\end{pmatrix}_{\mathbf{a}}\right)(\boldsymbol{i}_0),\end{aligned}$$

where we take $\kappa := \lambda s^t\lambda - (\lambda q\,{}^t\nu + \nu\,{}^tq\,{}^t\lambda), \boldsymbol{p} := (\lambda,0,0)_{\mathbf{h}}\mathrm{diag}[q,\tilde{q}]_{\mathbf{h}}$ and $f_{\boldsymbol{p}}$ is as in Definition 3.3.

Since $f_{\boldsymbol{p}} \in M^n_{k,S}(\boldsymbol{G}(F)\cap \boldsymbol{p}K[\mathfrak{b},\mathfrak{c}]\boldsymbol{p}^{-1},\chi)$, it is invariant under the translations $\tau\mapsto\tau+b$ and $w\mapsto w+\mu$ for every $b\in\mathcal{L} := Sym_n(F)\cap q_{\mathbf{h}}X\,{}^tq_{\mathbf{h}}$ and $\mu\in\mathcal{L}_{l,n} := M_{l,n}(F)\cap (X_{l,n}\,{}^tq_{\mathbf{h}})$. Indeed, for each such b and μ the finite parts of the adelic elements

$$(0,0,\lambda b^t\lambda)\begin{pmatrix}1 & b\\ 0 & 1\end{pmatrix} = (\lambda,0,0)\mathrm{diag}[q,\tilde{q}](0,-\lambda b\tilde{q},0)\begin{pmatrix}1 & q^{-1}b\tilde{q}\\ 0 & 1\end{pmatrix}\mathrm{diag}[q^{-1},{}^tq](-\lambda,0,0)$$

and

$$(0,\mu,\lambda q\mu\tilde{q}+\mu^t\lambda) = (\lambda,0,0)\mathrm{diag}[q,\tilde{q}](0,\mu\tilde{q},0)\mathrm{diag}[q^{-1},{}^tq](-\lambda,0,0)$$

are in the finite part of the group $\boldsymbol{p}K[\mathfrak{b},\mathfrak{c}]\boldsymbol{p}^{-1}$. Hence, $f_{\boldsymbol{p}}$ has a Fourier expansion

$$f_{\boldsymbol{p}}(\tau,w)=\sum_{\substack{t\in L\\ t\geq 0}}\sum_{r\in M}c(\boldsymbol{p};t,r)\mathbf{e}_{\mathbf{a}}(\mathrm{tr}\,(t\tau+{}^t r w)),$$

where

$$L=\{x\in Sym_n(F):\mathbf{e}_{\mathbf{a}}(\mathrm{tr}\,(x\mathcal{L}))=1\},$$
$$M=\{x\in M_{l,n}(F):\mathbf{e}_{\mathbf{a}}(\mathrm{tr}\,({}^t x\mathcal{L}_{l,n}))=1\}.$$

In particular, $c(\boldsymbol{p};t,r)\neq 0$ only if at every $v\in\mathbf{h}$ and for every $x\in X_v, x_{l,n}\in (X_{l,n})_v$ we have $e(\mathrm{tr}\,({}^t q_v t_v q_v x))=1$ and $e(\mathrm{tr}\,({}^t q_v\,{}^t r_v(x_{l,n})))=1$. Further, if we put $\boldsymbol{r}:=(\lambda,\nu\,{}^t q,\lambda s^t\lambda)_{\mathbf{a}}\left(\begin{smallmatrix} q & qx\\ & \tilde{q}\end{smallmatrix}\right)_{\mathbf{a}}$, we have

$$\begin{aligned}
&\mathbf{f}((\lambda,\mu,0)\begin{pmatrix} q & \sigma\tilde{q}\\ & \tilde{q}\end{pmatrix})=\psi_S(\kappa_{\mathbf{h}})J_{k,S}(\boldsymbol{r},\boldsymbol{i}_0)^{-1}f_{\boldsymbol{p}}(\boldsymbol{r}\boldsymbol{i}_0)\\
&=\Psi_{\mathbf{h}}(\mathrm{tr}\,(S\kappa))\mathbf{e}_{\mathbf{a}}(\mathrm{tr}\,(S[\lambda]s)+\mathrm{tr}\,(S[\lambda](iq\,{}^t q+qx\,{}^t q)))(\det q)^k_{\mathbf{a}}\\
&\quad\cdot f_{\boldsymbol{p}}(iq\,{}^t q+qx\,{}^t q, i\lambda q\,{}^t q+\lambda qx\,{}^t q+\nu\,{}^t q)\\
&=\Psi_{\mathbf{h}}(\mathrm{tr}\,(S\kappa))\mathbf{e}_{\mathbf{a}}(\mathrm{tr}\,(S[\lambda](iq\,{}^t q+\sigma)))(\det q)^k_{\mathbf{a}}f_{\boldsymbol{p}}(iq\,{}^t q+qx\,{}^t q, i\lambda q\,{}^t q+\lambda qx\,{}^t q+\nu\,{}^t q),
\end{aligned}$$

Now note that

$$\begin{aligned}
\Psi_{\mathbf{h}}(\mathrm{tr}\,(S\kappa))&=\Psi_{\mathbf{h}}(\mathrm{tr}\,(S(\lambda s^t\lambda-(\lambda q\,{}^t\nu+\nu\,{}^t q\,{}^t\lambda))))=\Psi_{\mathbf{h}}(\mathrm{tr}\,(S(\lambda s^t\lambda))\\
&=\Psi_{\mathbf{h}}(\mathrm{tr}\,(S(\lambda\sigma^t\lambda))\Psi_{\mathbf{h}}(-\mathrm{tr}\,(S(\lambda qx^t q^t\lambda))=\Psi_{\mathbf{h}}(\mathrm{tr}\,(S(\lambda\sigma^t\lambda))).
\end{aligned}$$

Moreover, since $\mathbf{e}_{\mathbf{h}}(\mathrm{tr}\,(tqx\,{}^t q))=1=\mathbf{e}_{\mathbf{h}}({}^t r\lambda qx\,{}^t q+{}^t r\nu\,{}^t q))$ for $t\in L, r\in M$, we have

$$\mathbf{e}_{\mathbb{A}}(\mathrm{tr}\,(t\sigma))=\mathbf{e}_{\mathbb{A}}(\mathrm{tr}\,(ts+tqx\,{}^t q))=\mathbf{e}_{\mathbb{A}}(\mathrm{tr}\,(tqx\,{}^t q))=\mathbf{e}_{\mathbf{a}}(\mathrm{tr}\,(tqx\,{}^t q))$$

and

$$\mathbf{e}_{\mathbb{A}}(\mathrm{tr}\,({}^t r(\lambda\sigma+\mu)))=\mathbf{e}_{\mathbb{A}}(\mathrm{tr}\,({}^t r(m+\nu\,{}^t q)+{}^t r\lambda qx\,{}^t q))=\mathbf{e}_{\mathbf{a}}(\mathrm{tr}\,({}^t r\nu\,{}^t q+{}^t r\lambda qx\,{}^t q)).$$

Hence,

$$f_{\boldsymbol{p}}(\boldsymbol{r}\boldsymbol{i}_0)=\sum_{\substack{t\in L\\ t\geq 0}}\sum_{r\in M}c(\boldsymbol{p};t,r)\mathbf{e}_{\mathbf{a}}(i\mathrm{tr}\,(tq\,{}^t q+{}^t r\lambda q\,{}^t q))\mathbf{e}_{\mathbb{A}}(\mathrm{tr}\,(t\sigma))\mathbf{e}_{\mathbb{A}}(\mathrm{tr}\,({}^t r\lambda\sigma+{}^t r\mu)).$$

In this way we obtain Fourier expansion (6) that satisfies properties (1) and (3). The second property follows from the fact that $\mathbf{f}|_{k,S}\mathrm{diag}[a,\tilde{a}]=\chi(\det a)^{-1}\mathbf{f}$ for $a\in\mathrm{GL}_n(F)$. □

4. Jacobi Eisenstein series

In this section we introduce Klingen-type Jacobi Eisenstein series. We do this both from a classical and adelic point of view, and also explore the relation between the two in the spirit of the bijection (5) between classical and adelic Siegel-Jacobi forms, which was established in the previous section. First systematic study of Eisenstein series from a classical point of view was undertaken by Ziegler in [28]. Our contribution here is to extend his results to include non-trivial level, non-trivial nebentype and we also work over a general totally real field. Furthermore, we introduce the adelic point of view, which, to the best of our knowledge, a systematic study of which, has not appeared before in the literature in the Jacobi setting.

For an integer $r \in \{0, 1, \dots, n\}$, we let $P^{n,r}, \boldsymbol{P}^{n,r}$ be Klingen parabolic subgroups of G^n and $\boldsymbol{G}^{n,l}$ respectively, as defined in Section 2. We define the map $\lambda^n_{r,l} : \boldsymbol{G}^{n,l} \to F$ by

$$\lambda^n_{r,l}((\lambda, \mu, \kappa)g) := \lambda^n_r(g),$$

where $\lambda^n_r : \mathrm{Sp}_n \to F$ is the map defined as in [22] by

$$\lambda^n_r\left(\begin{pmatrix} a_1 & a_2 & b_1 & b_2 \\ a_3 & a_4 & b_3 & b_4 \\ c_1 & c_2 & d_1 & d_2 \\ c_3 & c_4 & d_3 & d_4 \end{pmatrix}\right) = \det(d_4),$$

where the matrices a_1, b_1, c_1, d_1 are of size r and the matrices a_4, b_4, c_4, d_4 of size $n-r$; we set $\lambda^n_n(g) := 1$. We extend this map to the adeles so that $\lambda^n_{r,l} : \boldsymbol{G}^{n,l}(\mathbb{A}) \to \mathbb{A}$.

Furthermore for $r > 0$ we define the map

$$\boldsymbol{\omega}_r : \mathcal{H}_{n,l} \to \mathcal{H}_{r,l}$$

by $\boldsymbol{\omega}_r(\tau, w) := (\tau_1, w_1)$, where τ_1 denotes the $r \times r$ upper left corner of the matrix τ and w_1 is the $l \times r$ matrix obtained from the first r columns of w. Note that $\tau_1 = \omega_r(\tau)$ for ω_r as in [22]; we extend this and write $\boldsymbol{\omega}_r(w) := w_1$.

Finally, we define a (set theoretic) map

$$\boldsymbol{\pi}_r : H^{n,l} \times M_{2n} \to H^{r,l} \times M_{2r}, \quad \boldsymbol{\pi}_r((\lambda, \mu, \kappa), g) := ((\lambda_1, \mu_1, \kappa), \pi_r(g)),$$

where λ_1 (resp μ_1) is the $l \times r$ matrix obtained by taking the first r columns of λ (resp. μ), and $\pi_r(g) := \left(\begin{smallmatrix} a_1(g) & b_1(g) \\ c_1(g) & d_1(g) \end{smallmatrix}\right)$ is the map defined in [22] with $\pi_0(g) := 1$.

As we pointed out above, the maps $\lambda^n_r, \boldsymbol{\omega}_r, \boldsymbol{\pi}_r$ generalize the maps defined in [22]. In a similar manner their properties generalize the ones of the symplectic setting.

Lemma 4.1. *Assume $r > 0$. Then for all $\boldsymbol{g} \in \boldsymbol{P}^{n,r}(\mathbb{A})$we have*

$$\boldsymbol{\omega}_r(\boldsymbol{g}z) = \boldsymbol{\pi}_r(\boldsymbol{g})\boldsymbol{\omega}_r(z) \tag{7}$$

and

$$J_{k,S}(\boldsymbol{g}, z) = (\lambda^n_{r,l}(\boldsymbol{g})_{\mathbf{a}})^k J_{k,S}(\boldsymbol{\pi}_r(\boldsymbol{g}), \boldsymbol{\omega}_r(z)). \tag{8}$$

Proof. Write $z = (\tau, w)$ and $\boldsymbol{g} = hg = (\lambda, \mu, \kappa)g$. Then, by [22, (1.24)], $\omega_r(g\tau) = \pi_r(g)\omega_r(\tau)$ and $j(g,\tau) = \lambda_r(g)_{\mathbf{a}} j(\pi_r(g), \omega_r(\tau))$. Thus, to show (7) it suffices to establish the equality

$$(w(c_g\tau + d_g)^{-1} + \lambda g\tau + \mu)_1 = w_1(c_{\pi_r(g)}\omega_r(\tau) + d_{\pi_r(g)})^{-1} + \lambda_1\pi_r(g)\omega_r(\tau) + \mu_1;$$

or, after using the fact that $\pi_r(g)\omega_r(\tau) = \omega_r(g\tau)$ for $g \in P^{n,r}$,

$$(w(c_g\tau + d_g)^{-1})_1 = w_1(c_{\pi_r(g)}\omega_r(\tau) + d_{\pi_r(g)})^{-1}, \qquad (\lambda g\tau)_1 = \lambda_1\omega_r(g\tau).$$

Set $c := c_g$, $d := d_g$ and observe that for $\boldsymbol{g} \in \boldsymbol{P}^{n,r}(\mathbb{A})$,

$$c\tau + d = \begin{pmatrix} c_1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \tau_1 & \tau_2 \\ {}^t\tau_2 & \tau_4 \end{pmatrix} + \begin{pmatrix} d_1 & d_2 \\ 0 & d_4 \end{pmatrix} = \begin{pmatrix} c_1\tau_1 + d_1 & * \\ 0 & d_4 \end{pmatrix},$$

where c_1, τ_1, d_1 are $r \times r$ matrices. Hence,

$$(w(c\tau+d)^{-1})_1 = ((w_1\, w_2)\begin{pmatrix}(c_1\tau_1+d_1)^{-1} & * \\ 0 & d_4^{-1}\end{pmatrix})_1 = (w_1(c_1\tau_1+d_1)^{-1}\ *)_1$$
$$= w_1(c_1\tau_1+d_1)^{-1} = w_1(c_{\pi_r(g)}\tau_1 + d_{\pi_r(g)})^{-1}.$$

Similarly,

$$\lambda g\tau = (\lambda_1\ 0)\begin{pmatrix}\omega_r(g\tau) & * \\ * & *\end{pmatrix} = (\lambda_1\omega_r(g\tau)\ *).$$

We will now sketch a proof of the equality (8). Because $\lambda^n_{r,l}(\mathbf{g})_{\mathbf{a}} = \lambda_r(g)_{\mathbf{a}}$ and $j(g,\tau) = \lambda_r(g)_{\boldsymbol{a}} j(\pi_r(g), \omega_r(\tau))$, it is enough to show that

$$\mathcal{J}_S(\boldsymbol{g}, z) = \mathcal{J}_S(\boldsymbol{\pi}_r(\boldsymbol{g}), \boldsymbol{\omega}_r(z)),$$

that is,

(1) $\mathrm{tr}\,(S[w](c_g\tau+d_g)^{-1}c_g) = \mathrm{tr}\,(S[w_1](c_{\pi_r(g)}\tau_1 + d_{\pi_r(g)})^{-1}c_{\pi_r(g)})$,
(2) $\mathrm{tr}\,({}^t\lambda S w(c_g\tau+d_g)^{-1}) = \mathrm{tr}\,({}^t\lambda_1 S w_1(c_{\pi_r(g)}\tau_1 + d_{\pi_r(g)})^{-1})$ and
(3) $\mathrm{tr}\,(S[\lambda]g\tau) = \mathrm{tr}\,(S[\lambda_1]\pi_r(g)\tau_1)$.

Write $w = (w_1\ w_2)$, so that

$$S[w] = \begin{pmatrix} {}^tw_1 \\ {}^tw_2\end{pmatrix} S(w_1\ w_2) = \begin{pmatrix} {}^tw_1 S \\ {}^tw_2 S\end{pmatrix}(w_1\ w_2) = \begin{pmatrix} S[w_1] & * \\ * & *\end{pmatrix}.$$

Moreover, as we have seen before, $(c_g\tau+d_g)^{-1} = \begin{pmatrix}(c_{\pi_r(g)}\omega_r(\tau)+d_{\pi_r(g)})^{-1} & * \\ 0 & *\end{pmatrix}$, $c = \begin{pmatrix} c_{\pi_r(g)} & 0 \\ 0 & 0\end{pmatrix}$, so that

$$(c_g\tau+d_g)^{-1}c_g = \begin{pmatrix}(c_{\pi_r(g)}\omega_r(\tau)+d_{\pi_r(g)})^{-1}c_{\pi_r(g)} & 0 \\ 0 & 0\end{pmatrix}.$$

Hence

$$\mathrm{tr}\,(S[w](c_g\tau+d_g)^{-1}c_g) = \mathrm{tr}\,\left(\begin{pmatrix} S[w_1] & * \\ * & *\end{pmatrix}\begin{pmatrix}(c_{\pi_r(g)}\tau_1+d_{\pi_r(g)})^{-1}c_{\pi_r(g)} & 0 \\ 0 & 0\end{pmatrix}\right)$$
$$= \mathrm{tr}\,(S[w_1](c_{\pi_r(g)}\tau_1+d_{\pi_r(g)})^{-1}c_{\pi_r(g)}).$$

Similar calculations with $\lambda = (\lambda_1\ 0)$ prove the remaining equalities. □

4.1. **Adelic Jacobi Eisenstein series of Klingen-type.** We are now ready to define adelic Jacobi Eisenstein series of Klingen type. Fix a weight $k \in \mathbb{Z}^{\mathbf{a}}$ and consider a Hecke character χ such that for a fixed integral ideal $\mathfrak{c}$ of F we have

(1) $\chi_v(x) = 1$ for all $x \in \mathfrak{o}_v^\times$ with $x - 1 \in \mathfrak{c}_v$, $v \in \mathbf{h}$,
(2) $\chi_{\mathbf{a}}(x_{\mathbf{a}}) = \mathrm{sgn}(x_{\mathbf{a}})^k := \prod_{v\in\mathbf{a}}\left(\frac{x_v}{|x_v|}\right)^{k_v}$, for $x_{\mathbf{a}} \in \mathbb{A}_{\mathbf{a}}$;

we will also write $\chi_{\mathfrak{c}} := \prod_{v|\mathfrak{c}}\chi_v$. We fix a fractional ideal $\mathfrak{b}$ and an integral ideal $\mathfrak{e}$ such that $\mathfrak{c} \subset \mathfrak{e}$ and $\mathfrak{e}$ is prime to $\mathfrak{e}^{-1}\mathfrak{c}$. Further, for $r \in \{1, \dots, n\}$ we set

$$K := K_{\mathbf{h}}[\mathfrak{b}, \mathfrak{c}](H^{n,l}_{\mathbf{a}} \rtimes D^{\mathbf{a}}_\infty),$$

$$K^{n,r} := \{\boldsymbol{x} = (\lambda, \mu, \kappa)x \in K : (a_1(x) - 1_r)_v \in M_{r,r}(\mathfrak{e}_v),$$
$$(a_2(x))_v \in M_{r,n-r}(\mathfrak{e}_v), (b_1(x))_v \in M_{r,r}(\mathfrak{b}_v^{-1}\mathfrak{e}_v) \text{ for every } v|\mathfrak{e}\},$$

where $x = \begin{pmatrix} a_x & b_x \\ c_x & d_x \end{pmatrix} = \begin{pmatrix} a_1(x) & a_2(x) & b_1(x) & b_2(x) \\ a_3(x) & a_4(x) & b_3(x) & b_4(x) \\ c_1(x) & c_2(x) & d_1(x) & d_2(x) \\ c_3(x) & c_4(x) & d_3(x) & d_4(x) \end{pmatrix}$, and

$$K^r := \{\boldsymbol{x} \in K^r[\mathfrak{b}^{-1}\mathfrak{e}, \mathfrak{b}\mathfrak{c}] : (a_x - 1_r)_v \in M_{r,r}(\mathfrak{e}_v) \text{ for every } v|\mathfrak{e}\}.$$

If $r = 0$, we put $K^{n,0} := K$.

For a cusp form $\mathbf{f} \in \mathcal{S}^r_{k,S}(K^r, \chi^{-1})$, $\mathbf{f} := 1$ if $r = 0$, we define a $\mathbb{C}$-valued function $\phi(x, s; \mathbf{f})$ with $x \in \boldsymbol{G}^n(\mathbb{A})$ and $s \in \mathbb{C}$ as follows. We set $\phi(x, s; \mathbf{f}) := 0$ if $x \notin \boldsymbol{P}^{n,r}(\mathbb{A})K^{n,r}$ and otherwise, if $x = \boldsymbol{p}\boldsymbol{w}$ with $\boldsymbol{p} \in \boldsymbol{P}^{n,r}(\mathbb{A})$ and $\boldsymbol{w} \in K^{n,r}$, we set

$$\phi(x, s; \mathbf{f}) := \chi(\lambda^n_{r,l}(\boldsymbol{p}))^{-1}\chi_{\mathfrak{c}}(\det(d_w)))^{-1}J_{k,S}(\boldsymbol{w}, \mathbf{i}_0)^{-1}\mathbf{f}(\pi_r(\boldsymbol{p}))|\lambda^n_{r,l}(\boldsymbol{p})|^{-2s}_{\mathbb{A}},$$

where $\boldsymbol{w} = hw$ with $w \in \mathrm{Sp}_n(\mathbb{A})$. We recall here that if we write p for the symplectic part of $\boldsymbol{p}$ then $\lambda^n_{r,l}(\boldsymbol{p}) = \lambda^n_r(p)$. Moreover, since at archimedean places $x_{\mathbf{a}} \in \boldsymbol{P}^{n,r}_{\mathbf{a}}K^{n,r}_{\mathbf{a}} = P^{n,r}_{\mathbf{a}}K^{n,r}_{\mathbf{a}}$ if and only if $x_{\mathbf{a}} \in P'_{\mathbf{a}}K^{n,r}_{\mathbf{a}}$, where $P' := \bigcap_{r=0}^{n-1} P^{n,r}$ ([22], Lemma 3.1), we always choose $\boldsymbol{p} \in \boldsymbol{P}^{n,r}(\mathbb{A})$ so that $\boldsymbol{p}_{\mathbf{a}} = p_{\mathbf{a}} \in P'_{\mathbf{a}}$. We now check that $\phi(x, s; \mathbf{f})$ is well-defined, i.e. that it is independent of the choice of p and $\boldsymbol{w}$.

Let $x = \boldsymbol{p}_1\boldsymbol{w}_1 = \boldsymbol{p}_2\boldsymbol{w}_2$, set $\boldsymbol{r} := \boldsymbol{p}_2^{-1}\boldsymbol{p}_1 = \boldsymbol{w}_2\boldsymbol{w}_1^{-1} \in \boldsymbol{P}^{n,r}(\mathbb{A}) \cap K^{n,r}$ and assume that $(p_1)_{\mathbf{a}}, (p_2)_{\mathbf{a}} \in P'_{\mathbf{a}}$. Observe that $\lambda^n_{r,l}(\boldsymbol{r})_v = (\det d_{p_2,4})^{-1}_v(\det d_{p_1,4})_v \in \mathfrak{o}^{\times}_v$ for every $v \in \mathbf{h}$, and $|\lambda^n_{r,l}(\boldsymbol{r})_v|_v = 1$ for all $v \in \mathbf{a}$. Hence, $|\lambda^n_{r,l}(\boldsymbol{p})|^{-2s}_{\mathbb{A}}$ is independent of choice of $\boldsymbol{p}$ and $\boldsymbol{w}$, and $\chi(\lambda^n_{r,l}(\boldsymbol{p}))^{-1} = \chi_{\mathfrak{c}}(\lambda^n_{r,l}(\boldsymbol{p}))^{-1}(\lambda^n_{r,l}(\boldsymbol{p})_{\mathbf{a}})^{-k}$. Because

$$\mathbf{f}(\pi_r(\boldsymbol{p}_1)) = \mathbf{f}(\pi_r(\boldsymbol{p}_2\boldsymbol{r})) = \mathbf{f}(\pi_r(\boldsymbol{p}_2)\pi_r(\boldsymbol{r})) = \mathbf{f}(\pi_r(\boldsymbol{p}_2))\chi_{\mathfrak{c}}(\det a_{\pi_r(\boldsymbol{r})})J_{k,S}(\pi_r(\boldsymbol{r}), \boldsymbol{i}_0)^{-1},$$

we have to prove that

$$\chi_{\mathfrak{c}}(\lambda^n_{r,l}(\boldsymbol{r}))^{-1}(\lambda^n_{r,l}(\boldsymbol{r})_{\mathbf{a}})^{-k}\chi_{\mathfrak{c}}(\det(d_{w_1}))^{-1}\chi_{\mathfrak{c}}(\det(d_{w_2}))\chi_{\mathfrak{c}}(\det a_{\pi_r(\boldsymbol{r})}) \\ = J_{k,S}(\pi(\boldsymbol{r}), \boldsymbol{i}_0)J_{k,S}(\boldsymbol{w}_1, \mathbf{i}_0)J_{k,S}(\boldsymbol{w}_2, \mathbf{i}_0)^{-1}$$

First of all, since $\boldsymbol{r}_{\mathbf{a}} \in P'_{\mathbf{a}}$,

$$(\lambda^n_{r,l}(\boldsymbol{r})_{\mathbf{a}})^k J_{k,S}(\pi(\boldsymbol{r}), \boldsymbol{i}_0)J_{k,S}(\boldsymbol{w}_1, \mathbf{i}_0)J_{k,S}(\boldsymbol{w}_2, \mathbf{i}_0)^{-1} = J_{k,S}(\boldsymbol{r}, \boldsymbol{i}_0)J_{k,S}(\boldsymbol{r}, \boldsymbol{w}_1 \cdot \mathbf{i}_0)^{-1} = 1.$$

Moreover, it is easy to check that

$$\chi_{\mathfrak{c}}(\lambda^n_{r,l}(\boldsymbol{r}))^{-1}\chi_{\mathfrak{c}}(\det(d_{w_1}))^{-1}\chi_{\mathfrak{c}}(\det(d_{w_2}))\chi_{\mathfrak{c}}(\det a_{\pi_r(\boldsymbol{r})}) \\ = \chi_{\mathfrak{c}}(\det d_{\pi_r(w_2)})\chi_{\mathfrak{c}}(\det d_{\pi_r(w_1)})^{-1}\chi_{\mathfrak{c}}(\det a_{\pi_r(\boldsymbol{r})}) = 1.$$

This proves the statement above.

We define the Eisenstein series of Klingen type by

$$E(x, s; \mathbf{f}) := E(x, s; \mathbf{f}, \chi, K^{n,r}) := \sum_{\gamma \in \boldsymbol{P}^{n,r}(F)\backslash \boldsymbol{G}^n(F)} \phi(\gamma x, s; \mathbf{f}), \quad \mathrm{Re}(s) \gg 0. \tag{9}$$

If $r = 0$ and $\mathbf{f} = 1$, then we say that $E(x, s) := E(x, s; 1)$ is an Eisenstein series of Siegel type.

It is clear from the above calculations that this is well defined, and for $\gamma \in \boldsymbol{P}^{n,r}(F)$, $\boldsymbol{w} \in K^{n,r}_{\mathbf{h}} \times K_{\infty}$,

$$\phi(\gamma x\boldsymbol{w}, s; \mathbf{f}) = \chi_{\mathfrak{c}}(\det(d_w))^{-1}J_{k,S}(\boldsymbol{w}, \mathbf{i}_0)^{-1}\phi(x, s; \mathbf{f}).$$

In particular, for $\kappa \in Sym_l(\mathbb{A})$, $\gamma \in \boldsymbol{G}^n(F)$, $x \in \boldsymbol{G}^n(\mathbb{A})$ and $\boldsymbol{w} \in K_{\mathbf{h}}^{n,r} \times K_\infty$,

$$E((0,0,\kappa)\gamma x\boldsymbol{w}, s; \mathbf{f}) = \psi_S(\kappa)\chi_{\mathfrak{c}}(\det(d_w))^{-1} J_{k,S}(\boldsymbol{w}, \mathbf{i}_0)^{-1} E(x, s; \mathbf{f}).$$

We will show in Proposition 4.3 below that the series above, evaluated at $s = k/2$ for $k \in \mathbb{Z}$, $k > n + r + l + 1$, is absolutely convergent and hence defines an adelic Siegel-Jacobi modular form of parallel weight $k\mathbf{a} := (k, k, \ldots, k) \in \mathbb{Z}^{\mathbf{a}}$.

We now investigate the relation of the adelic Eisenstein series (9) with the classical one. Write $K_{\mathbf{h}}^{n,r} = C_{\mathbf{h}}[\mathfrak{o}, \mathfrak{b}^{-1}, \mathfrak{b}^{-1}] \rtimes D_{\mathbf{h}}^{n,r}[\mathfrak{b}^{-1}, \mathfrak{bc}]$. Then it follows from [22, Lemma 3.2] and [20, Lemma 1.3] that

$$P^{n,r}(\mathbb{A}) = \bigsqcup_{x \in X} P^{n,r}(F) x (P^{n,r}(\mathbb{A}) \cap D_{\mathbf{h}}^{n,r}[\mathfrak{b}^{-1}, \mathfrak{bc}]) P^{n,r}(\mathbb{A}_{\mathbf{a}}),$$

where X is a finite subset of $P^{n,r}(\mathbb{A})$ such that $\{\mathfrak{a}_r(x) : x \in X\}$ forms a set of representatives for the ideal class group of F, where $\mathfrak{a}_r(x)$ is the ideal of F defined in [22, page 551] as the ideal corresponding to the idele $\lambda_r(x)$. In particular one may pick x's of a very specific form, namely $\mathrm{diag}[1_{n-1}, t^{-1}, 1_{n-1}, t]$ with $t \in \mathbb{A}_{\mathbf{h}}^{\times}$. Since $\boldsymbol{P}^{n,r} = H_r^{n,l} \rtimes P^{n,r}$ and the strong approximation holds for $H_r^{n,l}$ by the same argument as in Lemma 3.2,

$$\boldsymbol{P}^{n,r}(\mathbb{A}) = \bigsqcup_{x' \in X'} \boldsymbol{P}^{n,r}(F) x' (\boldsymbol{P}^{n,r}(\mathbb{A}) \cap K_{\mathbf{h}}^{n,r}[\mathfrak{b}, \mathfrak{c}]) \boldsymbol{P}^{n,r}(\mathbb{A}_{\mathbf{a}}),$$

where X' is the set X extended trivially to $\boldsymbol{G}^n$ by the canonical embedding $\mathrm{Sp}_n \hookrightarrow \boldsymbol{G}^n$. We can now establish that

$$\begin{aligned} \boldsymbol{P}^{n,r}(\mathbb{A}) K^{n,r} &= \bigsqcup_{x' \in X'} \boldsymbol{P}^{n,r}(F) x' K_{\mathbf{h}}^{n,r}[\mathfrak{b}, \mathfrak{c}] \boldsymbol{P}^{n,r}(\mathbb{A}_{\mathbf{a}}) K^{n,r}(\mathbb{A}_{\mathbf{a}}) \\ &= \bigsqcup_{x' \in X'} \boldsymbol{P}^{n,r}(F) x' K_{\mathbf{h}}^{n,r}[\mathfrak{b}, \mathfrak{c}] \boldsymbol{G}^n(\mathbb{A}_{\mathbf{a}}). \end{aligned}$$

Indeed, we only need to establish that the union is disjoint. Assume that the cosets determined by $x_1, x_2 \in X'$ are not disjoint, that is $x_1 = a x_2 b c$ for some $a \in \boldsymbol{P}^{n,r}(F)$, $b \in K_{\mathbf{h}}[\mathfrak{b}, \mathfrak{c}]$ and $c \in \boldsymbol{P}^{n,r}(\mathbb{A}_{\mathbf{a}}) K^{n,r}(\mathbb{A}_{\mathbf{a}})$. Since $x_1, x_2 \in \boldsymbol{G}_{\mathbf{h}}^n$, $x_1 = a_{\mathbf{h}} x_2 b$. Moreover, since $a \in \boldsymbol{P}^{n,r}(F)$ and x_1, x_2 are diagonal, $b \in \boldsymbol{P}^{n,r}(\mathbb{A}) \cap K_{\mathbf{h}}^{n,r}[\mathfrak{b}, \mathfrak{c}]$ and $c_{\mathbf{a}} \in \boldsymbol{P}^{n,r}(\mathbb{R})$. This implies that $x_1 \in \boldsymbol{P}^{n,r}(F) x_2 (\boldsymbol{P}^{n,r}(\mathbb{A}) \cap K_{\mathbf{h}}^{n,r}[\mathfrak{b}, \mathfrak{c}]) \boldsymbol{P}^{n,r}(\mathbb{A}_{\mathbf{a}})$, and thus $x_1 = x_2$.

Take the set X' to be of the particular form indicated above, that is let $x' \in X'$ be of the form $\mathrm{diag}[1_{n-1}, t^{-1}, 1_{n-1}, t] \in \mathrm{Sp}_n(\mathbb{A}) \hookrightarrow \boldsymbol{G}^n(\mathbb{A})$ with $t \in \mathbb{A}_{\mathbf{h}}^{\times}$. Observe that for any such x', $x' K_{\mathbf{h}}^{n,r}[\mathfrak{b}, \mathfrak{c}](\mathbb{A}_{\mathbf{a}}) \boldsymbol{G}^n(\mathbb{A}\mathbf{a}) \cap \boldsymbol{G}^n(F) \neq \emptyset$. Indeed, this follows from the fact that $\mathrm{diag}[1_{n-1}, t^{-1}, 1_{n-1}, t] D_{\mathbf{h}}^{n,r}[\mathfrak{b}^{-1}, \mathfrak{bc}] \mathrm{Sp}_n(\mathbb{R}) \cap \mathrm{Sp}_n(F) \neq \emptyset$. In particular, we can conclude the analogue of [22, Lemma 3.3] in the Jacobi setting:

Lemma 4.2. *Set* $Y := \bigcup_{t \in \mathbb{A}_{\mathbf{h}}^{\times}} \mathrm{diag}[1_{n-1}, t^{-1}, 1_{n-1}, t] K_{\mathbf{h}}[\mathfrak{b}, \mathfrak{c}] \boldsymbol{P}^{n,r}(\mathbb{A}_{\mathbf{a}}) K^{n,r}(\mathbb{A}_{\mathbf{a}})$. *Then there exists a subset* Z *of* $\boldsymbol{G}^n(F) \cap Y$ *such that*

$$\boldsymbol{P}^{n,r}(\mathbb{A}) K^{n,r} = \bigsqcup_{\zeta \in Z} \boldsymbol{P}^{n,r}(F) \zeta K_{\mathbf{h}}^{n,r}[\mathfrak{b}, \mathfrak{c}] \boldsymbol{P}^{n,r}(\mathbb{R}) K^{n,r}(\mathbb{A}_{\mathbf{a}}) = \bigsqcup_{\zeta \in Z} \boldsymbol{P}^{n,r}(F) \zeta K_{\mathbf{h}}^{n,r}[\mathfrak{b}, \mathfrak{c}] \boldsymbol{G}^n(\mathbb{R})$$

and

$$\begin{aligned}\boldsymbol{G}^n(F) \cap \boldsymbol{P}^{n,r}(\mathbb{A}) K^{n,r} &= \bigsqcup_{\zeta \in Z} \boldsymbol{P}^{n,r}(F) \zeta \left(K^{n,r}_{\mathbf{h}}[\mathfrak{b}, \mathfrak{c}] \boldsymbol{P}^{n,r}(\mathbb{A}_{\mathbf{a}}) K(\mathbb{A}_{\mathbf{a}}) \cap \boldsymbol{G}^n(F) \right) \\ &= \bigsqcup_{\zeta \in Z} \boldsymbol{P}^{n,r}(F) \zeta \left(K^{n,r}_{\mathbf{h}}[\mathfrak{b}, \mathfrak{c}] \boldsymbol{G}^n(\mathbb{A}_{\mathbf{a}}) \cap \boldsymbol{G}^n(F) \right).\end{aligned}$$

4.2. **Classical Jacobi Eisenstein series of Klingen-type.** We now associate a Siegel-Jacobi modular form to an adelic Eisenstein series defined in (9). We set $\Gamma := \boldsymbol{G}^n(F) \cap K^{n,r}_{\mathbf{h}}[\mathfrak{b}, \mathfrak{c}] \boldsymbol{G}^n(\mathbb{A}_{\mathbf{a}})$, and with Z as in Lemma 4.2 we define $R_\zeta := (\boldsymbol{P}^{n,r}(F) \cap \zeta\Gamma\zeta^{-1}) \setminus \zeta\Gamma$, for $\zeta \in Z$. Then, again by the same lemma, it follows that a set of representatives for $\boldsymbol{P}^{n,r}(F) \setminus (\boldsymbol{G}^n(F) \cap \boldsymbol{P}^{n,r}(\mathbb{A}) K^{n,r})$ is given by $R := \bigcup_{\zeta \in Z} R_\zeta$. In particular, we may write

$$E(x, s; \mathbf{f}) = \sum_{\gamma \in R} \phi(\gamma x, s; \mathbf{f}).$$

For any given $z \in \mathcal{H}_{n,l}$ there is an $y \in \boldsymbol{G}^n_{\mathbf{a}}$ such that $y \cdot \mathbf{i}_0 = z$. Moreover, we can always pick y such that the symmetric matrix in the Heisenberg part of y is zero, i.e. $\kappa_y = 0$. A Siegel-Jacobi modular form that corresponds to $E(x, s; \mathbf{f})$ via the bijection (5) with $\boldsymbol{g} = 1$ is the Eisenstein series,

$$E(z, s; \mathbf{f}) = J_{k,S}(y, \mathbf{i}_0) \sum_{\gamma \in R} \phi(\gamma y, s; \mathbf{f}).$$

We will write it down in terms of f and z using the bijection (5) again. For some $\zeta \in Z$ and $\gamma \in R_\zeta$ we may write $\gamma y = \tau \boldsymbol{w}$, where $\tau_{\mathbf{h}} = \mathrm{diag}[1_{n-1}, t^{-1}, 1_{n-1}, t]$ as in Lemma 4.2, $\tau_{\mathbf{a}} \in \cap_{r=0}^{n-1} P^{n,r}_{\mathbf{a}}$ and $\boldsymbol{w} \in K^{n,r}$. This is because $H^{n,l}_{\mathbf{a}} \subset K^{n,r}_{\mathbf{a}}$ and, by [22, Lemma 3.1], $G^n(\mathbb{A}) = \cap_{r=0}^{n-1} P^{n,r}(\mathbb{A}) D^{\mathbf{a}}_\infty D_{\mathbf{h}}[\mathfrak{b}^{-1}, \mathfrak{b}]$. Therefore

$$\phi(\tau \boldsymbol{w}, s; \mathbf{f}) = \chi_{\mathbf{h}}(t)^{-1} \chi_{\mathbf{a}}(\lambda^n_{r,l}(\tau)_{\mathbf{a}})^{-1} \chi_{\mathfrak{c}}(\det(d_w))^{-1} J_{k,S}(\boldsymbol{w}, \mathbf{i}_0)^{-1} \mathbf{f}(\pi_r(\tau_{\mathbf{a}})) |\lambda^n_{r,l}(\tau)|_{\mathbb{A}}^{-2s}.$$

Observe further that, in case $r > 0$,

(1) $\mathbf{f}(\boldsymbol{\pi}_r(\tau_{\mathbf{a}})) = J_{k,S}(\boldsymbol{\pi}_r(\tau_{\mathbf{a}}), \boldsymbol{i}_0)^{-1} f(\boldsymbol{\pi}_r(\tau_{\mathbf{a}})) \overset{(7),(8)}{=} J_{k,S}(\tau_{\mathbf{a}}, \boldsymbol{i}_0)^{-1} (\lambda^n_{r,l}(\tau)_{\mathbf{a}})^k f(\boldsymbol{\omega}_r(\gamma z))$;
(2) $|\lambda^n_{r,l}(\tau)_{\mathbf{a}}|_F = |\frac{j(\tau_{\mathbf{a}}, \boldsymbol{i})}{j(\pi_r(\tau_{\mathbf{a}}), \omega_r(\boldsymbol{i}))}|_F = \left(\frac{\delta(\pi_r(\tau_{\mathbf{a}}), \boldsymbol{i})}{\delta(\tau_{\mathbf{a}}, \boldsymbol{i})} \right)^{1/2} = \left(\frac{\delta(\boldsymbol{\omega}_r(\gamma z))}{\delta(\gamma z)} \right)^{1/2}$;
(3) $|\lambda^n_{r,l}(\tau)|_{\mathbb{A}} = |t|_F^{-1} |\lambda^n_{r,l}(\tau)_{\mathbf{a}}|_F$;
(4) $J_{k,S}(\gamma, z) J_{k,S}(y, \boldsymbol{i}_0) = J_{k,S}(\gamma y, \boldsymbol{i}_0) = J_{k,S}(\tau, \boldsymbol{w}\boldsymbol{i}_0) J_{k,S}(\boldsymbol{w}, \boldsymbol{i}_0) = J_{k,S}(\tau, \boldsymbol{i}_0) J_{k,S}(\boldsymbol{w}, \boldsymbol{i}_0)$.

Moreover, since the product $\chi_{\mathbf{h}}(t)^{-1} \chi_{\mathfrak{c}}(\det(d_w))^{-1}$ depends only on the symplectic part of γ, we can follow the reasoning in [22, Lemma 3.6] and denote it by $\chi[\gamma]$, which agrees with the definition of $\chi[\gamma]$ in [22, (3.11)]. Taking all these into account we obtain

$$\begin{aligned}E(z, s; \mathbf{f}) &= \sum_{\gamma \in R} \chi[\gamma] |t|_F^{2s} \left(\frac{\delta(\gamma z)}{\delta(\boldsymbol{\omega}_r(\gamma z))} \right)^{s - k/2} f(\boldsymbol{\omega}_r(\gamma z)) J_{k,S}(\gamma, z)^{-1} \\ &= \sum_{\zeta \in Z} |\lambda^n_{r,l}(\zeta)|_F^{2s} \sum_{\gamma \in R_\zeta} \chi[\gamma] \left(\frac{\delta(z)}{\delta(\boldsymbol{\omega}_r(z))} \right)^{s - k/2} f(\boldsymbol{\omega}_r(z))|_{k,S}\gamma. \qquad (10)\end{aligned}$$

Analogously, if $r = 0$ (and $\mathbf{f} = 1$), we obtain the Siegel type Jacobi Eisenstein series,

(11)
$$E(z,s) = \sum_{\zeta \in Z} |\lambda^n_{0,l}(\zeta)|_F^{2s} \sum_{\gamma \in R_\zeta} \chi[\gamma]\delta(z)^{s-k/2}|_{k,S}\gamma = \sum_{\zeta \in Z} N(\mathfrak{a}(\zeta))^{2s} \sum_{\gamma \in R_\zeta} \chi[\gamma]\delta(z)^{s-k/2}|_{k,S}\gamma.$$

We finish this section with a result regarding the absolute convergence of the series.

Proposition 4.3. *The Eisenstein series $E(z,s;\mathbf{f})$ is absolutely convergent for* $\mathrm{Re}(2s) > n+r+l+1$. *In particular for $k\mathbf{a} \in \mathbb{Z}^{\mathbf{a}}$ with $k > n+r+l+1$ the series $E(z,k/2;\mathbf{f})$ is a Siegel-Jacobi form of parallel weight k.*

Proof. This follows from the calculations of Ziegler in [28, pages 204-207]. The difference with his Theorem 2.5 is the different normalisation of our Eisentein series as well as the introduction of the complex parameter s, but it is easy to see that his calculations lead to the range of absolute convergence stated above. $\square$

Later in the paper we will explore analytic properties of the Klingen-type Eisenstein series, such as analytic continuation and possible poles regarding the parameter s. This will be done in section 8. Furthermore, in the last section of this paper we will study the analytic properties of $E(z,s;\mathbf{f})$ with respect to the variable z for some particular values of s. Namely, we will try to establish whether this series, even if it fails to be holomorphic in z, still has some good algebraic properties. To do this, we will introduce in the last section the notion of nearly holomorphic Siegel-Jacobi forms, and we will see that for particular values of s the Jacobi Eisenstein series are of this kind.

5. The Doubling Method

As it was discussed in the introduction of this paper one of the most fruitful methods for studying various L-functions attached to (classical, i.e. Siegel, Hermitian, orthogonal) automorphic forms is, what is often called, the doubling method. It is perhaps not surprising that the same method can be used to study also L-functions attached to Siegel-Jacobi forms. We will introduce the latter a bit later in the paper, after developing necessary background for the doubling method. Actually there are two, rather different, ways to use this method.

(1) **Method I.** This is the original approach of Murase [15, 16], where he used a homomorphism (actually an injection)
$$\boldsymbol{G}^{n,l} \times \boldsymbol{G}^{n,l} \to \mathrm{Sp}_{l+2n}.$$
One of the main advantages of this approach is the fact that analytic properties of the L-function can be read off from analytic properties of (classical) Siegel Eisenstein series of Sp_{2n+l}, which are well-understood. On the other hand, it is not quite clear how one could translate the picture classically, i.e. pulling back the Siegel Eisenstein series to the Jacobi symmetric space, which makes the method less attractive for other applications (differential operators, algebraicity, study of Klingen-type Eisenstein series and others).

(2) **Method II.** The second approach, which we follow in this paper, was first employed by Arakawa [3]. It uses a homomorphism (shortly to be made explicit),
$$\boldsymbol{G}^{m,l} \times \boldsymbol{G}^{n,l} \to \boldsymbol{G}^{n+m,l}.$$
This seems to be a more natural approach and closer to the spirit of the doubling method, since one "doubles" the same "kind" of a group. Moreover, it is quite clear what happens on the corresponding symmetric spaces. However, this method calls for a study of analytic and algebraic properties of Siegel-type Jacobi Eisenstein series introduced in the previous section, a task that will be taken upon later in this paper.

In this section we will develop technical results which will be necessary to apply the doubling method. The main result here is Lemma 5.3, which will be used in the next section to study a particular pullback of a Siegel-type Eisenstein series. Our approach is modeled on the work of Shimura in [22] where the symplectic case is considered, and our results here generalize those of Shimura to the Jacobi setting.

We define first the map mentioned above. Let
$$\iota_A \colon \boldsymbol{G}^{m,l} \times \boldsymbol{G}^{n,l} \to \boldsymbol{G}^{m+n,l},$$
$$\iota_A((\lambda,\mu,\kappa)g) \times (\lambda',\mu',\kappa')g')) := ((\lambda\,\lambda'),(\mu\,\mu'),\kappa+\kappa';\iota_S(g\times g')),$$
where
$$\iota_S : G^m \times G^n \hookrightarrow G^{m+n}, \quad \iota_S\left(\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) \times \left(\begin{smallmatrix} a' & b' \\ c' & d'\end{smallmatrix}\right)\right) := \begin{pmatrix} a & & b & \\ & a' & & b' \\ c & & d & \\ & c' & & d' \end{pmatrix}.$$
In what follows we will often write $\boldsymbol{g} \times \boldsymbol{g'}$ for $\iota_A(\boldsymbol{g}\times\boldsymbol{g'})$. Sometimes it will be useful to view elements of $\boldsymbol{G}^{m+n,l}$ as elements of G^{l+m+n} via the embedding in equation (1). Denote by $H^{n,l}_r$ the Heisenberg subgroup of $\boldsymbol{P}^{n,r}$, that is, put
$$H^{n,l}_r(F) := \{((\lambda\, 0_{l,n-r}),\mu,\kappa) \in H^{n,l}(F)\}.$$

We will now adapt a method presented in [22] to find good coset representatives for $\boldsymbol{P}^{m+n}(F)\backslash\boldsymbol{G}^{m+n}(F)$. Let $n \leq m$ and define $\boldsymbol{\tau}_r := 1_H\tau_r \in \boldsymbol{G}^{m+n}(F)$, where
$$\tau_r := \begin{pmatrix} 1_m & & & \\ & 1_n & & \\ & e_r & 1_m & \\ {}^te_r & & & 1_n \end{pmatrix}, \qquad e_r := \begin{pmatrix} 1_r & \\ & 0 \end{pmatrix} \in M_{m,n}(F).$$

Lemma 5.1. *If $n \leq m$,*
$$\boldsymbol{G}^{m+n}(F) = \bigsqcup_{0\leq r\leq n} \boldsymbol{P}^{m+n}(F)\boldsymbol{\tau}_r\iota_A(\boldsymbol{G}^m(F)\times\boldsymbol{G}^n(F)).$$

Proof. Let $\boldsymbol{G}^{m+n}(F) = \bigsqcup_i \boldsymbol{P}^{m+n}(F)\boldsymbol{g}_i\iota_A(\boldsymbol{G}^m(F)\times\boldsymbol{G}^n(F))$ be a double coset decomposition. There exist unique $g_i \in G^{m+n}(F)$ and $h_i \in H^{m+n,l}(F)$ such that $\boldsymbol{g}_i = g_ih_i$. Note also that $\iota_A(\boldsymbol{G}^m(F)\times\boldsymbol{G}^n(F)) = H^{m+n,l}(F) \rtimes \iota_A(G^m(F)\times G^n(F))$. We have
$$\begin{aligned} \boldsymbol{G}^{m+n}(F) &= \bigsqcup_i \boldsymbol{P}^{m+n}(F)g_ih_iH^{m+n,l}(F)\iota_A(G^m(F)\times G^n(F)) \\ &= \bigsqcup_i H^{m+n,l}_0(F)P^{m+n}(F)H^{m+n,l}(F)g_i\iota_S(G^m(F)\times G^n(F))) \end{aligned}$$

$$= \bigsqcup_i H_0^{m+n,l}(F)H^{m+n,l}(F)P^{m+n}(F)g_i\iota_S(G^m(F)\times G^n(F)).$$

Since $\boldsymbol{G}^{m+n}(F) = H^{m+n,l}(F)G^{m+n}(F)$ and $G^{m+n}(F) = \bigsqcup_{0\le r\le n} P^{m+n}(F)\tau_r\iota_S(G^m(F)\times G^n(F))$ by [22, Lemma 4.2], we can take $\{g_i\}_i = \{\tau_r : 0\le r\le n\}$ and thus $\{\boldsymbol{g}_i\}_i = \{\boldsymbol{\tau}_r : 0 \le r\le n\}$. $\square$

Lemma 5.2.

$$\boldsymbol{P}^{m+n}(F)\boldsymbol{\tau}_r(\boldsymbol{G}^m(F)\times\boldsymbol{G}^n(F)) = \bigsqcup_{\boldsymbol{\xi},\boldsymbol{\beta},\boldsymbol{\gamma}} \boldsymbol{P}^{m+n}(F)\boldsymbol{\tau}_r((\boldsymbol{\xi}\times 1_H 1_{2m-2r})\boldsymbol{\beta}\times\boldsymbol{\gamma}),$$

where $\boldsymbol{\xi}$ runs over $Sym_l(F)\backslash\boldsymbol{G}^r(F)$, $\boldsymbol{\beta}$ over $\boldsymbol{P}^{m,r}(F)\backslash\boldsymbol{G}^m(F)$, and $\boldsymbol{\gamma}$ over $\boldsymbol{P}^{n,r}(F)\backslash\boldsymbol{G}^n(F)$.

Proof. By previous lemma and Lemma 4.3 from [22],

$$\begin{aligned}
&\boldsymbol{P}^{m+n}(F)\boldsymbol{\tau}_r\iota_A(\boldsymbol{G}^m(F)\times\boldsymbol{G}^n(F))\\
&\qquad= \bigsqcup_{\xi,\beta,\gamma} H_0^{m+n,l}(F)H^{m+n,l}(F)P^{m+n}(F)\tau_r\iota_S(\iota_S(\xi\times 1_{2m-2r})\beta\times\gamma)),
\end{aligned}$$

where ξ,β,γ run over $G^r(F), P^{m,r}(F)\backslash G^m(F), P^{n,r}(F)\backslash G^n(F)$ respectively. Note that

$$H_0^{m+n,l}(F)H^{m+n,l}(F) = \bigcup_{\substack{\lambda\in M_{l,m}(F)\\ \lambda'\in M_{l,n}(F)}} H_0^{m+n,l}(F)((\lambda,0,0)1_{2m}\times(\lambda',0,0)1_{2n}),$$

and for $g = \left(\begin{smallmatrix} A & B\\ & D\end{smallmatrix}\right)\in P^{m+n}(F)$,

$$((\lambda,0,0)1_{2m}\times(\lambda',0,0)1_{2n})1_H g \in H_0^{m+n,l}(F)P^{m+n}(F)((\lambda\,\lambda')A,0,0)1_{2(m+n)}.$$

Indeed, if we view it as an element of G^{l+m+n}, we obtain

$$\begin{aligned}
&\begin{pmatrix} 1_l & \lambda & \lambda' & & & \\ & 1_m & & & & \\ & & 1_n & & & \\ & & & 1_l & & \\ & & & -{}^t\lambda & 1_m & \\ & & & -{}^t\lambda' & & 1_n \end{pmatrix}\begin{pmatrix} 1_l & & & \\ & A & & B\\ & & 1_l & \\ & & & D\end{pmatrix}\\
&\quad= \begin{pmatrix} 1_l & & & \\ & A & & B\\ & & 1_l & \\ & & & D\end{pmatrix}\begin{pmatrix} 1_l & & \kappa & (\lambda\,\lambda')B\\ & 1_{m+n} & {}^tB\,{}^t(\lambda\,\lambda') & \\ & & 1_l & \\ & & & 1_{m+n}\end{pmatrix}\begin{pmatrix} 1_l & (\lambda\,\lambda')A & & \\ & 1_{m+n} & & \\ & & 1_l & \\ & & -{}^tA\,{}^t(\lambda\,\lambda') & 1_{m+n}\end{pmatrix}\\
&\quad= \begin{pmatrix} 1_l & & \kappa & (\lambda\,\lambda')B\,{}^tA\\ & 1_{m+n} & A\,{}^tB\,{}^t(\lambda\,\lambda') & \\ & & 1_l & \\ & & & 1_{m+n}\end{pmatrix}\begin{pmatrix} 1_l & & & \\ & A & & B\\ & & 1_l & \\ & & & D\end{pmatrix}\begin{pmatrix} 1_l & (\lambda\,\lambda')A & & \\ & 1_{m+n} & & \\ & & 1_l & \\ & & -{}^tA\,{}^t(\lambda\,\lambda') & 1_{m+n}\end{pmatrix},
\end{aligned}$$

where $\kappa = (\lambda\,\lambda')B\,{}^tA\,{}^t(\lambda\,\lambda')$. Moreover, because $\boldsymbol{\tau}_r$ commutes with $((\lambda,0,0)1_{2m}\times(\lambda',0,0)1_{2n})$, we have

$$\begin{aligned}
\boldsymbol{P}^{m+n}(F)\boldsymbol{\tau}_r\iota_A(\boldsymbol{G}^m(F)\times\boldsymbol{G}^n(F)) = \bigsqcup_{\xi,\beta,\gamma}\ \bigcup_{\lambda\in M_{l,m}(F)\lambda'\in M_{l,n}(F)} & H_0^{m+n,l}(F)P^{m+n}\boldsymbol{\tau}_r\\
&\iota_A((\lambda,0,0)1_{2m}\times(\lambda',0,0)1_{2n})\iota_S(\iota_S(\xi\times 1_{2m-2r})\beta\times\gamma).
\end{aligned}$$

Write $\lambda = (\lambda_1\,\lambda_2)$ and $\lambda' = (\lambda_1'\,\lambda_2')$ as concatenation of matrices $\lambda_1\in M_{l,r}(F), \lambda_2\in M_{l,m-r}(F), \lambda_1'\in M_{l,r}(F), \lambda_2'\in M_{l,n-r}(F)$. Because $H_0^{m+n,l}(F)$ and $P^{m+n}(F)$ commute (as follows from the above computation) and

$$H_0^{m+n,l}(F)\boldsymbol{\tau}_r = \boldsymbol{\tau}_r\{(\mu'\,{}^te_r, \mu, \kappa)1_{2m} \times (\mu e_r, \mu', \kappa')1_{2n} : \\ \mu \in M_{l,n}(F), \mu' \in M_{l,n}(F), \kappa, \kappa' \in Sym_l(F)\},$$

we can include $(0, (\lambda'_1\, 0), 0)1_{2m} \times ((\lambda'_1\, 0), 0, 0)1_{2n}$ in the set above for each λ', and so we are left with

$$(\lambda, (-\lambda'_1\, 0), 0)\iota_S(\xi \times 1_{2m-2r})\beta \times ((0\, \lambda'_2), 0, 0)\gamma.$$

In fact,

$$(\lambda, (-\lambda'_1\, 0), 0)\iota_S(\xi \times 1_{2m-2r})\beta = ((\lambda_1, -\lambda'_1, 0)\xi \times 1_H 1_{2m-2r})((0\, \lambda_2), 0, 0)\beta.$$

Therefore we can exchange the representatives

$$\boldsymbol{\tau}_r\iota_A(\iota_A((\lambda_1, -\lambda'_1, 0)\xi \times 1_H 1_{2m-2r})((0\, \lambda_2), 0, 0)\beta \times ((0\, \lambda'_2), 0, 0)\gamma)$$

with $\boldsymbol{\tau}_r\iota_A((\iota_A(\boldsymbol{\xi} \times 1_H 1_{2m-2r})\boldsymbol{\beta} \times \boldsymbol{\gamma})$, where $\boldsymbol{\xi}, \boldsymbol{\beta}, \boldsymbol{\gamma}$ are as in the hypothesis. Reversing the process described above, it is easy to see that the cosets are distinct. □

We are now ready to prove the main result of this section. The following lemma is the generalization of [22, Lemma 4.4].

Lemma 5.3. *Let $\mathfrak{e}, \mathfrak{b}, \mathfrak{c}$ be as in Section 4.1, and $\boldsymbol{\sigma}$ an element of $\boldsymbol{G}^{m+n}_{\boldsymbol{h}}$ given by*

$$\boldsymbol{\sigma}_v := \begin{cases} 1_H \mathrm{diag}[1_m, \theta_v^{-1}1_n, 1_m, \theta_v 1_n] & \text{if } v \nmid \mathfrak{e}, \\ 1_H \mathrm{diag}[1_m, \theta_v^{-1}1_n, 1_m, \theta_v 1_n]\boldsymbol{\tau}_n & \text{if } v | \mathfrak{e}, \end{cases}$$

where θ is an element of $F^\times_{\boldsymbol{h}}$ such that $\theta\mathfrak{o} = \mathfrak{b}$. Let $\boldsymbol{D}^{m+n} := K^{m+n}[\mathfrak{b}, \mathfrak{c}] \subset \boldsymbol{G}^{m+n}(\mathbb{A})$. Assume that $n \le m$. Then

$$\boldsymbol{P}^{m+n}(F)\boldsymbol{\tau}_n(\boldsymbol{G}^m(F) \times \boldsymbol{G}^n(F)) \cap (\boldsymbol{P}^{m+n}(\mathbb{A})\boldsymbol{D}^{m+n}\boldsymbol{\sigma}) \\ = \bigsqcup_{\boldsymbol{\xi} \in \boldsymbol{X}, \boldsymbol{\beta} \in \boldsymbol{B}} \boldsymbol{P}^{m+n}(F)\boldsymbol{\tau}_n((1_H\iota_S(\boldsymbol{\xi} \times 1_{2m'})\boldsymbol{\beta} \times 1_H 1_{2n}),$$

where $m' = m - n$, $\boldsymbol{B}$ is a subset of $\boldsymbol{G}^m(F) \cap Y$ as in Lemma 4.2, which represents $\boldsymbol{P}^{m,n}(F)\backslash(\boldsymbol{G}^m(F) \cap \boldsymbol{P}^{m,n}(\mathbb{A})\boldsymbol{D}^m)$, and $\boldsymbol{X} = \boldsymbol{G}^n(F) \cap \boldsymbol{G}^n_{\mathbf{a}} \prod_{v \in \mathbf{h}} \boldsymbol{X}_v$ with

$$\boldsymbol{X}_v = \begin{cases} \{(\lambda, \mu, \kappa)x \in C_v[\mathfrak{o}, \mathfrak{b}^{-1}, \mathfrak{b}^{-1}]D^n_v[\mathfrak{b}^{-1}\mathfrak{c}, \mathfrak{b}\mathfrak{c}] : a_x - 1 \in M_{n,n}(\mathfrak{e}_v)\} & \text{if } v | \mathfrak{e}, \\ C_v[\mathfrak{o}, \mathfrak{b}^{-1}, \mathfrak{b}^{-1}]D^n_v[\mathfrak{b}^{-1}\mathfrak{c}, \mathfrak{b}]W_v C_v[\mathfrak{o}, \mathfrak{b}^{-1}, \mathfrak{b}^{-1}]D^n_v[\mathfrak{b}^{-1}, \mathfrak{b}\mathfrak{c}] & \text{if } v | \mathfrak{e}^{-1}\mathfrak{c}, \\ C_v[\mathfrak{o}, \mathfrak{b}^{-1}, \mathfrak{b}^{-1}]G^n(F_v)C_v[\mathfrak{o}, \mathfrak{b}^{-1}, \mathfrak{b}^{-1}] & \text{if } v \nmid \mathfrak{c}, \end{cases}$$

$$W_v = \{\mathrm{diag}[q, \tilde{q}] : q \in \mathrm{GL}_n(F_v) \cap M_{n,n}(\mathfrak{c}_v)\};$$

if $m = n$, we take $\boldsymbol{B} = \{1_H 1_{2m}\}$.

Remark 5.4. Before we proceed to the proof of the lemma we should stress a significant difference between this result and the symplectic case. In [22, Lemma 4.4], at the places v which do not divide $\mathfrak{c}$, one obtains that the set X_v (with the notation there) is the entire symplectic group $G^n(F_v) = \mathrm{Sp}_n(F_v)$. However, this is not the case here as the set $\boldsymbol{X}_v$ above is not equal to the group $\boldsymbol{G}^n(F_v)$. This is one of the main differences between the Jacobi and the symplectic group regarding their Hecke theory at the "good places". It will become even more apparent later in this paper when we will consider the theory of Hecke operators.

Proof of Lemma 5.3. We will divide the proof into two parts: the case where v does not divide $\mathfrak{c}$ (a good place) and when it does (a bad place). We first consider the case of v being good.

We first obtain a description of the set $C_v[\mathfrak{o},\mathfrak{b}^{-1},\mathfrak{b}^{-1}]G^n(F_v)C_v[\mathfrak{o},\mathfrak{b}^{-1},\mathfrak{b}^{-1}]$. First note that a set of representatives for $G^n(F_v)/D_v[\mathfrak{b}^{-1},\mathfrak{b}]$ consists of

$$m(g,h,\sigma) := \begin{pmatrix} g^{-1}h & g^{-1}\sigma{}^t h^{-1} \\ 0 & {}^t g {}^t h^{-1} \end{pmatrix}$$

where $(g,h) \in \mathrm{GL}_n(\mathfrak{o}_v)\backslash W/(\mathrm{GL}_n(\mathfrak{o}_v)\times 1_n)$, $\sigma \in Sym_n(F_v)/gSym_n(\mathfrak{b}_v^{-1}){}^t g$ and $W = \{(g,h) \in B\times B : gL + hL = L\}$, where $L = M_{n,1}(\mathfrak{o}_v)$, and $B = GL_n(F_v)\cap M_n(\mathfrak{o}_v)$. In particular, if we write $\boldsymbol{D}_v^{m+n} = C_v D_v$, then

$$\begin{aligned} C_v G^n(F_v) C_v &= \bigcup_{g,h,\sigma} C_v m(g,h,\sigma) D_v C_v = \bigcup_{g,h,\sigma} C_v m(g,h,\sigma) C_v D_v \\ &= \bigcup_{\substack{g,h,\sigma\\ \lambda,\mu}} C_v(\lambda h^{-1} g, -\lambda h^{-1}\sigma {}^t g^{-1} + \mu {}^t h {}^t g^{-1}, *) m(g,h,\sigma) D_v. \end{aligned} \tag{12}$$

Consider now the set $\boldsymbol{P}^{m+n}(F_v)\boldsymbol{D}_v^{m+n}$ and write $\boldsymbol{P}^{m+n}(F_v) = H^0(F_v)P^{m+n}(F_v)$. Since

$$\begin{pmatrix} a_p & b_p \\ 0 & d_p \end{pmatrix} (\lambda,\mu,*) = (\lambda a_p^{-1}, \lambda a_p^{-1} b_p d_p^{-1} + \mu d_p^{-1}, *) \begin{pmatrix} a_p & b_p \\ 0 & d_p \end{pmatrix},$$

we can conclude that

$$\boldsymbol{P}^{m+n}(F_v)\boldsymbol{D}_v^{m+n} = \{(\lambda,\mu,\kappa)g : \lambda \in M_{l,n+m}(\mathfrak{o}_v)a_p^{-1}, \mu \in M_{l,n+m}(F_v), g = pk \in \mathrm{Sp}_{n+m}(F_v)\}.$$

Note that this is well defined. Indeed, if we write $g = p_1k_1 = p_2k_2$ then $p_1^{-1}p_2 \in D_v$ and in particular $a_{p_1}^{-1}a_{p_2} \in M_{n+m}(\mathfrak{o}_v)\cap \mathrm{GL}_{n+m}(F_v)$, and similarly $a_{p_2}^{-1}a_{p_1} \in M_{n+m}(\mathfrak{o}_v)\cap \mathrm{GL}_{n+m}(F_v)$; that is, $a_{p_1}^{-1}a_{p_2} \in \mathrm{GL}_{n+m}(\mathfrak{o}_v)$.

Consider now $\boldsymbol{\alpha} = \iota_A(\boldsymbol{\xi}\times 1_H 1_{2m'})\boldsymbol{\beta}$ with $\boldsymbol{\xi} \in Sym_l(F)\backslash \boldsymbol{G}^n(F)$, $\boldsymbol{\beta} \in \boldsymbol{P}^{m,n}(F)\backslash \boldsymbol{G}^m(F)$, and write $\boldsymbol{\xi} = (\lambda_1,\mu_1,0)\xi$, $\boldsymbol{\beta} = ((0\,\lambda_2),0,0)\beta$, where $\lambda_2 \in M_{r,m-n}(F)$. Then

$$\begin{aligned} \boldsymbol{\alpha} &= \iota_A((\lambda_1,\mu_1,0)\xi\times 1_H 1_{2m'})((0\,\lambda_2),0,0)\beta = ((\lambda_1\,0),(\mu_1\,0),0)(\xi\times 1_{2m'})((0\,\lambda_2),0,0)\beta \\ &= ((\lambda_1\,0),(\mu_1\,0),0)((0\,\lambda_2),0,0)(\xi\times 1_{2m'})\beta = ((\lambda_1\,\lambda_2),(\mu_1\,0),0)(\xi\times 1_{2m'})\beta, \end{aligned}$$

and so

$$\iota_A(\boldsymbol{\alpha}\times 1_H 1_{2n}) = ((\lambda_1\,\lambda_2\,0),(\mu_1\,0\,0),0)((\xi\times 1_{2m'})\beta\times 1_{2n}).$$

Now we see that

$$\begin{aligned} \boldsymbol{\tau}_n\iota_A(\boldsymbol{\alpha}\times 1_H 1_{2n})\boldsymbol{\sigma}^{-1} &= ((\lambda_1\,\lambda_2\,(-\mu_1)),(\mu_1\,0\,0),0)\tau_n((\xi\times 1_{2m'})\beta\times 1_{2n})\boldsymbol{\sigma}^{-1} \\ &= ((\lambda_1\,\lambda_2\,(-\mu_1)),(\mu_1\,0\,0),0)\tau_n((\xi\times 1_{2m'})\beta\times 1_{2n})\sigma^{-1}. \end{aligned}$$

Put $g := \tau_n((\xi\times 1_{m-n})\beta\times 1_{2n})\sigma^{-1}$ and write $g = pk \in P^{m+n}D^{m+n}$. Then by [22, Lemma 4.4] we may take β to be of the form hw, where $h = \mathrm{diag}[1_{m-1},t^{-1},1_{m-1},t]$ and w is in the congruence subgroup D^m. Moreover, we may take

$$\xi = \begin{pmatrix} g^{-1}h & g^{-1}\sigma{}^t h^{-1} \\ 0 & {}^t g {}^t h^{-1} \end{pmatrix} d,$$

where g, h, σ are in the sets as above, and $d \in D^n$. In particular,

$$(\xi \times 1_{2m'})\beta \times 1_{2n} = \begin{pmatrix} A & 0 & B & 0 \\ 0 & 1_n & 0 & 0 \\ 0 & 0 & D & 0 \\ 0 & 0 & 0 & 1_n \end{pmatrix} d_1,$$

where d_1 is some element in D^{n+m},

$$A := \begin{pmatrix} g^{-1}h & 0 \\ 0 & \tilde{h} \end{pmatrix}, \ B := \begin{pmatrix} g^{-1}\sigma {}^th^{-1} & 0 \\ 0 & 0 \end{pmatrix}, \ D := \begin{pmatrix} {}^tg{}^th^{-1} & 0 \\ 0 & \tilde{h}^{-1} \end{pmatrix}$$

and $\tilde{h} = \mathrm{diag}[1_{m-n-1}, t]$. In this way we obtain

$$\tau_n((\xi \times 1_{2m'})\beta \times 1_{2n})\sigma^{-1} = \begin{pmatrix} A & 0 & B & 0 \\ 0 & \theta_v 1_n & 0 & 0 \\ 0 & \theta_v e_n & D & 0 \\ {}^te_n A & 0 & {}^te_n B & \theta_v^{-1} 1_n \end{pmatrix} d'$$

for some d' in the congruence subgroup D^n. Furthermore, if we write

$$\begin{pmatrix} A & 0 & B & 0 \\ 0 & \theta_v 1_n & 0 & 0 \\ 0 & \theta_v e_n & D & 0 \\ {}^te_n A & 0 & {}^te_n B & \theta_v^{-1} 1_n \end{pmatrix} = pk$$

for some $p \in P^{n+m}(F_v)$ and $k = \begin{pmatrix} k_1 & k_2 \\ k_3 & k_4 \end{pmatrix} \in D_v^{n+m}[\mathfrak{b}^{-1}, \mathfrak{b}\mathfrak{c}]$, then we can conclude that

$${}^ta_p^{-1} k_3 = \begin{pmatrix} 0 & \theta_v e_n \\ {}^te_n A & 0 \end{pmatrix} \quad \text{and} \quad {}^ta_p^{-1} k_4 = \begin{pmatrix} D & 0 \\ {}^te_n B & \theta_v^{-1} 1_n \end{pmatrix}.$$

Since the matrix $[k_3 \ k_4]$ extends to an element in the congruence subgroup $D_v^{n+m}[\mathfrak{b}^{-1}, \mathfrak{b}\mathfrak{c}]$, it follows that

$$\theta_v^{-1} k_3 \Lambda + k_4 \Lambda = \Lambda,$$

where now $\Lambda = M_{n+m,l}(\mathfrak{o})$. That is, for any given $\ell \in \Lambda$ there exist $\ell_1, \ell_2 \in \Lambda$ such that $\theta_v^{-1} k_3 \ell_1 + k_4 \ell_2 = \ell$. Write $\Lambda = {}^t[\Lambda_1, \Lambda_2, \Lambda_3]$ with $\Lambda_1, \Lambda_3 \in M_{l,n}$ and $\Lambda_2 \in M_{l,m-n}$. Then the relation ${}^ta_p^{-1} \theta_v^{-1} k_3 \Lambda + {}^ta_p^{-1} k_4 \Lambda = {}^ta_p^{-1} \Lambda$, which can be also written as

$$\begin{pmatrix} 0 & e_n \\ \theta_v^{-1} {}^te_n A & 0 \end{pmatrix} \Lambda + \begin{pmatrix} D & 0 \\ {}^te_n B & \theta_v^{-1} 1_n \end{pmatrix} \Lambda = {}^ta_p^{-1} \Lambda,$$

means that the set ${}^ta_p^{-1} \Lambda$ can be described as

$$\begin{pmatrix} 0 & e_n \\ \theta_v^{-1} {}^te_n A & 0 \end{pmatrix} {}^t[\ell_1, \ell_2, \ell_3] + \begin{pmatrix} D & 0 \\ {}^te_n B & \theta_v^{-1} 1_n \end{pmatrix} {}^t[\ell_1', \ell_2', \ell_3'],$$

where $\ell_1, \ell_1' \in \Lambda_1$, $\ell_3, \ell_3' \in \Lambda_3$, $\ell_2, \ell_2' \in \Lambda_2$ and, recall, $e_n = \binom{1_n}{0} \in M_{m,n}$. Therefore, since ${}^te_n A = \begin{pmatrix} g^{-1}h & 0 \end{pmatrix}$ and ${}^te_n B = \begin{pmatrix} g^{-1}\sigma {}^th^{-1} & 0 \end{pmatrix}$, we get

$$\begin{pmatrix} 0 & e_n \\ \theta_v^{-1} {}^te_n A & 0 \end{pmatrix} {}^t[\ell_1, \ell_2, \ell_3] = \begin{pmatrix} {}^t\ell_3 \\ 0 \\ \theta_v^{-1} g^{-1} h {}^t\ell_1 \end{pmatrix}$$

and

$$\begin{pmatrix} D & 0 \\ {}^te_nB & \theta_v^{-1}1_n \end{pmatrix} {}^t[\ell_1', \ell_2', \ell_3'] = \begin{pmatrix} {}^tg{}^th^{-1}{}^t\ell_1' \\ \tilde{h}{}^t\ell_2' \\ g^{-1}\sigma{}^th^{-1}{}^t\ell'_1 + \theta_v^{-1}{}^t\ell'_3 \end{pmatrix}.$$

Hence,

$${}^ta_p^{-1}\Lambda = \begin{pmatrix} {}^t\ell_3 + {}^tg{}^th^{-1}{}^t\ell_1' \\ \tilde{h}{}^t\ell_2' \\ g^{-1}h\theta_v^{-1}{}^t\ell_1 + g^{-1}\sigma{}^th^{-1}{}^t\ell_1' + \theta_v^{-1}{}^t\ell_3' \end{pmatrix},$$

and after taking a transposition

$${}^t\Lambda a_p^{-1} = \begin{pmatrix} \ell_3 + \ell_1'h^{-1}g & \ell_2'{}^t\tilde{h} & \theta_v^{-1}\ell_1{}^th{}^tg^{-1} + \ell_1'h^{-1}\sigma{}^tg^{-1} + \theta_v^{-1}\ell_3' \end{pmatrix}.$$

In particular, we see that the element

$$\boldsymbol{\tau}_n\iota_A(\boldsymbol{\alpha} \times 1_H 1_{2n})\boldsymbol{\sigma}^{-1} = ((\lambda_1\, \lambda_2\, (-\mu_1)), (\mu_1\, 0\, 0), 0)\tau_n((\xi \times 1_{m-n})\beta \times 1_{2n})\sigma^{-1}$$

belongs to $\mathbf{P}^{n+m}(F_v)\mathbf{D}_v^{m+n}$ if and only if λ_1 is of the form $\ell_3 + \ell_1'h^{-1}g$, and μ_1 is of the form $-(\theta_v^{-1}\ell_1{}^th{}^tg^{-1} + \ell_1'h^{-1}\sigma{}^tg^{-1} + \theta_v^{-1}\ell_3')$. This together with (12) concludes the proof of the lemma in the case of good places.

Now assume that v is a place in the support of $\mathfrak{c}$. First we consider the case when $v|\mathfrak{e}^{-1}\mathfrak{c}$. As above, we start with a description of the set

$$C_v[\mathfrak{o}, \mathfrak{b}^{-1}, \mathfrak{b}^{-1}]D_v^n[\mathfrak{b}^{-1}\mathfrak{c}, \mathfrak{b}]W_vC_v[\mathfrak{o}, \mathfrak{b}^{-1}, \mathfrak{b}^{-1}]D_v^n[\mathfrak{b}^{-1}, \mathfrak{b}\mathfrak{c}],$$

where $W_v = \{\mathrm{diag}[q, \tilde{q}] : q \in \mathrm{GL}_n(F_v) \cap M_{n,n}(\mathfrak{c}_v)\}$. As it was shown in [22, page 567],

$$D_v^n[\mathfrak{b}^{-1}\mathfrak{c}, \mathfrak{b}]\mathrm{diag}[q, \tilde{q}]D_v^n[\mathfrak{b}^{-1}, \mathfrak{b}\mathfrak{c}] = \bigcup_{f,g} \begin{pmatrix} f & g\tilde{f} \\ 0 & \tilde{f} \end{pmatrix} D_v^n[\mathfrak{b}^{-1}, \mathfrak{b}\mathfrak{c}],$$

where $f \in \mathrm{GL}_n(\mathfrak{o}_v)\backslash\mathrm{GL}_n(\mathfrak{o}_v)q\mathrm{GL}_n(\mathfrak{o}_v)$ and $g \in Sym_n(\mathfrak{b}_v^{-1}\mathfrak{c}_v)/{}^tfSym_n(\mathfrak{b}_v^{-1})f$. Set $C_v := C_v[\mathfrak{o}, \mathfrak{b}^{-1}, \mathfrak{b}^{-1}]$. Then:

$$\begin{aligned} C_vD_v^n[\mathfrak{b}^{-1}\mathfrak{c}, \mathfrak{b}]W_vC_vD_v^n[\mathfrak{b}^{-1}, \mathfrak{b}\mathfrak{c}] &= C_vD_v^n[\mathfrak{b}^{-1}\mathfrak{c}, \mathfrak{b}]W_vD_v^n[\mathfrak{b}^{-1}, \mathfrak{b}\mathfrak{c}]C_v \\ &= \bigcup_q \bigcup_{f_q,g_q} C_v \begin{pmatrix} f_q & g_q\tilde{f}_q \\ 0 & \tilde{f}_q \end{pmatrix} D_v^n[\mathfrak{b}^{-1}, \mathfrak{b}\mathfrak{c}]C_v = \bigcup_q \bigcup_{f_q,g_q} C_v \begin{pmatrix} f_q & g_q\tilde{f}_q \\ 0 & \tilde{f}_q \end{pmatrix} C_vD_v^n[\mathfrak{b}^{-1}, \mathfrak{b}\mathfrak{c}] \\ &= \bigcup_q \bigcup_{f_q,g_q,\lambda,\mu} C_v(\lambda f_q^{-1}, -\lambda f_q^{-1}g_q + \mu{}^tf_q, *] \begin{pmatrix} f_q & g_q\tilde{f}_q \\ 0 & \tilde{f}_q \end{pmatrix} D_v^n[\mathfrak{b}^{-1}, \mathfrak{b}\mathfrak{c}], \end{aligned} \tag{13}$$

where $f_q \in \mathrm{GL}_n(\mathfrak{o}_v)\backslash\mathrm{GL}_n(\mathfrak{o}_v)q\mathrm{GL}_n(\mathfrak{o}_v)$ and $g_q \in Sym_n(\mathfrak{b}_v^{-1}\mathfrak{c}_v)/{}^tf_qSym_n(\mathfrak{b}_v^{-1})f_q$.

Further we argue as in the case of good places. In particular, we may write as before

$$\boldsymbol{\tau}_n\iota_A(\boldsymbol{\alpha} \times 1_H 1_{2n})\boldsymbol{\sigma}^{-1} = ((\lambda_1\, \lambda_2\, (-\mu_1)), (\mu_1\, 0\, 0), 0)\tau_n((\xi \times 1_{m-n})\beta \times 1_{2n})\sigma^{-1}$$

with $\boldsymbol{\xi} = (\lambda_1, \mu_1, 0)\xi \in Sym_l(F)\backslash\boldsymbol{G}^n(F)$, $\boldsymbol{\beta} = ((0\, \lambda_2), 0, 0)\beta \in \boldsymbol{P}^{m,n}(F)\backslash\boldsymbol{G}^{m,l}(F)$. Moreover, using [22, Lemma 4.4] again, we may take $\xi = \begin{pmatrix} f_q & g_q\tilde{f}_q \\ 0 & \tilde{f}_q \end{pmatrix} d$ for some $q \in M_n(\mathfrak{c}_v)\cap\mathrm{GL}_n(F_v)$, $f_q \in \mathrm{GL}_n(\mathfrak{o}_v)\backslash\mathrm{GL}_n(\mathfrak{o}_v)q\mathrm{GL}_n(\mathfrak{o}_v)$, $g_q \in Sym_n(\mathfrak{b}_v^{-1}\mathfrak{c}_v)/{}^tf_qSym_n(\mathfrak{b}_v^{-1})f_q$ and $d \in D_v[\mathfrak{b}^{-1}, \mathfrak{b}\mathfrak{c}]$. Then we obtain

$$\tau_n((\xi \times 1_{2m'})\beta \times 1_{2n})\sigma^{-1} = \begin{pmatrix} A & 0 & B & 0 \\ 0 & \theta_v 1_n & 0 & 0 \\ 0 & \theta_v e_n & D & 0 \\ {}^t e_n A & 0 & {}^t e_n B & \theta_v^{-1} 1_n \end{pmatrix} d'$$

for some $d' \in D_v^{m+n}$, where this time

$$A := \begin{pmatrix} f_q & 0 \\ 0 & \tilde{f}_q \end{pmatrix}, \; B := \begin{pmatrix} g_q {}^t f_q^{-1} & 0 \\ 0 & 0 \end{pmatrix}, \; D := \begin{pmatrix} {}^t f_q^{-1} & 0 \\ 0 & \tilde{h}^{-1} \end{pmatrix}$$

As before, write $\begin{pmatrix} A & 0 & B & 0 \\ 0 & \theta_v 1_n & 0 & 0 \\ 0 & \theta_v e_n & D & 0 \\ {}^t e_n A & 0 & {}^t e_n B & \theta_v^{-1} 1_n \end{pmatrix}$ as a product of an element in P^{m+n} and D^{m+n}. Then, after the same computations and with notation as above, we obtain

$${}^t\Lambda a_p^{-1} = \begin{pmatrix} \ell_3 + \ell_1' f_q^{-1} & \ell_2' {}^t\tilde{h} & \theta_v^{-1} \ell_1 {}^t f_q + \ell_1' f_q^{-1} g_q + \theta_v^{-1} \ell_3' \end{pmatrix}$$

In particular, we see that the element

$$\boldsymbol{\tau}_n \iota_A(\boldsymbol{\alpha} \times 1_H 1_{2n})\boldsymbol{\sigma}^{-1} = ((\lambda_1 \, \lambda_2 \, (-\mu_1)), (\mu_1 \, 0 \, 0), 0)\tau_n((\xi \times 1_{m-n})\beta \times 1_{2n})\sigma^{-1}$$

belongs to $\mathbf{P}^{n+m}(F_v)\mathbf{D}_v^{m+n}$ if and only if λ_1 is of the form $\ell_3 + \ell_1' f_q^{-1}$, and μ_1 is of the form $-(\theta_v^{-1}\ell_1 {}^t f_q + \ell_1' f_q^{-1} g_q + \theta_v^{-1} \ell_3')$. This requirement matches the decomposition (13), and thus finishes the proof of the second case.

Finally, we consider the case of $v|\mathfrak{e}$. In this situation we also argue as before, but note that now

$$\tau_n((\xi \times 1_{2m'})\beta \times 1_{2n})\sigma^{-1} = \begin{pmatrix} A & 0 & B & 0 \\ 0 & \theta_v 1_n & 0 & 0 \\ 0 & \theta_v e_n & D & 0 \\ {}^t e_n A & 0 & {}^t e_n B & \theta_v^{-1} 1_n \end{pmatrix} d',$$

where

$$d' \in D_v^{m+n}, \; A := \begin{pmatrix} 1_n & 0 \\ 0 & 1_n \end{pmatrix}, \; B := \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \; D := \begin{pmatrix} 1_n & 0 \\ 0 & \tilde{h}^{-1} \end{pmatrix}.$$

Hence, doing exactly the same computations as before, we see that the element

$$\boldsymbol{\tau}_n \iota_A(\boldsymbol{\alpha} \times 1_H 1_{2n})\boldsymbol{\sigma}^{-1} = ((\lambda_1 \, \lambda_2 \, (-\mu_1)), (\mu_1 \, 0 \, 0), 0)\tau_n((\xi \times 1_{m-n})\beta \times 1_{2n})\sigma^{-1}$$

belongs to $\mathbf{P}^{n+m}(F_v)\mathbf{D}_v^{m+n}$ if and only if λ_1 is of the form $\ell_3 + \ell_1'$, and μ_1 is of the form $-(\theta_v^{-1}\ell_1 + \ell_1' + \theta_v^{-1}\ell_3')$, which gives the set we claimed in the lemma. $\square$

6. Diagonal Restriction of Eisenstein Series

The map $\boldsymbol{G}^{m,l} \times \boldsymbol{G}^{n,l} \to \boldsymbol{G}^{m+n,l}$ introduced in the previous section induces an embedding

$$\mathcal{H}_{m,l} \times \mathcal{H}_{n,l} \hookrightarrow \mathcal{H}_{n+m,l}, \;\; z_1 \times z_2 \mapsto \mathrm{diag}[z_1, z_2],$$

defined by

$$(\tau_1, w_1) \times (\tau_2, w_2) \mapsto (\mathrm{diag}[\tau_1, \tau_2], (w_1 \, w_2)).$$

The aim of this section is to obtain the main identity (21), that is, to compute the Petersson inner product of a cuspidal Siegel-Jacobi modular form against a pull-backed

Siegel-type Eisenstein series. This identity should be seen as a generalization of the identity [22, equation (4.11)] from the Siegel to the Jacobi setting.

6.1. **The factor of automorphy.** We start with a study of the behavior of the factor of automorphy under diagonal restriction. First we compute $J_{k,S}(\boldsymbol{\tau}_r, z)$ for $0 \leq r \leq n$.

Lemma 6.1. *Let $z = \mathrm{diag}[z_1, z_2]$ be as above, and $\boldsymbol{\tau}_r$ as in the previous section. Then*

$$J_{k,S}(\boldsymbol{\tau}_r, z) = \mathbf{e}_{\mathbf{a}}(-\mathrm{tr}\,(S[\boldsymbol{\omega}_r(w_2)\omega_r(\tau_2)^{-1} - \boldsymbol{\omega}_r(w_1)](\omega_r(\tau_2)^{-1} - \omega_r(\tau_1))^{-1}))$$
$$\cdot J_{k,S}(\boldsymbol{\eta}_r, \boldsymbol{\omega}_r(z_2))) \det(\omega_r(\tau_1) - \omega_r(\tau_2)^{-1})^k,$$

where, recall, we write $\boldsymbol{\omega}_r(z_i) = \boldsymbol{\omega}_r(\tau_i, w_i) = (\omega_r(\tau_i), \boldsymbol{\omega}_r(w_i))$ for $i = 1, 2$.

Proof. Similar calculations have been done in [3, page 191]; a difference in the formulae comes from a difference between $\boldsymbol{\tau}_r$ and $t^*_{m,n,r}(D)$. First we find that

$$\lambda(\tau_r, \begin{pmatrix}\tau_1 & \\ & \tau_2\end{pmatrix})^{-1}\begin{pmatrix} & e_r \\ {}^t e_r & \end{pmatrix} = \begin{pmatrix} -\omega_r(\tau_2)(1_r - \omega_r(\tau_1)\omega_r(\tau_2))^{-1} & 0 & (1_r - \omega_r(\tau_2)\omega_r(\tau_1))^{-1} & 0 \\ 0 & 0 & 0 & 0 \\ (1_r - \omega_r(\tau_1)\omega_r(\tau_2))^{-1} & 0 & -\omega_r(\tau_1)(1_r - \omega_r(\tau_2)\omega_r(\tau_1))^{-1} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Then we compute the trace, so that

$$J_{k,S}(\boldsymbol{\tau}_r, z) = \mathbf{e}_{\mathbf{a}}(-\mathrm{tr}\,(S[\boldsymbol{\omega}_r(w_2)\omega_r(\tau_2)^{-1} - \boldsymbol{\omega}_r(w_1)](\omega_r(\tau_2)^{-1} - \omega_r(\tau_1))^{-1})))$$
$$\cdot\, \mathbf{e}_{\mathbf{a}}(\mathrm{tr}\,(S[\boldsymbol{\omega}_r(w_2)\omega_r(\tau_2)^{-1}]\omega_r(\tau_2)))j(\tau_r, \mathrm{diag}[\tau_1, \tau_2])^k.$$

But $j(\tau_r, \mathrm{diag}[\tau_1, \tau_2]) = \det(1_r - \omega_r(\tau_1)\omega_r(\tau_2)) = \det(\omega_r(\tau_1) + \boldsymbol{\eta}_r\omega_r(\tau_2))\det(-\omega_r(\tau_2))$, where $\boldsymbol{\eta}_r = 1_H\begin{pmatrix} & -1_r \\ 1_r & \end{pmatrix}$, and so the second factor is equal to

$$J_{k,S}(\boldsymbol{\eta}_r, (\omega_r(\tau_2), \boldsymbol{\omega}_r(w_2)))\det(\omega_r(\tau_1) - \omega_r(\tau_2)^{-1})^k.$$

□

Now, with the notation of Lemma 5.3, we compute $J_{k,S}(\boldsymbol{\tau_r}((\boldsymbol{\xi}\times 1_{2m-2r})\boldsymbol{\beta}\times\boldsymbol{\gamma}), \mathrm{diag}[z_1\, z_2])$.

Lemma 6.2. *With notation as above,*

$$J_{k,S}(\boldsymbol{\tau_r}((\boldsymbol{\xi} \times 1_{2m-2r})\boldsymbol{\beta} \times \boldsymbol{\gamma}), \mathrm{diag}[z_1\, z_2]) = \tag{14}$$
$$= J_{k,S}(\boldsymbol{\xi}, \boldsymbol{\omega}_r(\boldsymbol{\beta}z_1))J_{k,S}(\boldsymbol{\beta}, z_1)J_{k,S}(\boldsymbol{\gamma}, z_2)J_{k,S}(\boldsymbol{\eta}_r, \boldsymbol{\omega}_r(\boldsymbol{\gamma}z_2))\det(\omega_r(\tau'_1) - \omega_r(\tau'_2)^{-1})^k$$
$$\cdot\, \mathbf{e}_{\mathbf{a}}(-\mathrm{tr}\,(S[\omega_r(w'_2)\omega_r(\tau'_2)^{-1} - \omega_r(w'_1)](\omega_r(\tau'_2)^{-1} - \omega_r(\tau'_1))^{-1}))$$

and

$$\delta(\boldsymbol{\tau_r}((\boldsymbol{\xi} \times 1_{2m-2r})\boldsymbol{\beta} \times \boldsymbol{\gamma}), \mathrm{diag}[z_1\, z_2]) = \delta(\tau_r((\xi \times 1_{2m-2r})\beta \times \gamma), \mathrm{diag}[\tau_1\, \tau_2]) \tag{15}$$
$$= \delta(\beta\tau_1)\delta(\gamma\tau_2)|j(\xi, \omega_r(\beta\tau_1))j(\eta_r, \omega_r(\gamma\tau_2))\det(\xi\omega_r(\beta\tau_1) - \omega_r(\gamma\tau_2)^{-1})|^{-2}.$$

Proof. By the cocycle relation,

$$J_{k,S}(\boldsymbol{\tau_r}((\boldsymbol{\xi} \times 1_{2m-2r})\boldsymbol{\beta} \times \boldsymbol{\gamma}), \mathrm{diag}[z_1\, z_2])$$
$$J_{k,S}(\boldsymbol{\tau_r}, ((\boldsymbol{\xi} \times 1_{2m-2r})\boldsymbol{\beta} \times \boldsymbol{\gamma}) \cdot \mathrm{diag}[z_1\, z_2]) \cdot J_{k,S}((\boldsymbol{\xi} \times 1_{2m-2r})\boldsymbol{\beta} \times \boldsymbol{\gamma}), \mathrm{diag}[z_1\, z_2]).$$

Note that

$$((\boldsymbol{\xi} \times 1_{2m-2r})\boldsymbol{\beta} \times \boldsymbol{\gamma}) \cdot \mathrm{diag}[z_1\, z_2] = \mathrm{diag}[(\boldsymbol{\xi} \times 1_{2m-2r})\boldsymbol{\beta}z_1, \boldsymbol{\gamma}z_2],$$

and thus we find that

$$J_{k,S}((\boldsymbol{\xi} \times 1_{2m-2r})\boldsymbol{\beta} \times \boldsymbol{\gamma}), \mathrm{diag}[z_1\, z_2]) \\ = J_{k,S}((\boldsymbol{\xi} \times 1_{2m-2r}), \boldsymbol{\beta} z_1) J_{k,S}(\boldsymbol{\beta}, z_1) J_{k,S}(\boldsymbol{\gamma}, z_2).$$

Since $\boldsymbol{\xi} \times 1_{2m-2r} \in \mathbf{P}^{m,r}$,

$$J_{k,S}((\boldsymbol{\xi}\times 1_{2m-2r}), \boldsymbol{\beta} z_1) \overset{(8)}{=} (\lambda^m_{r,l}(\boldsymbol{\xi}\times 1_{2m-2r}))^k J_{k,S}(\boldsymbol{\pi}_r(\boldsymbol{\xi}\times 1_{2m-2r}), \boldsymbol{\omega}_r(\boldsymbol{\beta} z_1)) = J_{k,S}(\boldsymbol{\xi}, \boldsymbol{\omega}_r(\boldsymbol{\beta} z_1)).$$

Moreover, by Lemma 6.1,

$$J_{k,S}(\boldsymbol{\tau_r}, \mathrm{diag}[(\boldsymbol{\xi} \times 1_{2m-2r})\boldsymbol{\beta} z_1, \boldsymbol{\gamma} z_2]) = \\ = \mathbf{e_a}(-\mathrm{tr}\,(S[\boldsymbol{\omega}_r(w_2')\omega_r(\tau_2')^{-1} - \boldsymbol{\omega}_r(w_1')](\omega_r(\tau_2')^{-1} - \omega_r(\tau_1'))^{-1})) \\ \cdot J_{k,S}(\boldsymbol{\eta}_r, (\omega_r(\tau_2'), \boldsymbol{\omega}_r(w_2'))) \det(\omega_r(\tau_1') - \omega_r(\tau_2')^{-1})^k,$$

where we have set $(\boldsymbol{\xi} \times 1_{2m-2r})\boldsymbol{\beta} z_1 = (\tau_1', w_1')$ and $\boldsymbol{\gamma} z_2 = (\tau_2', w_2')$. Putting everything together gives the equality (14).

The second formula follows from the identity

$$\delta(g\tau) = \delta(\tau)|j(g,\tau)|^{-2} \qquad \text{for } g \in G^n,\ \tau \in \mathbb{H}_n.$$

□

6.2. **Decomposing the Eisenstein series I; the non-full rank part.** Thanks to the strong approximation (Lemma 3.2) we can pick an element $\boldsymbol{\rho} = 1_H\rho \in \boldsymbol{G}^{m+n}(F) \cap K^{m+n}[\mathfrak{b},\mathfrak{c}]\boldsymbol{\sigma}$ such that $a_{\sigma_v\rho_v^{-1}} - 1 \in M_{m+n,m+n}(\mathfrak{c})_v$ for all $v|\mathfrak{c}$. If we now write $\boldsymbol{\rho} = \boldsymbol{w}\boldsymbol{\sigma}$ with $\boldsymbol{w} \in K^{m+n}[\mathfrak{b},\mathfrak{c}]$, then for $y \in \boldsymbol{G}_\mathbf{a}$ such that $y\mathbf{i}_0 = z$,

$$E(y\boldsymbol{\sigma}^{-1}) = E(\boldsymbol{\rho}^{-1}\boldsymbol{w}y) = E(\boldsymbol{w}y) = E(\boldsymbol{w}_\mathbf{h}\boldsymbol{w}_\mathbf{a}y) = \chi(\det(d_{\boldsymbol{w}_\mathbf{h}}))^{-1}E(\boldsymbol{w}_\mathbf{a}y) \\ = \chi(\det(d_{\boldsymbol{w}_\mathbf{h}}))^{-1}(E|_{k,S}\boldsymbol{w}_\mathbf{a}y)(\boldsymbol{i}_0).$$

But since $\boldsymbol{\sigma}_\mathbf{a}$ is trivial, $\boldsymbol{w}_\mathbf{a} = \boldsymbol{\rho}_\mathbf{a}$ and, by the condition on ρ, $\chi(\det(d_{\boldsymbol{w}_\mathbf{h}})) = \chi(\det(d_{\boldsymbol{\sigma}_\mathbf{h}})^{-1}$. In particular, we see that the adelic Eisenstein series $E(x\boldsymbol{\sigma}^{-1}, s)$ corresponds to the classical series $(E|_{k,S}\boldsymbol{\rho})(z,s)$.

Let $y, \boldsymbol{\rho}$ be as above and put

$$\varepsilon_r(z,s) := \sum_{\alpha\in A_r} p_\alpha(z), \qquad p_\alpha(z) := \phi(\alpha y\boldsymbol{\sigma}^{-1}, s)J_{k,S}(y, \boldsymbol{i}_0),$$

where $A_r := \boldsymbol{P}^{m+n}(F)\backslash\boldsymbol{P}^{m+n}(F)\boldsymbol{\tau}_r\iota_A(\boldsymbol{G}^m(F) \times \boldsymbol{G}^n(F))$. Then

$$(E|_{k,S}\boldsymbol{\rho})(z,s) = \sum_{0\le r\le n} \varepsilon_r(z,s),$$

and for a fixed r each $\alpha \in A_r$ is of the form $\alpha(\boldsymbol{\xi},\boldsymbol{\beta},\boldsymbol{\gamma}) := \boldsymbol{\tau}_r((\boldsymbol{\xi} \times 1_H 1_{2(m-r)})\boldsymbol{\beta} \times \boldsymbol{\gamma})$ for some $\boldsymbol{\xi},\boldsymbol{\beta},\boldsymbol{\gamma}$ as in Lemma 5.2.

The following Lemma is a straightforward generalization of Lemma 2.2 in [22] to the Jacobi case; we omit the proof.

Lemma 6.3. *Let f be a cuspidal Siegel-Jacobi form on $\mathcal{H}_{n,l}$ of weight $k \in \mathbb{Z}^{\mathbf{a}}$ and $g(z)$ a function on $\mathcal{H}_{n,l}$ depending only on $\boldsymbol{\omega}_r(z)$ and $\mathrm{Im}(z) := (\mathrm{Im}(\tau), \mathrm{Im}(w))$ for some $r \in \mathbb{N}$ with $0 \leq r < n$. If for a congruence subgroup $\mathbf{\Gamma}$ we have $g|_{k,S}\gamma = g$ for every $\gamma \in \boldsymbol{P}^{n,r}(F) \cap \tau\mathbf{\Gamma}\tau^{-1}$ with $\tau \in \boldsymbol{G}^{n,l}(F)$, then*

$$< \sum_{\gamma \in R} g|_{k,S}\gamma, f >= 0$$

for any set R of representatives for $\boldsymbol{P}^{n,r}(F) \cap \tau\mathbf{\Gamma}\tau^{-1} \setminus \tau\Gamma$.

Proposition 6.4. *Let $n \leq m$, $z_1 \in \mathcal{H}_{m,l}$ and $z_2 \in \mathcal{H}_{n,l}$. For a cusp form f on $\mathcal{H}_{n,l}$ of weight k, $0 \leq r < n$ and for s large enough, we have*

$$< \varepsilon_r(\mathrm{diag}[z_1, z_2], s), f(z_2) >= 0.$$

Proof. Let $z = \mathrm{diag}[z_1, z_2] \in \mathcal{H}_{m+n,l}$ and fix $r \in \{0, 1, \ldots, n-1\}$. Put

$$\boldsymbol{D}' := \{x \in K^{m+n}[\mathfrak{b}, \mathfrak{c}] : \det(d_x)_v - 1 \in \mathfrak{c}_v \text{ for every } v|\mathfrak{c}\}.$$

Let $\mathbf{\Gamma}$ be a congruence subgroup of $\boldsymbol{G}^n(F)$ such that $\iota_A(1_H 1_{2m} \times \mathbf{\Gamma}) \subset \boldsymbol{\sigma}^{-1}\boldsymbol{D}'\boldsymbol{\sigma}$. By the definition of ϕ, for any $d' \in K^{m+n}[\mathfrak{b}, \mathfrak{c}]$

$$\phi(xd', s) = \chi_{\mathfrak{c}}(\det(d_{d'}))^{-1} J_{k,S}(d', \boldsymbol{i}_0)^{-1}\phi(x, s),$$

and thus $p_\alpha|_k\alpha' = p_{\alpha\alpha'}$ for $\alpha' \in \boldsymbol{G}^{m+n}(F) \cap \boldsymbol{\sigma}^{-1}\boldsymbol{D}'\boldsymbol{\sigma}$. Further, write $\boldsymbol{G}^n(F) = \bigsqcup_{\boldsymbol{\tau} \in T} \boldsymbol{P}^{n,r}(F)\boldsymbol{\tau}\mathbf{\Gamma}$, so that

$$\varepsilon_r = \sum_{\boldsymbol{\xi},\boldsymbol{\beta},\boldsymbol{\gamma}} p_{\alpha(\boldsymbol{\xi},\boldsymbol{\beta},\boldsymbol{\gamma})} = \sum_{\boldsymbol{\xi},\boldsymbol{\beta}} \sum_{\boldsymbol{\tau} \in T} \sum_{\gamma \in R_{\boldsymbol{\tau}}} p_{\alpha(\boldsymbol{\xi},\boldsymbol{\beta},\boldsymbol{\tau})}|_k\iota_A(1_H 1_{2m} \times \boldsymbol{\tau}^{-1})|_k\iota_A(1_H 1_{2m} \times \gamma),$$

where $R_{\boldsymbol{\tau}} := (\boldsymbol{P}^{n,r}(F) \cap \boldsymbol{\tau}\mathbf{\Gamma}\boldsymbol{\tau}^{-1})\backslash\boldsymbol{\tau}\mathbf{\Gamma}$. We will check that for each $\boldsymbol{\tau} \in T$,

$$g_{\boldsymbol{\tau}} := \sum_{\boldsymbol{\xi},\boldsymbol{\beta}} p_{\alpha(\boldsymbol{\xi},\boldsymbol{\beta},\boldsymbol{\tau})}|_k\iota_A(1_H 1_{2m} \times \boldsymbol{\tau}^{-1})$$

satisfies the conditions of Lemma 6.3.

Fix $\boldsymbol{\tau} \in T$ and take $\boldsymbol{\eta} \in \boldsymbol{P}^{n,r}(F) \cap \boldsymbol{\tau}\mathbf{\Gamma}\boldsymbol{\tau}^{-1}$. We will show that

$$\sum_{\boldsymbol{\xi},\boldsymbol{\beta}} p_{\alpha(\boldsymbol{\xi},\boldsymbol{\beta},\boldsymbol{\tau})}|_k\iota_A(1_H 1_{2m} \times \boldsymbol{\tau}^{-1}\boldsymbol{\eta}\boldsymbol{\tau}) = \sum_{\boldsymbol{\xi},\boldsymbol{\beta}} p_{\alpha(\boldsymbol{\xi},\boldsymbol{\beta},\boldsymbol{\tau})}, \tag{16}$$

which in turn immediately implies

$$\sum_{\boldsymbol{\xi},\boldsymbol{\beta}} p_{\alpha(\boldsymbol{\xi},\boldsymbol{\beta},\boldsymbol{\tau})}|_k\iota_A(1_H 1_{2m} \times \boldsymbol{\tau}^{-1}\boldsymbol{\eta}) = \sum_{\boldsymbol{\xi},\boldsymbol{\beta}} p_{\alpha(\boldsymbol{\xi},\boldsymbol{\beta},\boldsymbol{\tau})}|_k\iota_A(1_H 1_{2m} \times \boldsymbol{\tau}^{-1}).$$

First of all, because $\boldsymbol{\tau}^{-1}\boldsymbol{\eta}\boldsymbol{\tau} \in \mathbf{\Gamma}$, $p_{\alpha(\boldsymbol{\xi},\boldsymbol{\beta},\boldsymbol{\tau})}|_k\iota_A(1_H 1_{2m} \times \boldsymbol{\tau}^{-1}\boldsymbol{\eta}\boldsymbol{\tau}) = p_{\alpha(\boldsymbol{\xi},\boldsymbol{\beta},\boldsymbol{\eta}\boldsymbol{\tau})}$, where

$$\alpha(\boldsymbol{\xi}, \boldsymbol{\beta}, \boldsymbol{\eta}\boldsymbol{\tau}) = \boldsymbol{\tau}_r((\boldsymbol{\xi} \times 1_H 1_{2(m-r)})\boldsymbol{\beta} \times \boldsymbol{\eta}\boldsymbol{\tau}) = \boldsymbol{\tau}_r(1_H 1_{2m} \times \boldsymbol{\eta})((\boldsymbol{\xi} \times 1_H 1_{2(m-r)})\boldsymbol{\beta} \times \boldsymbol{\tau}).$$

Because p_α depends only on $\boldsymbol{P}^{m+n}(F)\alpha$, in order to prove (16) it suffices to show that there exists $\boldsymbol{\zeta} \in \boldsymbol{G}^r(F)$ such that

$$\alpha(\boldsymbol{\xi}, \boldsymbol{\beta}, \boldsymbol{\eta}\boldsymbol{\tau}) \in \boldsymbol{P}^{m+n}(F)\alpha(\boldsymbol{\zeta}\boldsymbol{\xi}, \boldsymbol{\beta}, \boldsymbol{\tau}). \tag{17}$$

Write $\boldsymbol{\eta} = ((\lambda_1'\, 0), \mu', \kappa')\eta$. By the same calculation as in the proof of Lemma 5.2,

$$\boldsymbol{\tau}_r(1_H 1_{2m} \times \boldsymbol{\eta}) \in \boldsymbol{P}^{m+n}(F)\boldsymbol{\tau}_r((-\mu'\, {}^t e_r, (-\lambda_1'\, 0), 0)1_{2m} \times 1_H\eta)$$

$$= \boldsymbol{P}^{m+n}(F)\boldsymbol{\tau}_r(1_H 1_{2m} \times 1_H \eta)((-\mu'\,{}^t e_r, (-\lambda'_1\, 0), 0)1_{2m} \times 1_H 1_{2n}).$$

On the other hand, by [22, Lemma 4.3], there is $\zeta \in G^r(F)$ such that $\tau_r \iota_S(1_{2m} \times \eta) \in P^{m+n}(F)\tau_r \iota_S(\iota_S(\zeta \times 1_{2(m-r)}) \times 1_{2n})$. Hence, (17) holds for $\boldsymbol{\zeta} = \zeta(-\mu'\binom{1_r}{0}, -\lambda'_1, 0)$. This proves (16), and thus also an invariance property for $g_{\boldsymbol{\tau}}$.

It remains to show that $g_{\boldsymbol{\tau}}(\mathrm{diag}[z_1, z_2], s)$ depends only on $s, z_1, \mathrm{Im}(z_2)$ and $\boldsymbol{\omega}_r(z_2)$. Observe that whenever $\alpha y \boldsymbol{\sigma}^{-1} = pw$ for some $p \in \boldsymbol{P}^{n,0}(\mathbb{A}), w \in K^{n,0}$, then

$$\begin{aligned}\phi(\alpha y \boldsymbol{\sigma}^{-1}, s) J_{k,S}(y, \boldsymbol{i}_0) &= \chi(\det d_p)^{-1}\chi_{\mathfrak{c}}(\det(d_w)_{\mathfrak{c}})^{-1} J_{k,S}(w, \boldsymbol{i}_0)^{-1} |\det d_p|_{\mathbb{A}}^{-2s} J_{k,S}(y, \boldsymbol{i}_0) \\ &= \mu(\alpha_{\mathbf{h}}\boldsymbol{\sigma}^{-1})\chi_{\mathbf{a}}(\det(d_p)_{\mathbf{a}})^{-1} J_{k,S}(p, \boldsymbol{i}_0) J_{k,S}(\alpha, z)^{-1} |\det d_p|_{\mathbb{A}}^{-2s},\end{aligned}$$

where we put $\mu(\alpha_{\mathbf{h}}\boldsymbol{\sigma}^{-1}) := \chi_{\mathbf{h}}(\det(d_p)_{\mathbf{h}})^{-1}\chi_{\mathfrak{c}}(\det(d_w)_{\mathfrak{c}})^{-1}$. Moreover, because

$$J_{k,S}(p, \boldsymbol{i}_0) = \chi_{\mathbf{a}}(\det(d_p)_{\mathbf{a}})|\det d_p|_{\mathbf{a}}^{k} \quad \text{and} \quad |\det d_p|_{\mathbb{A}}^{-2s} = \delta(\alpha_{\mathbf{a}} z)^s N(\mathfrak{a}_0(\alpha\boldsymbol{\sigma}^{-1}))^{2s},$$

we get

$$\begin{aligned}(18) \quad (E|_{k,S}\boldsymbol{\rho})(z, s) &= \sum_{0 \le r \le n} \sum_{\alpha \in A_r} \phi(\alpha y \boldsymbol{\sigma}^{-1}, s) J_{k,S}(y, \boldsymbol{i}_0) \\ &= \sum_r \sum_\alpha N(\mathfrak{a}_0(\alpha\boldsymbol{\sigma}^{-1}))^{2s} \mu(\alpha_{\mathbf{h}}\boldsymbol{\sigma}^{-1}) J_{k,S}(\alpha_{\mathbf{a}}, \mathrm{diag}[z_1, z_2])^{-1} \delta(\alpha_{\mathbf{a}} \mathrm{diag}[z_1, z_2])^{s-k/2}.\end{aligned}$$

From this and the formulas (14), (15) we see that $g_{\boldsymbol{\tau}}$ depends only on $s, z_1, \mathrm{Im}(z_2)$ and $\boldsymbol{\omega}_r(z_2)$. This finishes the proof. ☐

6.3. Decomposing the Eisenstein series II; the full rank part.

Lemma 6.5 (Reproducing Kernel). *Let f be a holomorphic function on $\mathcal{H}_{n,l}$ of weight $k \in \mathbb{Z}^{\mathbf{a}}$ such that $\Delta_{S,k}(z) f(z)^2$ is bounded. Then for $s \in \mathbb{C}^{\mathbf{a}}$ satisfying $\mathrm{Re}(s_\nu) \geq 0$, $\mathrm{Re}(s_\nu) + k_\nu - l/2 > 2n$ for each $\nu \in \mathbf{a}$, and for $(\zeta, \rho) \in \mathcal{H}_{n,l}$ we have*

$$\tilde{c}_{S,k}(s) \det(\mathrm{Im}(\zeta))^{-s} f(\zeta, \rho) =$$

$$\int_{\mathcal{H}_{n,l}} f(\tau, w) \overline{\mathbf{e}_{\mathbf{a}}(-\mathrm{tr}\,(S[w - \bar{\rho}](\tau - \bar{\zeta})^{-1})) \det(\tau - \bar{\zeta})^{-k}} |\det(\tau - \bar{\zeta})|^{-2s} \det(\mathrm{Im}(\tau))^s \Delta_{S,k}(z) d(\tau, w),$$

where

$$\tilde{c}_{S,k}(s) = \prod_{\nu \in \mathbf{a}} \det(2S_\nu)^{-n} (-1)^{n(l+k_\nu/2)} 2^{n(n+3)/2 - 4s_\nu - nk_\nu} \pi^{n(n+1)/2} \frac{\Gamma_n(s_\nu + k_\nu - \frac{l}{2} - \frac{n+1}{2})}{\Gamma_n(s_\nu + k_\nu - \frac{l}{2})}$$

and $\Gamma_n(s) := \pi^{n(n-1)/4} \prod_{i=0}^{n-1} \Gamma(s - \frac{i}{2})$.

Proof. We remark that a very similar integral was computed in the proof of [3, Lemma 2.8]. The main difference in the formula comes from a choice of parametrization for w.

The proof bases on the identity:

$$\begin{aligned}&\int_{\mathbb{R}^{l \times n}} \exp(a \mathrm{tr}\,(-S[X]A + RXA)) dX \\ &\qquad = (\det A)^{-l/2} \left(\frac{\pi}{a}\right)^{nl/2} (\det S)^{-n/2} \exp\left(\frac{a}{4} \mathrm{tr}\,(S^{-1}[{}^t R]A)\right),\end{aligned}$$

where $S \in Sym_l(\mathbb{R})$ is a symmetric positive definite matrix, $X \in M_{l,n}(\mathbb{R}), A \in Sym_n(\mathbb{C})$ and $a \in \mathbb{C}^\times$.

For $f(\tau,w)=\sum_{T,R}c(T,R)\mathbf{e_a}(\mathrm{tr}\,(T\tau+Rw))$, we obtain

$$\begin{aligned}\int_{\mathcal{H}_{n,l}} & f(\tau,w)\overline{\mathbf{e_a}(-tr(S[w-\bar\rho](\tau-\bar\zeta)^{-1}))\det(\tau-\bar\zeta)^{-k}}|\det(\tau-\bar\zeta)|^{-2s}\det(\mathrm{Im}(\tau))^s\\ &\cdot\Delta_{S,k}(z)d(\tau,w)\\ &=2^{-nl/2}\det(2S)^{-n}\sum_R\mathbf{e_a}(\mathrm{tr}\,\left(R\rho+\frac{1}{4}S^{-1}[{}^tR]\zeta\right))\\ &\cdot\int_{\mathbb{H}_n^{\mathbf{a}}}\det(\zeta-\bar\tau)^{l/2-k}(-1)^{n(k+l/2+l/4)}|\det(\zeta-\bar\tau)|^{-2s}\det(\mathrm{Im}(\tau))^{s+k-l/2}\\ &\cdot\mathbf{e_a}\left(-\frac{1}{4}\mathrm{tr}\,(S^{-1}[{}^tR]\tau)\right)\sum_T c(T,R)\mathbf{e_a}(\mathrm{tr}\,(T\tau))\det(\mathrm{Im}(\tau))^l d\tau.\end{aligned}$$

By the "classical" reproducing kernel formula for holomorphic functions on the Siegel upper half space as stated for example in [22, Lemma 4.7], the last integral equals

$$\frac{\tilde c_{S,k}(s)}{2^{-nl/2}\det(2S)^{-n}}\mathbf{e_a}\left(-\frac{1}{4}\mathrm{tr}\,(S^{-1}[{}^tR]\zeta)\right)\det(\mathrm{Im}(\zeta))^{-s}\sum_T c(T,R)\mathbf{e_a}(\mathrm{tr}\,(T\zeta)),$$

where $\tilde c_{S,k}(s)$ is as in the hypothesis. This concludes the proof. $\square$

In order to proceed further we introduce the following notation, taken from [22, equation (4.5)]. We have that $G^n(\mathbb{A})=D^n[\mathfrak{b}^{-1},\mathfrak{b}]WD^n[\mathfrak{b}^{-1},\mathfrak{b}]$ with

$$W=\left\{\mathrm{diag}[q,\tilde q]:\ q\in GL_n(\mathbb{A}_{\mathbf{h}})\cap\prod_{v\in\mathbf{h}}GL_n(\mathfrak{o}_v)\right\},$$

that is, any element $x\in G^n(\mathbb{A})$ may be written as $x=\gamma_1\mathrm{diag}[q,\tilde q]\gamma_2$ with $\gamma_1,\gamma_2\in D^n[\mathfrak{b}^{-1},\mathfrak{b}]$ and $q\in W$. We define $\ell_0(x)$ to be the ideal associated to $\det(q)$, $\ell_1(x):=\prod_{v\nmid\mathfrak{c}}\ell_0(x)_v$ and set $\ell(x)$ for the norm of the ideal $\ell_0(x)$. With this notation we have,

Lemma 6.6. *For $z_1\in\mathcal{H}_{m,l}$ and $z_2\in\mathcal{H}_{n,l}$,*

$$\begin{aligned}\varepsilon_n(\mathrm{diag}[z_1,z_2],s)=&\sum_{\boldsymbol\beta\in\mathbf{B}}\sum_{\boldsymbol\xi\in\boldsymbol{X}}N(\mathfrak{b})^{-2ns}N(\mathfrak{a}_0(\beta))^{2s}\ell(\xi)^{-2s}\chi_{\mathbf{h}}(\theta^n)\chi[\beta]\chi^*(\ell_1(\xi))\chi_{\mathfrak{c}}(\det(d_\xi))^{-1}\\ &\cdot J_{k,S}(\boldsymbol\xi,\boldsymbol\omega_n(\boldsymbol\beta z_1))^{-1}J_{k,S}(\boldsymbol\beta,z_1)^{-1}J_{k,S}(\boldsymbol\eta_n,z_2)^{-1}\det(\omega_n(\tau_1')-\tau_2^{-1})^{-k}\\ &\cdot\mathbf{e_a}(\mathrm{tr}\,(S[w_2\tau_2^{-1}-\boldsymbol\omega_n(w_1')](\tau_2^{-1}-\omega_n(\tau_1'))^{-1}))(\delta(\beta\tau_1)\delta(\tau_2))^{s-k/2}\\ &\cdot|j(\xi,\omega_n(\beta\tau_1))j(\eta_n,\tau_2)\det(\omega_n(\tau_1')-\tau_2^{-1})|^{-2s+k},\end{aligned}$$

where we have set $(\boldsymbol\xi\times 1_{2m-2n})\boldsymbol\beta z_1=(\tau_1',w_1')$.

Proof. The statement follows from the explicit computation of the factors occurring in the formula (18). Recall that we have already computed the values of the automorphy factor and δ in (14), (15). Therefore it suffices to find $\mathfrak{a}_0(\alpha\boldsymbol\sigma^{-1})$ and $\mu(\alpha_{\mathbf{h}}\boldsymbol\sigma^{-1})$ for $\alpha=\boldsymbol\tau_n\iota_A(\iota_A(\boldsymbol\xi\times 1_H1_{2(m-n)})\boldsymbol\beta\times\boldsymbol\gamma)$ with $\boldsymbol\xi\in\boldsymbol{X},\boldsymbol\beta\in\boldsymbol{B}$ as in Lemma 5.3. Observe though that neither $\mathfrak{a}_0$ nor μ depends on the elements from Heisenberg group. Moreover, because for any symplectic matrix g we have $gH=Hg$, the symplectic factors of the representatives given in Lemma 5.3 are exactly the same as the representatives provided in [22, Lemma 4.4]. Hence, it is clear that the formulas for $\mathfrak{a}_0$ and μ have to be the same as the ones computed in [22, Lemma 4.6]. That is:

$$\mathfrak{a}_0(\alpha\boldsymbol{\sigma}^{-1}) = \mathfrak{b}^{-n}\mathfrak{a}_0(\beta)\ell_0(\xi)^{-1}, \quad \mu(\alpha_{\mathbf{h}}\boldsymbol{\sigma}^{-1}) = \chi_{\mathbf{h}}(\theta^n)\chi[\beta]\chi^*(\ell_1(\xi))\chi_{\mathfrak{c}}(\det(d_\xi))^{-1}.$$

□

We now consider an $f \in S_k(\mathbf{\Gamma}, \chi^{-1})$ where $\mathbf{\Gamma} := \boldsymbol{G}^n \cap \boldsymbol{D}$ with

$$\boldsymbol{D} := \{(\lambda, \mu, \kappa)x \in C[\mathfrak{o}, \mathfrak{b}^{-1}, \mathfrak{b}^{-1}]D[\mathfrak{b}^{-1}\mathfrak{c}, \mathfrak{b}\mathfrak{c}] : (a_x - 1_n)_v \in M_{n,n}(\mathfrak{c}_v) \text{ for every } v|\mathfrak{c}\}.$$

We set $\nu_{\mathfrak{c}} = 2$ if $\mathfrak{c}|2$, and 1 otherwise. Then by using the standard unfolding trick regarding the z_2 variable and setting $A := \mathbf{\Gamma} \setminus \mathcal{H}_{n,l}$, we obtain

$$\begin{aligned}
&<\varepsilon_n(\operatorname{diag}[z_1, z_2], s), f(z_2)> \\
&\quad = \nu_{\mathfrak{c}} vol(A)^{-1} \sum_{\boldsymbol{\beta}\in\mathbf{B}} \sum_{\boldsymbol{\xi}\in\boldsymbol{X}} N(\mathfrak{b})^{-2ns} N(\mathfrak{a}_0(\beta))^{2s} \ell(\xi)^{-2s} \chi_{\mathbf{h}}(\theta^n)\chi[\beta]\chi^*(\ell_1(\xi))\chi_{\mathfrak{c}}(\det(d_\xi))^{-1} \\
&\quad \cdot J_{k,S}(\boldsymbol{\xi}, \boldsymbol{\omega}_n(\boldsymbol{\beta} z_1))^{-1} J_{k,S}(\boldsymbol{\beta}, z_1)^{-1} \delta(\beta\tau_1)^{s-k/2} |j(\xi, \omega_n(\beta\tau_1))|^{-2s+k} \\
&\quad \int_{\mathcal{H}_{n,l}} J_{k,S}(\boldsymbol{\eta}_n, z_2)^{-1} \det(\omega_n(\tau_1') - \tau_2^{-1})^{-k} \mathbf{e}_{\mathbf{a}}(\operatorname{tr}(S[w_2\tau_2^{-1} - \boldsymbol{\omega}_n(w_1')](\tau_2^{-1} - \omega_n(\tau_1'))^{-1})) \\
&\quad \cdot \delta(\tau_2)^{s-k/2} |j(\eta_n, \tau_2)\det(\omega_n(\tau_1') - \tau_2^{-1})|^{-2s+k} \overline{f(z_2)} \Delta_{S,k}(\tau_2, w_2) d(\tau_2, w_2).
\end{aligned}$$

It is easy to show that the integral on the right of the above formula is equal to

$$\begin{aligned}
&\int_{\mathcal{H}_{n,l}} \overline{f|_{k,S}\boldsymbol{\eta}_n(z_2)} \det(\tau_2 + \omega_n(\tau_1'))^{-k} \mathbf{e}_{\mathbf{a}}(-\operatorname{tr}(S[w_2 + \boldsymbol{\omega}_n(w_1')](\tau_2 + \omega_n(\tau_1'))^{-1})) \\
&\qquad \cdot (-1)^{n(s+k/2)} \delta(\tau_2)^{s-k/2} |\det(\tau_2 + \omega_n(\tau_1'))|^{-2(s-k/2)} \Delta_{S,k}(\tau_2, w_2) d(\tau_2, w_2),
\end{aligned}$$

and by Lemma 6.5, this further equals

$$(-1)^{n(s+k/2)} \overline{\tilde{c}_{S,k}(\bar{s} - k/2)\delta(\xi\omega_n(\beta\tau_1))^{-\bar{s}+k/2} f|_{k,S}\boldsymbol{\eta}_n(-\overline{\boldsymbol{\xi}\boldsymbol{\omega}_n(\boldsymbol{\beta} z_1)})}. \tag{19}$$

Put $\delta_{n,k} := \prod_{v\in\mathbf{a}} \delta_{v,n,k}$, where $\delta_{v,n,k}$ is equal to 1 if nk_v even and -1 otherwise, and let $c_{S,k}(s) := \delta_{n,k}\tilde{c}_{S,k}(s)$. Then, because $\overline{\Gamma(\bar{s})} = \Gamma(s)$, the quantity (19) equals

$$(-1)^{n(s+k/2)} c_{S,k}(s - k/2)\delta(\xi\omega_n(\beta\tau_1))^{-s+k/2} \overline{f|_{k,S}\boldsymbol{\eta}_n(-\overline{\boldsymbol{\xi}\boldsymbol{\omega}_n(\boldsymbol{\beta} z_1)})}.$$

Hence, if we set $f^c(z) := \overline{f(-\bar{z})}$, where $-\bar{z} := (-\bar{\tau}, -\bar{w})$ for $z = (\tau, w)$, then

$$\begin{aligned}
&N(\mathfrak{b})^{2ns}\chi_{\mathbf{h}}(\theta)^{-n}(-1)^{n(s-k/2)} c_{S,k}(s - k/2)^{-1} vol(A) < \varepsilon_n(\operatorname{diag}[z_1, z_2], s), f(z_2) > \\
&\quad = \nu_{\mathfrak{c}} \sum_{\boldsymbol{\beta}\in\mathbf{B}} \sum_{\boldsymbol{\xi}\in\boldsymbol{X}} N(\mathfrak{a}_0(\beta))^{2s} \ell(\xi)^{-2s} \chi[\beta]\chi^*(\ell_1(\xi))\chi_{\mathfrak{c}}(\det(d_\xi))^{-1} J_{k,S}(\boldsymbol{\beta}, z_1)^{-1} \\
&\qquad J_{k,S}(\boldsymbol{\xi}, \boldsymbol{\omega}_n(\boldsymbol{\beta} z_1))^{-1} \delta(\beta\tau_1)^{s-k/2} |j(\xi, \omega_n(\beta\tau_1))|^{-2s+k} \delta(\xi\omega_n(\beta\tau_1))^{-s+k/2} \\
&\qquad ((f|_{k,S}\boldsymbol{\eta}_n)^c|_{k,S}\boldsymbol{\xi})(\boldsymbol{\omega}_n(\boldsymbol{\beta} z_1)) J_{k,S}(\boldsymbol{\xi}, \boldsymbol{\omega}_n(\boldsymbol{\beta} z_1)) \\
&\quad = \sum_{\boldsymbol{\beta}\in\mathbf{B}} N(\mathfrak{a}_0(\beta))^{2s} \chi[\beta] J_{k,S}(\boldsymbol{\beta}, z_1)^{-1} \left(\frac{\delta(\beta\tau_1)}{\delta(\omega_n(\beta\tau_1))}\right)^{s-k/2} \\
&\qquad \sum_{\boldsymbol{\xi}\in\boldsymbol{X}} \ell(\xi)^{-2s} \chi^*(\ell_1(\xi))\chi_{\mathfrak{c}}(\det(d_\xi))^{-1} ((f|_{k,S}\boldsymbol{\eta}_n)^c|_{k,S}\boldsymbol{\xi})(\boldsymbol{\omega}_n(\boldsymbol{\beta} z_1)).
\end{aligned}$$

It is not hard to see that $\boldsymbol{\eta}_n^{-1}\boldsymbol{X} = \boldsymbol{Y}\boldsymbol{\eta}_n^{-1}$, where $\boldsymbol{Y} = \boldsymbol{G}^n(F) \cap \boldsymbol{G}^n_{\mathbf{a}} \prod_{v \in \mathbf{h}} \boldsymbol{Y}_v$ with

$$\boldsymbol{Y}_v = \begin{cases} \{(\lambda,\mu,\kappa)y \in C_v[\mathfrak{b}^{-1},\mathfrak{o},\mathfrak{b}^{-1}]D_v^n[\mathfrak{b}\mathfrak{c},\mathfrak{b}^{-1}\mathfrak{c}] : a_y - 1 \in M_{n,n}(\mathfrak{e}_v)\} & \text{if } v|\mathfrak{e}, \\ C_v[\mathfrak{b}^{-1},\mathfrak{o},\mathfrak{b}^{-1}]D_v^n[\mathfrak{b},\mathfrak{b}^{-1}\mathfrak{c}]Z_vC_v[\mathfrak{b}^{-1},\mathfrak{o},\mathfrak{b}^{-1}]D_v^n[\mathfrak{b}\mathfrak{c},\mathfrak{b}^{-1}] & \text{if } v|\mathfrak{e}^{-1}\mathfrak{c}, \\ C_v[\mathfrak{b}^{-1},\mathfrak{o},\mathfrak{b}^{-1}]G^n(F_v)C_v[\mathfrak{b}^{-1},\mathfrak{o},\mathfrak{b}^{-1}] & \text{if } v \nmid \mathfrak{c}, \end{cases}$$

$$Z_v = \{\mathrm{diag}[\tilde{q},q] : q \in \mathrm{GL}_n(F_v) \cap M_{n,n}(\mathfrak{c}_v)\}.$$

Moreover, it follows from Proposition 7.9 which we prove later that $(f|_{k,S}\boldsymbol{\eta}_n)^c = f^c|_{k,S}\boldsymbol{\eta}_n^{-1}$. Set

$$\mathcal{D}(z,s,g) := \sum_{\boldsymbol{\xi} \in \boldsymbol{Y}} \ell'(\xi)^{-s}\chi^*(\ell_1'(\xi))\chi_{\mathfrak{c}}(\det(a_\xi))^{-1}(g|_{k,S}\boldsymbol{\xi})(z), \tag{20}$$

where $\ell'(\xi) := \ell(\eta_n\xi\eta_n^{-1}), \ell_1'(\xi) := \ell_1'(\eta_n\xi\eta_n^{-1})$. Then, using Proposition 6.4, formula (10) and the fact that $N(\mathfrak{a}(\beta)) = |\lambda_{n,l}^m(\beta)|_F$, we obtain

$$N(\mathfrak{b})^{2ns}\chi_{\mathbf{h}}(\theta)^{-n}(-1)^{n(s-k/2)}c_{S,k}(s-k/2)^{-1}vol(A) < (E|_{k,S}\boldsymbol{\rho})(\mathrm{diag}[z_1,z_2],s), f(z_2) >$$

$$= \nu_{\mathfrak{e}} \sum_{\boldsymbol{\beta} \in \mathbf{B}} N(\mathfrak{a}_0(\beta))^{2s}\chi[\beta]J_{k,S}(\boldsymbol{\beta},z_1)^{-1}\left(\frac{\delta(\beta\tau_1)}{\delta(\omega_n(\beta\tau_1))}\right)^{s-k/2}\mathcal{D}(\boldsymbol{\omega}_n(\boldsymbol{\beta}z_1),2s,f^c)|_{k,S}\boldsymbol{\eta}_n^{-1}. \tag{21}$$

7. Shintani's Hecke Algebras and the standard L-function attached to Siegel-Jacobi modular forms

In this section we define Hecke operators acting on the space of Siegel-Jacobi modular forms. These operators were studied in the higher index case first by Shintani (unpublished), Murase [15, 16] and Murase and Sugano [17]. As we have indicated in the introduction this was done in the case of trivial level, and one of our contributions in this section is to define such operators also for non-trivial level. Furthermore, in this section we introduce the standard Dirichlet series which can be attached to a Hecke eigenform. Our main result here is an Euler product representation for this series, which extends previous results in [17] from index one to higher indices.

We start by fixing some notation. For the usual fractional ideals $\mathfrak{b},\mathfrak{c},\mathfrak{e}$ let

$$\boldsymbol{D} := \{(\lambda,\mu,\kappa)x \in C[\mathfrak{o},\mathfrak{b}^{-1},\mathfrak{b}^{-1}]D[\mathfrak{b}^{-1}\mathfrak{e},\mathfrak{b}\mathfrak{c}] : (a_x - 1_n)_v \in M_n(\mathfrak{e}_v) \text{ for every } v|\mathfrak{e}\},$$

$$\boldsymbol{\Gamma} := \boldsymbol{G}^n(F) \cap \boldsymbol{D},$$

$$Q(\mathfrak{e}) := \{r \in \mathrm{GL}_n(\mathbb{A}_{\mathbf{h}}) \cap \prod_{v \in \mathbf{h}} M_n(\mathfrak{o}_v) : r_v = 1_n \text{ for every } v|\mathfrak{e}\},$$

$$R(\mathfrak{e}) := \{\mathrm{diag}[\tilde{r},r] : r \in Q(\mathfrak{e})\}.$$

For $r \in Q(\mathfrak{e})$ and $f \in M^n_{k,S}(\boldsymbol{\Gamma},\psi)$ we define a linear operator $T_{r,\psi} : M^n_{k,S}(\boldsymbol{\Gamma},\psi) \to M^n_{k,S}(\boldsymbol{\Gamma},\psi)$ by

$$f|T_{r,\psi} := \sum_{\boldsymbol{\alpha} \in \boldsymbol{A}} \psi_{\mathfrak{c}}(\det(a_\alpha)_{\mathfrak{c}})^{-1} f|_{k,S}\boldsymbol{\alpha}, \tag{22}$$

where $\boldsymbol{A} \subset \boldsymbol{G}^n(F)$ is such that $\boldsymbol{G}^n(F) \cap \boldsymbol{D}\mathrm{diag}[\tilde{r}, r]\boldsymbol{D} = \coprod_{\boldsymbol{\alpha} \in \boldsymbol{A}} \boldsymbol{\Gamma}\boldsymbol{\alpha}$. Further, for an integral ideal $\mathfrak{a}$ of F we put

$$f|T_\psi(\mathfrak{a}) := \sum_{\substack{r \in Q(\mathfrak{c}) \\ \det(r)\mathfrak{o} = \mathfrak{a}}} f|T_{r,\psi},$$

where we sum over all those r for which the cosets ErE are distinct, where $E := \prod_{v \in \mathbf{h}} \mathrm{GL}_n(\mathfrak{o}_v)$.

We also note here that if we let $\mathbf{f}|T_{r,\psi}$ be the adelic Siegel-Jacobi form associated to $f|T_{r,\psi}$ by the bijection given in (5) with $\mathbf{g} = 1$, then

$$(\mathbf{f}|T_{r,\psi})(x) = \sum_{\boldsymbol{\alpha} \in \boldsymbol{A}} \psi_{\mathfrak{c}}(\det(a_\alpha)_{\mathfrak{c}})^{-1} \mathbf{f}(x\boldsymbol{\alpha}^{-1}), \qquad x \in \boldsymbol{G}^n(\mathbb{A}),$$

where $\boldsymbol{D}\mathrm{diag}[\tilde{r}, r]\boldsymbol{D} = \coprod_{\boldsymbol{\alpha} \in \boldsymbol{A}} \boldsymbol{D}\boldsymbol{\alpha}$ with $\boldsymbol{A} \subset \boldsymbol{G}_{\mathbf{h}}$. As above we may also define $\mathbf{f}|T_\psi(\mathfrak{a})$.

We now consider a nonzero $\mathbf{f} \in \mathcal{S}^n_{k,S}(\boldsymbol{D}, \psi)$ such that $\mathbf{f}|T_\psi(\mathfrak{a}) = \lambda(\mathfrak{a})\mathbf{f}$ for all integral ideals $\mathfrak{a}$ of F. For a Hecke character χ of F we define the series

$$D(s, \mathbf{f}, \chi) := \sum_{\mathfrak{a}} \lambda(\mathfrak{a})\chi^*(\mathfrak{a})N(\mathfrak{a})^{-s}, \quad \mathrm{Re}(s) \gg 0,$$

where for a Hecke character χ we write χ^* for the corresponding ideal character. Of course, for a prime ideal $\mathfrak{q}$ that divides the conductor $\mathfrak{f}_\chi$ we set $\chi^*(\mathfrak{q}) = 0$. A similar argument to [3, Lemma 2.2] extended to the totally real field case shows that the function $D(s, \mathbf{f}, \chi)$ is absolutely convergent for $\mathrm{Re}(s) > 2n + l + 1$.

We now impose a condition on the matrix S. We follow [15, page 142]. Consider any prime ideal $\mathfrak{p}$ of F such that $(\mathfrak{p}, \mathfrak{c}) = 1$ and write v for the corresponding finite place of F. We say that the lattice $L := \mathfrak{o}_v^l \subset F_v^l$ is an $\mathfrak{o}_v$-maximal lattice with respect to a symmetric matrix $2S$ if for every $\mathfrak{o}_v$ lattice M of F_v^l that contains L and satisfies $S[x] \in \mathfrak{o}_v$ for all $x \in M$, we have $M = L$. For any uniformiser π of F_v we now set

$$L' := \{x \in (2S)^{-1}L : \pi S[x] \in \mathfrak{o}_v\} \subset F_v^l.$$

We say that the matrix S satisfies the condition $M_{\mathfrak{p}}^+$ if L is an $\mathfrak{o}_v$-maximal lattice with respect to the symmetric matrix $2S$ and $L = L'$. The main aim of this section is to prove the following theorem.

Theorem 7.1. *Let $0 \neq \mathbf{f} \in \mathcal{S}^n_{k,S}(\boldsymbol{D}, \psi)$ be such that $\mathbf{f}|T_\psi(\mathfrak{a}) = \lambda(\mathfrak{a})\mathbf{f}$ for all integral ideals $\mathfrak{a}$ of F. Assume that the matrix S satisfies the condition $M_{\mathfrak{p}}^+$ for every prime ideal $\mathfrak{p}$ with $(\mathfrak{p}, \mathfrak{c}) = 1$. Then*

$$\mathfrak{L}(\chi, s)D(s + n + l/2, \mathbf{f}, \chi) = L(s, \mathbf{f}, \chi) := \prod_{\mathfrak{p}} L_{\mathfrak{p}}(\chi^*(\mathfrak{p})N(\mathfrak{p})^{-s})^{-s},$$

where for every prime ideal $\mathfrak{p}$ of F

$$L_{\mathfrak{p}}(X) = \begin{cases} \prod_{i=1}^n \Big((1 - \mu_{\mathfrak{p},i}X)(1 - \mu_{\mathfrak{p},i}^{-1}X)\Big), \ \ \mu_{\mathfrak{p},i} \in \mathbb{C}^\times & \textit{if } (\mathfrak{p}, \mathfrak{c}) = 1, \\ \prod_{i=1}^n (1 - \mu_{\mathfrak{p},i}X) \ \ \mu_{\mathfrak{p},i} \in \mathbb{C} & \textit{if } (\mathfrak{p}, \mathfrak{e}^{-1}\mathfrak{c}) \neq 1 \\ 1 & \textit{if } (\mathfrak{p}, \mathfrak{e}) \neq 1. \end{cases}$$

Moreover, $\mathfrak{L}(\chi, s) = \prod_{(\mathfrak{p},\mathfrak{c})=1} \mathfrak{L}_{\mathfrak{p}}(\chi, s)$*, where*

$$\mathfrak{L}_{\mathfrak{p}}(\chi, s) := G_{\mathfrak{p}}(\chi, s) \cdot \begin{cases} \prod_{i=1}^{n} L_{\mathfrak{p}}(2s + 2n - 2i, \chi^2) & \text{if } l \in 2\mathbb{Z} \\ \prod_{i=1}^{n} L_{\mathfrak{p}}(2s + 2n - 2i + 1, \chi^2) & \text{if } l \notin 2\mathbb{Z} \end{cases}$$

and $G_{\mathfrak{p}}(\chi, s)$ *is a ratio of Euler factors which for almost all* $\mathfrak{p}$ *is equal to one. (Below, in Theorem 7.6 we make* $G_{\mathfrak{p}}(\chi, s)$ *very precise.) In particular, the function* $L(s, \mathbf{f}, \chi)$ *is absolutely convergent for* $\mathrm{Re}(s) > n + l/2 + 1$*.*

Remark 7.2. It is worth to notice that the factor $G_{\mathfrak{p}}(\chi, s)$ does not appear in the works of [17] and [3]. It is because in the case of $l = 1$ considered there, the condition $M_{\mathfrak{p}}^{+}$ is equivalent to the condition that the matrix S is regular (see for example [15, Remark 4.3]), which implies that the factor $G_{\mathfrak{p}}(\chi, s)$ is equal to one for all good primes.

Before we proceed to the proof of the above theorem, we state an immediate corollary regarding the vanishing of the L-function defined above.

Corollary 7.3. *With notation and assumptions as in Theorem 7.1,*

$$L(s, \mathbf{f}, \chi) \neq 0$$

whenever $\mathrm{Re}(s) > n + l/2 + 1$*.*

Proof. This follows from the fact that the function $L(s, \mathbf{f}, \chi)$ is absolutely convergent for $\mathrm{Re}(s) > n + l/2 + 1$ and has an Euler product representation. For the formal argument see [24, Lemma 22.7]. □

The rest of this section is devoted to a proof of Theorem 7.1. Note that if we fix a prime ideal $\mathfrak{p}$ of F and consider the series

$$D_{\mathfrak{p}}(s, \mathbf{f}, \chi) := \sum_{j=0}^{\infty} \lambda(\mathfrak{p}^j) \chi^*(\mathfrak{p})^j N(\mathfrak{p})^{-js}, \quad \mathrm{Re}(s) \gg 0,$$

then

$$D(s, \mathbf{f}, \chi) = \prod_{\mathfrak{p}} D_{\mathfrak{p}}(s, \mathbf{f}, \chi) = \prod_{(\mathfrak{p}, \mathfrak{f}_\chi)=1} D_{\mathfrak{p}}(s, \mathbf{f}, \chi),$$

which means that it suffices to prove the theorem locally place by place.

Local Notation. For the rest of this section we fix the following notation. We fix a finite place $v \in \mathbf{h}$ of F. We abuse the notation and write F for F_v, $\mathfrak{o}$ for $\mathfrak{o}_v$, and just $\mathfrak{p}$ for the corresponding maximal ideal in $\mathfrak{o}_v$. Moreover, we denote by $\pi \in \mathfrak{p}$ any uniformiser of this place. We further set $q := [\mathfrak{o} : \mathfrak{p}]$ and denote by $|\cdot|$ the absolute value of F normalised so that $|\pi| = q^{-1}$. We also write $\boldsymbol{G}, G, \boldsymbol{D}, D$ for $\boldsymbol{G}(F_v), G(F_v), \boldsymbol{D}_v$ and D_v. Finally, in this part of the paper we denote by ψ_S the v-component of the additive adelic character ψ_S introduced in section 3.

7.1. **The good places.** We consider first a finite place v which is not in the support of $\mathfrak{c}\mathfrak{f}_\chi$. We assume that the matrix S_v satisfies condition $M_{\mathfrak{p}}^{+}$. As we have indicated at the beginning of this section we will extend the results of [17] from the case $l = 1$ to any l, and also introduce the twisting by a finite character χ. Here we use (more or less) the notation from [15, 16, 17].

We define a local Hecke algebra $\mathfrak{X}$ as in [15, page 142]. That is, let $\mathfrak{X}$ be the $\mathbb{C}$-module consisting of $\mathbb{C}$-valued functions ϕ on $\boldsymbol{G}$ which satisfy

$$\phi((00,\kappa)\boldsymbol{d}\boldsymbol{g}\boldsymbol{d}') = \psi_S(\kappa)\phi(\boldsymbol{g}), \quad \boldsymbol{d},\boldsymbol{d}' \in \boldsymbol{D}, \boldsymbol{g} \in \boldsymbol{G}, \;\kappa \in Sym_l(F)$$

and have compact support modulo $\mathcal{Z} := Sym_l(F) \subset \boldsymbol{G}$. As it is explained in [15], one can give to this module the structure of an algebra by defining multiplication through convolution of functions. Moreover, it is shown in [15, Lemma 4.4] that the assumption $M_{\mathfrak{p}}^+$ implies that a function $\phi \in \mathfrak{X}$ has support in

$$\bigcup_{\alpha\in\Lambda^+} \boldsymbol{D} d_n(\pi_\alpha)\boldsymbol{D}\mathcal{Z},$$

where $\Lambda^+ := \{(a_1,a_2,\dots,a_n) \in \mathbb{Z}^n : a_1 \geq a_2 \geq \dots \geq a_n \geq 0\}$,

$$d_n : GL_n \hookrightarrow G \subset \boldsymbol{G}, \qquad d_n(a) := \mathrm{diag}[a, {}^t a^{-1}],$$

and $\pi_\alpha := \mathrm{diag}[\pi^{a_1},\pi^{a_2},\dots,\pi^{a_n}] \in GL_n(F)$.

Let

$$T := T(F) := \{d_n(\mathrm{diag}[t_1,\dots,t_n]) : t_i \in F^\times\} \in G$$

and

$$X_0(T) := \{\xi \in Hom(T,\mathbb{C}^\times) : \xi \text{ is trivial on } T(\mathfrak{o})\}.$$

For a character $\xi \in X_0(T)$ and $\phi \in \mathfrak{X}$ set

$$\lambda_\xi(\phi) := \sum_{\alpha\in\mathbb{Z}^n} \xi^{-1}(d_n(\pi_\alpha))\hat{\phi}(d_n(\pi_\alpha)),$$

where for a function $\phi \in \mathfrak{X}$, $\hat{\phi}(t)$ is defined as in [15, equation (4.8)], that is,

$$\hat{\phi}(t) := \delta_{\mathbf{N}_0}(t)^{-1/2}\int_{\mathbf{N}_0}\phi(\mathbf{n}_0 t)d\mathbf{n}_0,$$

where $\mathbf{N}_0 := V_0 N_0 \subset \mathbf{G}$, N_0 is the unipotent radical of the Siegel parabolic P_0 of Sp_n, $V_0 := \{(0,\mu,0) : \mu \in M_{l,n}\}$, and $\delta_{\mathbf{N}_0}$ and the Haar measure $d\mathbf{n}_0$ are normalized as in [15, page 144].

For an $\alpha \in \Lambda^+$ we define $\phi_\alpha \in \mathfrak{X}$ by

$$\phi_\alpha(\boldsymbol{g}) := \begin{cases} \psi_S(\kappa) & \text{if } \boldsymbol{g} = (0,0,\kappa)\boldsymbol{d}d_n(\pi_\alpha)\boldsymbol{d}' \in \mathcal{Z}\boldsymbol{D}d_n(\pi_\alpha)\boldsymbol{D}, \\ 0 & \text{otherwise,} \end{cases}$$

and for a finite unramified character χ of $F^\times$ we define the function $\nu_{s,\chi}$ on $\mathbf{G}$, $s \in \mathbb{C}$, by the conditions

$$\nu_{s,\chi}((0,0,\kappa)\boldsymbol{d}g\boldsymbol{d}') = \psi_S(-\kappa)\nu_{s,\chi}(g), \;\; g \in G, \;\boldsymbol{d},\boldsymbol{d}' \in \mathbf{D}$$

and

$$\nu_{s,\chi}(\pi_\alpha) := \chi(\pi_v)^{\ell(\alpha)}q^{-\ell(\alpha)s},$$

where $\ell(\alpha) = \sum_{i=1}^n a_i$. It is shown in [17] that these two conditions uniquely determine the function $\nu_{s,\chi}$. Now, given a character $\xi \in X_0(T)$ and an unramified character χ of $F^\times$, we introduce the series

$$B(\xi,\chi,s) := \sum_{\alpha\in\Lambda_n^+}\lambda_\xi(\phi_\alpha)\chi(\pi)^{\ell(\alpha)}q^{-\ell(\alpha)s}.$$

Given a $\xi \in X_0(T)$ we define the function ϕ_ξ on $\mathbf{G}$ following [15, equation (4.11)] by

$$\phi_\xi((0,0,\kappa)\mathbf{n}_0 t(\lambda,0,0)\mathbf{d}) = \psi_S(\kappa)(\xi\delta_{\mathbf{n}_0}^{1/2})(t)\Phi_L(\lambda),\ \ \mathbf{d}\in\mathbf{D},\, t\in T,\ \mathbf{n}_0\in\mathbf{N}_0,$$

where Φ_L is the characteristic function of $L := M_{l,n}(\mathfrak{o})$. The following lemma ([3], Lemma 5.2) gives an important integral representation of the series $B(\xi,\chi,s)$.

Lemma 7.4 (Murase)**.** *For $\xi \in X_0(T)$ and a finite unramified character χ of $F^\times$ we have*

$$B(\xi,\chi,s) = \int_{\mathcal{Z}\backslash\mathbf{G}} \nu_{\chi,s}(g)\phi_\xi(g)dg.$$

Remark 7.5. The original lemma in [3] is stated without a twist by χ, but it is easy to see that the arguments there extend easily to include also the case of twisting by an unramified character.

For a finite unramified character χ and a character $\xi = (\xi_1,\dots,\xi_n) \in X_0(T)$, where ξ_i are unramified characters of $F^\times$, we define the local L-function

$$L(\xi,\chi,s) := \prod_{i=1}^{n}(1-\xi_i(\pi)\chi(\pi)q^{-s})^{-1}(1-\xi_i^{-1}(\pi)\chi(\pi)q^{-s})^{-1}.$$

In order to state the main theorem of this section we need to introduce a bit more notation. We write $\alpha_S(s,\chi)$ for the Siegel series attached to the symmetric matrix S and to the character χ, as defined for example in [23, Chapter III]. Moreover, by [23, Theorem 13.6], we have

$$\alpha_S(s,\chi) = \left(L(s,\chi)\prod_{i=1}^{[l/2]}L(2s-2i,\chi^2)\right)^{-1} g_S(s,\chi) \tag{23}$$

for some analytic function $g_S(s,\chi)$ of the form $g_S(s,\chi) = G(\chi(\pi)q^{-s})$ for some polynomial $G(X)\in\mathbb{Z}[X]$ of constant term one. Moreover if S is regular, that is, $\det(2S) = \mathfrak{o}^\times$ for l even and $\det(2S) = 2\mathfrak{o}^\times$ for l odd, then $g_S(s,\chi) = 1$.

The following theorem generalizes a result due to Murase and Sugano [17], where the case of $l = 1$ and χ trivial is considered.

Theorem 7.6. *With the notation as above,*

$$L(\xi,\chi,s) = \frac{g_S(s+n+l/2,\chi)}{g_S(s+l/2,\chi)}\Lambda(\chi,s)\int_{\boldsymbol{Z}\backslash\mathbf{G}}\nu_{\chi,s+n+l/2}(\boldsymbol{g})\phi_\xi(\boldsymbol{g})d\boldsymbol{g}\Lambda(\chi,s),$$

where

$$\Lambda(\chi,s) := \begin{cases}\prod_{i=1}^{n}L(2s+2n-2i,\chi^2) & \text{if } l\in 2\mathbb{Z},\\ \prod_{i=1}^{n}L(2s+2n-2i+1,\chi^2) & \text{if } l\notin 2\mathbb{Z}.\end{cases}$$

In particular,

$$L(\xi,\chi,s) = B(\xi,\chi,s+n+l/2)\frac{g_S(s+n+l/2,\chi)}{g_S(s+l/2,\chi)}\Lambda(\chi,s).$$

The rest of this subsection is devoted to a proof of this theorem. First we extend some calculations of Murase and Sugano [17]. Denote by σ_{n_1,n_2} the characteristic function of $M_{n_1,n_2}(\mathfrak{o})$ and let

$$F(s,\chi,\boldsymbol{g}) := F(s,\chi,hg) :=$$
$$\int_{GL_{2n+l}(F_v)} \sigma_{2n+l,4n+2l}\left(\left(y\begin{pmatrix}1_l & 0\\ 0 & g\end{pmatrix}, y\alpha(h)\right)\right)\chi(\det(y))|\det(y)|^{s+n+l/2}d^*y,$$

where for $h=(\lambda,\mu,\kappa)\in H$ we set

$$\alpha(h) := \begin{pmatrix}\kappa-\lambda{}^t\mu & -\lambda & -\mu\\ {}^t\mu & 1_n & 0\\ {}^t\lambda & 0 & 1_n\end{pmatrix}.$$

Define also

$$\mathcal{F}(s,\chi,\boldsymbol{g}) := \int_{\mathcal{Z}} F(s,\chi,(0,0,\kappa)\boldsymbol{g})\psi_S(\kappa)d\kappa.$$

We now recall a theorem of Murase in [16, Theorem 2.12].

Theorem 7.7 (Murase)**.** *We have the equality:*

$$L(\xi,\chi,s)=\alpha_S\left(s+\frac{l}{2},\chi\right)^{-1}L\left(s+\frac{l}{2},\chi\right)^{-1}\frac{\prod_{i=1}^{n}L(2s+2n+l-2i,\chi^2)}{\prod_{i=1}^{2n+l-1}L(s+n+l/2-i,\chi)}\int_{\mathcal{Z}\backslash\boldsymbol{G}}\mathcal{F}(s,\chi,\boldsymbol{g})\phi_\xi(\boldsymbol{g})d\boldsymbol{g}.$$

The following lemma extends a result of Murase and Sugano in [17, Lemma 6.8] from the case of index one ($l=1$) to any index.

Lemma 7.8. *We have the following equality:*

$$\mathcal{F}(s,\chi,\boldsymbol{g}) = \left(\prod_{i=1}^{l}L(s+n+l/2-i+1,\chi)\right)\alpha_S(s+n+l/2,\chi)$$
$$\cdot\left(\prod_{i=1}^{2n}L(s+n-l/2-i+1,\chi)\right)\nu_{s+n+l/2,\chi}(\boldsymbol{g}).$$

Proof. We recall first a result of Shimura. By [23, Lemma 3.13], for any $g\in M_m(F)$,

$$\int_{GL_m(F)}\sigma_{m,2m}(yg,y)\chi(\det(y))|\det(y)|^s d^*y = \prod_{i=1}^{m}L(s-i+1,\chi)\chi(\nu_0(g))\nu(g)^{-s}, \tag{24}$$

where $\nu_0(g)$ and $\nu(g)$ denote the denominator ideal of g and its norm respectively, as defined for example in [23, page 19].

By [16, Proposition 2.3],

$$\mathcal{F}(s,\chi,(0,0,\kappa)\boldsymbol{d}\boldsymbol{g}\boldsymbol{d}') = \psi_S(-\kappa)\mathcal{F}(s,\chi,\boldsymbol{g})$$

for all $\kappa\in\mathcal{Z}$ and $\boldsymbol{d},\boldsymbol{d}'\in\boldsymbol{D}$. That is, thanks to [15, Lemma 4.4], for a fixed s the function $\mathcal{F}(s,\chi,\boldsymbol{g})$ is supported on $\bigcup_{m\in\Lambda_n^+}\mathcal{Z}\boldsymbol{D}\pi_m\boldsymbol{D}$. Hence, it is enough to prove the equality of the Lemma for $\boldsymbol{g}=\pi_m$ for an $m\in\Lambda_n^+$. We have

$$\mathcal{F}(s,\chi,\pi_m) = \int_{GL_{2n+l}(F)}\sigma_{2n+l,4n+2l}\left(y\begin{pmatrix}1_l & \\ & \pi_m\end{pmatrix}, y\begin{pmatrix}\kappa & \\ & 1_{2n}\end{pmatrix}\right)\chi(\det(y))|\det(y)|^{s+n+l/2}d^*y$$

$$\cdot \int_{\mathcal{Z}} \psi_S(\kappa) d\kappa$$

Write $y = k \left(\begin{smallmatrix} a & b \\ & d \end{smallmatrix}\right)$, where $k \in \mathrm{GL}_{2n+l}(\mathfrak{o})$, $a \in \mathrm{GL}_l(F)$, $d \in \mathrm{GL}_{2n}(F)$ and $b \in M_{l,2n}(F)$. Then $\mathcal{F}(s, \chi, \pi_m) = I_1 \cdot I_2 \cdot I_3$, where

$$I_1 = \int_{\mathcal{Z}} \psi_S(\kappa) \int_{GL_l(F)} \sigma_{l,l}(a)\sigma_{l,l}(a\kappa)\chi(\det(a))|\det(a)|^{s+n+l/2} d^*a,$$

$$I_2 = \int_{M_{l,2n}(F)} \sigma_{l,2n}(b\pi_m)\sigma_{l,2n}(b) db$$

and

$$I_3 = \int_{\mathrm{GL}_{2n}(F)} \sigma_{2n,2n}(d)\sigma_{2n,2n}(d\pi_m)\chi(\det(d))|\det(d)|^{s+n+l/2}|\det(d)|^{-l} d^*d.$$

We compute first the integral I_1. By the equation (24),

$$\int_{\mathrm{GL}_l(F)} \sigma_{l,l}(a)\sigma_{l,l}(a\kappa)\chi(\det(a))|\det(a)|^{s+n+l/2} d^*a$$
$$= \prod_{i=1}^{l} L(s+n+l/2-i+1, \chi)\chi(\nu_0(\kappa))\nu(\kappa)^{-s-n-l/2},$$

and hence

$$I_1 = \prod_{i=1}^{l} L(s+n+l/2-i+1, \chi) \int_{\mathcal{Z}} \psi_S(\kappa)\chi(\nu_0(\kappa))\nu(\kappa)^{-s-n-l/2} d\kappa.$$

But the last integral is nothing else than the Siegel series $\alpha_S(s+n+l/2, \chi)$, and thus

$$I_1 = \prod_{i=1}^{l} L(s+n+l/2-i+1, \chi)\alpha_S(s+n+l/2, \chi).$$

Finally, it is easy to see that $I_2 = q^{-(m_1+\ldots+m_n)l}$, and that by the equation (24) again,

$$I_3 = \prod_{i=1}^{2n} L(s+n-l/2-i+1, \chi)\chi(\nu_0(\pi_m))\nu(\pi_m)^{-s-n-l/2}.$$

$\square$

Proof of Theorem 7.6. By Lemma 7.8,

$$L(\xi, \chi, s) = \alpha_S(s+l/2, \chi)^{-1} L(s+l/2, \chi)^{-1} \left(\prod_{i=1}^{2n+l-1} L(s+n+l/2-i, \chi) \right)^{-1}$$
$$\cdot \prod_{i=1}^{n} L(2s+2n+l-2i, \chi^2) \prod_{i=1}^{l} L(s+n+l/2-i+1, \chi)\alpha_S(s+n+l/2, \chi)$$
$$\cdot \prod_{i=1}^{2n} L(s+n-l/2-i+1, \chi) \int_{\boldsymbol{Z}\backslash\boldsymbol{G}} \nu_{s+n+l/2,\chi}(\boldsymbol{g})\phi_\xi(\boldsymbol{g}) d\boldsymbol{g}$$

$$
\begin{aligned}
&= \alpha_S(s+l/2,\chi)^{-1}L(s+l/2,\chi)^{-1}\prod_{i=1}^{n}L(2s+2n+l-2i,\chi^2)\\
&\quad\cdot L(s+n+l/2,\chi)\alpha_S(s+n+l/2,\chi)\int_{\boldsymbol{Z}\backslash\boldsymbol{G}}\nu_{s+n+l/2,\chi}(\boldsymbol{g})\phi_\xi(\boldsymbol{g})d\boldsymbol{g}\\
&= \frac{\alpha_S(s+n+l/2,\chi)}{\alpha_S(s+l/2,\chi)}\frac{L(s+n+l/2,\chi)}{L(s+l/2,\chi)}\prod_{i=1}^{n}L(2s+2n+l-2i,\chi^2)\\
&\quad\cdot\int_{\boldsymbol{Z}\backslash\boldsymbol{G}}\nu_{s+n+l/2,\chi}(\boldsymbol{g})\phi_\xi(\boldsymbol{g})d\boldsymbol{g}.
\end{aligned}
$$

If we now plug in the expression (23) for the Siegel series, we obtain

$$
\begin{aligned}
L(\xi,\chi,s) &= \frac{g_S(s+n+l/2,\chi)}{g_S(s+l/2,\chi)}\frac{\prod_{i=1}^{[l/2]}L(2s+l-2i,\chi^2)}{\prod_{i=1}^{[l/2]}L(2s+2n+l-2i,\chi^2)}\prod_{i=1}^{n}L(2s+2n+l-2i,\chi^2)\\
&\quad\cdot\int_{\boldsymbol{Z}\backslash\boldsymbol{G}}\nu_{s+n+l/2,\chi}(\boldsymbol{g})\phi_\xi(\boldsymbol{g})d\boldsymbol{g}\\
&= \frac{g_S(s+n+l/2,\chi)}{g_S(s+l/2,\chi)}\prod_{i=[l/2]+1}^{[n+l/2]}L(2s+2n+l-2i,\chi^2)\int_{\boldsymbol{Z}\backslash\boldsymbol{G}}\nu_{s+n+l/2,\chi}(\boldsymbol{g})\phi_\xi(\boldsymbol{g})d\boldsymbol{g},
\end{aligned}
$$

which finishes the proof. □

Given a cusp form $0\neq\mathbf{f}\in S^n_{k,S}(\boldsymbol{D},\psi)$ we can define an action of an element ϕ in the Hecke algebra $\mathfrak{X}$ by

$$(\mathbf{f}\star\phi)(\boldsymbol{g})=\int_{\mathcal{Z}\backslash\boldsymbol{G}}\mathbf{f}(\boldsymbol{g}x^{-1})\phi(x)dx.$$

If now $\mathbf{f}$ is a common eigenform for all $\phi\in\mathfrak{X}$, that is, $\mathbf{f}\star\phi=\lambda_{\mathbf{f}}(\phi)\mathbf{f}$ for all ϕ, then we obtain a $\mathbb{C}$-algebra homomorphism $\lambda_{\mathbf{f}}:\mathfrak{X}\to\mathbb{C}$. Thanks to [15, Theorem 4.15] we know that this homomorphism is of the form

$$\lambda_{\mathbf{f}}(\phi)=\lambda_{\xi_{\mathbf{f}}}(\phi)$$

for some character $\xi_{\mathbf{f}}\in X_0(T)$, and thus, as it is explained in [3, Lemma 5.4],

$$\mathbf{f}\star\phi_\alpha=\mathbf{f}|T_{\pi_\alpha^{-1},\psi_S}\quad\text{for every }\alpha\in\Lambda_n^+.$$

Note here that since $\boldsymbol{D}d_n(\pi_\alpha)\boldsymbol{D}=\boldsymbol{D}d_n(\pi_\alpha^{-1})\boldsymbol{D}$, we obtain

$$B(\xi_{\mathbf{f}},\chi,s)=D_{\mathfrak{p}}(s,\mathbf{f},\chi).$$

In this way we can conclude Theorem 7.1 in the case when v is a good prime by taking $\mu_{\mathfrak{p},i}:=\xi_i(\pi)$ if $\xi_{\mathbf{f}}=(\xi_1,\dots,\xi_n)$.

7.2. **The bad places.** We now consider the case of $(\mathfrak{p},\mathfrak{c})\neq1$. If $(\mathfrak{p},\mathfrak{e})\neq1$, then there is nothing to show, because in this case each Hecke operator is just the identity. Hence we consider the case of $(\mathfrak{p},\mathfrak{e}^{-1}\mathfrak{c})\neq1$. In this section we set $E:=\mathrm{GL}_n(\mathfrak{o})$ and $\mathcal{S}:=S(\mathfrak{b}^{-1}):=Sym_n(F)\cap M_n(\mathfrak{b}_v^{-1})$.

First we work out the decomposition of the double cosets $\boldsymbol{D}\mathrm{diag}[\tilde{\xi},\xi]\boldsymbol{D}$. Recall that we write $\boldsymbol{D} = CD$ with $C = C_v[\mathfrak{o},\mathfrak{b}^{-1},\mathfrak{b}^{-1}] \subset H$ and $D = D_v[\mathfrak{b}^{-1},\mathfrak{bc}] \subset G$. By [24, Lemma 19.2] we know that

$$D\mathrm{diag}[\tilde{\xi},\xi]D = \bigsqcup_{d,b} D\begin{pmatrix}\tilde{d} & \tilde{d}b\\ & d\end{pmatrix},$$

where $d \in E \setminus E\xi E$ and $b \in \mathcal{S}/{}^t d\mathcal{S}d$, and thus

$$\boldsymbol{D}\mathrm{diag}[\tilde{\xi},\xi]\boldsymbol{D} = CD\mathrm{diag}[\tilde{\xi},\xi]DC = \bigsqcup_{d,b}\boldsymbol{D}\begin{pmatrix}\tilde{d} & \tilde{d}b\\ & d\end{pmatrix}C.$$

Observe that for elements $(\lambda,\mu,\kappa) \in C$ and $\begin{pmatrix}\tilde{d} & \tilde{d}b\\ & d\end{pmatrix}$ as above we have

$$\begin{pmatrix}\tilde{d} & \tilde{d}b\\ & d\end{pmatrix}(\lambda,\mu,\kappa) = (\lambda{}^t d, (-\lambda b+\mu)d^{-1}, \kappa + \lambda{}^t d{}^t d^{-1}{}^t(-\lambda b+\mu) - \lambda{}^t\mu)\begin{pmatrix}\tilde{d} & \tilde{d}b\\ & d\end{pmatrix}.$$

In particular,

$$\boldsymbol{D}\mathrm{diag}[\tilde{\xi},\xi]\boldsymbol{D} = \bigsqcup_{d,b,\mu}\boldsymbol{D}(0,\mu,0)\begin{pmatrix}\tilde{d} & \tilde{d}b\\ & d\end{pmatrix}, \tag{25}$$

where $d \in E \setminus E\xi E$, $b \in \mathcal{S}/{}^t d\mathcal{S}d$ and $\mu \in M_{l,n}(\mathfrak{b}_v^{-1})d^{-1}/M_{l,n}(\mathfrak{b}_v^{-1})$.

We will show that the set $\boldsymbol{DXD}$, with $X = \{\mathrm{diag}(\tilde{\xi},\xi) : \xi \in M_n(\mathfrak{o}_v) \cap GL_n(F_v)\}$ is closed under multiplication. For $\boldsymbol{D}\mathrm{diag}[\tilde{\xi}_i,\xi_i]\boldsymbol{D} = \bigsqcup_{d_i,b_i,\mu_i}(0,\mu_i,0)\begin{pmatrix}\tilde{d}_i & \tilde{d}_i b_i\\ & d_i\end{pmatrix}$, $i = 1,2$, we have

$$\begin{aligned}
&\boldsymbol{D}\mathrm{diag}[\tilde{\xi}_1,\xi_1]\boldsymbol{D}\mathrm{diag}[\tilde{\xi}_2,\xi_2]\boldsymbol{D}\\
&= \bigsqcup_{d_1,b_1,\mu_1,d_2,b_2,\mu_2}\boldsymbol{D}(0,\mu_1,0)\begin{pmatrix}\tilde{d}_1 & \tilde{d}_1 b_1\\ & d_1\end{pmatrix}(0,\mu_2,0)\begin{pmatrix}\tilde{d}_2 & \tilde{d}_2 b_2\\ & d_2\end{pmatrix}\\
&= \bigsqcup_{d_1,b_1,\mu_1,d_2,b_2,\mu_2}\boldsymbol{D}(0,\mu_1,0)\mathrm{diag}[\tilde{d}_1\tilde{d}_2, d_1d_2]\begin{pmatrix}1 & b_2 + {}^t d_2 b_1 d_2\\ & 1\end{pmatrix}(0,\mu_2 d_2,0)\\
&= \bigsqcup_{d_1,b_1,\mu_1,d_2,b_2,\mu_2}\boldsymbol{D}\mathrm{diag}[\tilde{d}_1\tilde{d}_2, d_1d_2](0,\mu_1 d_1 d_2,0)\begin{pmatrix}1 & b_2 + {}^t d_2 b_1 d_2\\ & 1\end{pmatrix}(0,\mu_2 d_2,0).
\end{aligned}$$

Hence, because $(0,\mu_1 d_1 d_2,0),(0,\mu_2 d_2,0) \in C$, $\begin{pmatrix}1 & b_2 + {}^t d_2 b_1 d_2\\ & 1\end{pmatrix} \in D$ and $\tilde{d}_1\tilde{d}_2 = \widetilde{d_1 d_2}$, we have shown that

$$\boldsymbol{D}\mathrm{diag}[\tilde{\xi}_1,\xi_1]\boldsymbol{D}\mathrm{diag}[\tilde{\xi}_2,\xi_2]\boldsymbol{D} \subset \boldsymbol{DXD}.$$

We define the Hecke algebra $\mathfrak{X} := \mathfrak{X}_v$ for $v|\mathfrak{e}^{-1}\mathfrak{c}$ to be the algebra generated by the double cosets $\boldsymbol{DXD}$.

In order to define the Satake parameters associated to an eigenform of this algebra we need to define an injective algebra homomorphism $\omega : \mathfrak{X} \to \mathbb{Q}[t_1,\ldots,t_n]$. We will do this by reducing everything to the theory of GL_n, very much in the spirit of Shimura in [24, Theorem 19.8].

Given an element

$$\boldsymbol{D}\mathrm{diag}[\tilde{\xi},\xi]\boldsymbol{D} = \bigsqcup_{d,b,\mu} (0,\mu,0)\begin{pmatrix} \tilde{d} & \tilde{d}b \\ & d \end{pmatrix},$$

where $d \in E \setminus E\xi E$, $b \in \mathcal{S}/{}^t d\mathcal{S}d$ and $\mu \in M_{l,n}(\mathfrak{b}_v^{-1})d^{-1}/M_{l,n}(\mathfrak{b}_v^{-1})$, we set

$$\omega_0\left((0,\mu,0)\begin{pmatrix} \tilde{d} & \tilde{d}b \\ & d \end{pmatrix}\right) := \omega_0(Ed),$$

where ω_0 is the classical map of the spherical Hecke algebra of GL_n defined as $\omega_0(Ed) := \prod_{i=1}^n (\xi^{-i}t_i)^{e_i}$ if an upper triangular representative of Ed has the diagonal entries $\pi^{e_1}, \pi^{e_2}, \ldots, \pi^{e_n}$ with $e_i \in \mathbb{Z}$. Further, let

$$\omega(\boldsymbol{D}\mathrm{diag}[\tilde{\xi},\xi]\boldsymbol{D}) := \sum_{d,b,\mu} \omega_0\left((0,\mu,0)\begin{pmatrix} \tilde{d} & \tilde{d}b \\ & d \end{pmatrix}\right).$$

An identical argument to the one in [23, Proposition 16.14] shows that $\omega : \mathfrak{X} \to \mathbb{Q}[[t_1^{\pm}, t_2^{\pm}, \ldots, t_n^{\pm}]]$ is an injective algebra homomorphism.

For a finite unramified character χ and for $s \in \mathbb{C}$ consider the formal series

$$B(\chi,s) := \sum_{\xi \in E\setminus B/E} (\boldsymbol{D}\mathrm{diag}[\tilde{\xi},\xi]\boldsymbol{D})\chi(\det(\xi))N(\det(\xi))^{-s},$$

where $B := \mathrm{GL}_n(F) \cap M_n(\mathfrak{o})$. Then, if we define

$$\omega(B(\chi,s)) := \sum_{\xi \in E\setminus B/E} \omega(\boldsymbol{D}\mathrm{diag}[\tilde{\xi},\xi]\boldsymbol{D})\chi(\det(\xi))N(\det(\xi))^{-s},$$

we have that

$$\omega(B(\chi,s)) = \sum_{d \in E\setminus B} \omega_0(Ed)|\det(d)|^{-n-l}\chi(\det(d))N(\det(d))^{-s}.$$

Hence, by an argument similar to the one in [24, Theorem 19.8], we get

$$\omega(B(\chi,s)) = \prod_{i=1}^n (1 - q^{n+l}t_i\chi(\pi)q^{-s})^{-1} \in \mathbb{Q}[[t_1,\ldots,t_n]].$$

Now [24, Lemma 19.9] states that if we have a $\mathbb{Q}$-linear homomorphism $\lambda\colon \mathfrak{X} \to \mathbb{C}$ which maps the identity element to 1, then there exist Satake parameters $\mu_1,\ldots,\mu_n \in \mathbb{C}$ such that

$$\sum_{\xi \in E\setminus B/E} \lambda(\boldsymbol{D}\mathrm{diag}[\tilde{\xi},\xi]\boldsymbol{D})\chi(\det(\xi))N(\det(\xi))^{-s} = \prod_{i=1}^n (1 - q^{n+l}\mu_i\chi(\pi)q^{-s})^{-1}$$

or, equivalently,

$$\sum_{\xi \in E\setminus B/E} \lambda(\boldsymbol{D}\mathrm{diag}[\tilde{\xi},\xi]\boldsymbol{D})\chi(\det(\xi))N(\det(\xi))^{-(s+n+l/2)} = \prod_{i=1}^n (1 - q^{-l/2}\mu_i\chi(\pi)q^{-s})^{-1}$$

as an equality of formal series in $\mathbb{C}[[q^{-s}]]$. Hence, if we take as λ the homomorphism obtained from the eigenform $\mathbf{f}$ and let $\mu_{\mathfrak{p},i} := \mu_i q^{-l/2}$, we establish the rest of Theorem 7.1, as in this case

$$D_{\mathfrak{p}}(s,\mathbf{f},\chi) = \sum_{\xi \in E\backslash B/E} \lambda(\boldsymbol{D}\mathrm{diag}[\tilde{\xi},\xi]\boldsymbol{D})\chi(\det(\xi))N(\det(\xi))^{-s}.$$

7.3. **A ψ-twisted L-function.** To an eigenform $\mathbf{f} \in S^n_{k,S}(\boldsymbol{D},\psi)$ we can associate yet another L-function. It appears naturally in the doubling method when the form $\mathbf{f}$ has a non-trivial nebentype. For a character χ of conductor $\mathfrak{f}$ we define

$$\begin{aligned} L_{\psi}(s,\mathbf{f},\chi) &:= \prod_{\mathfrak{p}} L_{\mathfrak{p}}(\chi^*(\mathfrak{p})(\psi/\psi_{\mathfrak{c}})(\pi_{\mathfrak{p}})N(\mathfrak{p})^{-s}) \\ &= \left(\prod_{(\mathfrak{p},\mathfrak{c})=1} L_{\mathfrak{p}}((\chi\psi)^*(\mathfrak{p})N(\mathfrak{p})^{-s})\right)\left(\prod_{\mathfrak{p}|\mathfrak{c}} L_{\mathfrak{p}}(\chi^*(\mathfrak{p})N(\mathfrak{p})^{-s})\right), \end{aligned}$$

where $\psi_{\mathfrak{c}} = \prod_{v|\mathfrak{c}} \psi_v$, $\pi_{\mathfrak{p}} \in \mathfrak{o}_{\mathfrak{p}}$ is a uniformizer of the ring of integers $\mathfrak{o}_{\mathfrak{p}}$, and the factors $L_{\mathfrak{p}}(X)$ are as in Theorem 7.1. We also define the series

$$D_{\psi}(s,\mathbf{f},\chi) := \sum_{\mathfrak{a}} \lambda(\mathfrak{a})\chi^*(\mathfrak{a})\psi(\mathfrak{a}')N(\mathfrak{a})^{-s},$$

where for an ideal $\mathfrak{a}$ with prime decomposition $\prod_{\mathfrak{p}} \mathfrak{p}^{n_{\mathfrak{p}}}$ we put $\mathfrak{a}' := \prod_{(\mathfrak{p},\mathfrak{c})=1} \mathfrak{p}^{n_{\mathfrak{p}}}$. Then:

$$D_{\psi}(s,\mathbf{f},\chi) = \prod_{(\mathfrak{p},\mathfrak{c})=1} D_{\mathfrak{p}}(s,\mathbf{f},\chi\psi) \prod_{\mathfrak{p}|\mathfrak{c}} D_{\mathfrak{p}}(s,\mathbf{f},\chi).$$

In particular, by Theorem 7.1,

$$\mathfrak{L}_{\psi}(\chi,s)D_{\psi}(s+n+l/2,\mathbf{f},\chi) = L_{\psi}(s,\mathbf{f},\chi),$$

where $\mathfrak{L}_{\psi}(\chi,s) = \prod_{(\mathfrak{p},\mathfrak{c})=1} \mathfrak{L}_{\mathfrak{p}}(\chi\psi,s)$, and

$$\mathfrak{L}_{\mathfrak{p}}(\chi\psi,s) := G_{\mathfrak{p}}(\chi\psi,s) \begin{cases} \prod_{i=1}^n L_{\mathfrak{p}}(2s+2n-2i,(\chi\psi)^2) & \text{if } l \in 2\mathbb{Z} \\ \prod_{i=1}^n L_{\mathfrak{p}}(2s+2n-2i+1,(\chi\psi)^2) & \text{if } l \notin 2\mathbb{Z} \end{cases}.$$

Finally, for any given integral ideal $\mathfrak{x}$ we define the function

$$L_{\psi,\mathfrak{x}}(s,\mathbf{f},\chi) := \prod_{(\mathfrak{p},\mathfrak{x})=1} L_{\mathfrak{p}}(\chi^*(\mathfrak{p})(\psi/\psi_{\mathfrak{c}})(\pi_{\mathfrak{p}})N(\mathfrak{p})^{-s}),$$

that is, we remove the Euler factors at the primes which divide $\mathfrak{x}$.

7.4. **The global Hecke algebra.** Now let $\mathfrak{X} := \bigotimes_v \mathfrak{X}_v$ be the global Hecke algebra. Since every local Hecke algebra $\mathfrak{X}_v$ can be embedded in a power series ring (for the good places this has been established in [15, Theorem 4.14] and for the bad places above), and thus is commutative, we can conclude that the global Hecke algebra $\mathfrak{X}$ is also commutative. Moreover, if $T_{r,\psi}$ is the Hecke operator where $r_v = 1_n$ at $v|\mathfrak{c}$, then

$$< f|T_{r,\psi}, g > = < f, g|T_{r,\psi} > .$$

Indeed, this follows from the fact that $< f|_{S,k}\alpha, g|_{S,k}\alpha > = < f, g >$ for any $\alpha \in G^n$ and that for any r as above we have

$$\boldsymbol{D}\mathrm{diag}[\tilde{r}, r]\boldsymbol{D} = CD\mathrm{diag}[\tilde{r}, r]DC = CD\mathrm{diag}[{}^t r, r^{-1}]CD = \boldsymbol{D}\mathrm{diag}[{}^t r, r^{-1}]\boldsymbol{D},$$

where the second equality follows from [23, Remark on page 89]. In particular, it follows that the Hecke operators $T(\mathfrak{a})$ with $(\mathfrak{a}, \mathfrak{c}) = 1$ are normal, and thus can be simultaneously diagonalized.

We finish this section with a result which will be useful for our later considerations. First recall that we have defined $f^c(z) = \overline{f(-\overline{z})}$. Now set $\epsilon := \mathrm{diag}[1_n, -1_n]$ and define

$$\boldsymbol{\epsilon}((\lambda, \mu, \kappa)\gamma)\boldsymbol{\epsilon} := (\lambda, -\mu, -\kappa)\epsilon\gamma\epsilon. \tag{26}$$

It is easy to check that this map is an automorphism of the Jacobi group $\boldsymbol{G}^n$.

Proposition 7.9. *Let $\boldsymbol{\gamma} = (\lambda, \mu, \kappa)\gamma \in \boldsymbol{G}$. Then*

$$(f|_{k,S}\boldsymbol{\gamma})^c = f^c|_{k,S}\boldsymbol{\epsilon\gamma\epsilon}.$$

Moreover, if f is an eigenform with $f|T_\psi(\mathfrak{a}) = \lambda(\mathfrak{a})f$ for all fractional ideals $\mathfrak{a}$ prime to $\mathfrak{c}$, then so is f^c. In particular, $f^c|T_\psi(\mathfrak{a}) = \lambda(\mathfrak{a})f^c$ and $L_{\psi,\mathfrak{c}}(s, f, \chi) = L_{\psi,\mathfrak{c}}(s, f^c, \chi)$.

Proof. The first equality easily follows from a direct computation.

Now assume that f is an eigenform of $T(\mathfrak{a})$ with eigenvalues $\lambda(\mathfrak{a})$ for all integral ideals $\mathfrak{a}$. Because the map (26) is a group automorphism, we see that for any $r \in Q(\mathfrak{e})$ if $\boldsymbol{G}^n(F) \cap \boldsymbol{D}\mathrm{diag}[\tilde{r}, r]\boldsymbol{D} = \coprod_{\boldsymbol{\gamma}} \boldsymbol{\Gamma\gamma}$, then also $\boldsymbol{G}^n(F) \cap \boldsymbol{D}\mathrm{diag}[\tilde{r}, r]\boldsymbol{D} = \coprod_{\boldsymbol{\gamma}} \boldsymbol{\Gamma\epsilon\gamma\epsilon}$. This means that $f^c|T_{r,\psi} = (f|T_{r,\psi})^c$. In particular,

$$f^c|T_\psi(\mathfrak{a}) = (f|T_\psi(\mathfrak{a}))^c = (\lambda(\mathfrak{a})f)^c = \overline{\lambda(\mathfrak{a})}f^c$$

for all integral ideals $\mathfrak{a}$. However, since $0 \neq f$, then $< f, f > \neq 0$ and thus the equality

$$\lambda(\mathfrak{a}) < f, f > = < f|T_\psi(\mathfrak{a}), f > = < f, f|T_\psi(\mathfrak{a}) > = < f, f > \overline{\lambda(\mathfrak{a})}$$

implies that the eigenvalues $\lambda(\mathfrak{a})$ are totally real. The last statement regarding the L-functions is now obvious. □

8. Analytic properties of Siegel-type Jacobi Eisenstein series

In the previous section we introduced the standard L-function attached to a Siegel-Jacobi eigenfunction. Our first aim is to study its analytic properties using the identity (21). However, in order to do this we need to establish first the analytic properties of the Siegel-type Jacobi Eisenstein series with respect to the parameter s. This is the subject of this section. More precisely, we will establish the analytic continuation and detect possible poles of this Eisenstein series. The main idea of our method goes back to Böcherer [4], which was further extended by Heim in [12], and its aim is to relate Jacobi Eisenstein series of Siegel type to symplectic Eisenstein series (of Siegel type). We extend their results to include level, character and - more importantly - we deal also with the case of totally real field. This last generalization requires development of some new techniques in case the class number is not trivial. In this section the Jacobi Eisenstein series is denoted by a bold $\boldsymbol{E}$, and the symplectic by a normal E.

We start with the following lemma, which gives us good representatives for the sets $(\boldsymbol{P}^n \cap \zeta\boldsymbol{\Gamma}\zeta^{-1}) \setminus \zeta\boldsymbol{\Gamma}$, where $\zeta \in \mathrm{Sp}_n(F)$, and $\boldsymbol{\Gamma}$ is a congruent subgroup of the form $H \rtimes \Gamma_0(\mathfrak{b},\mathfrak{c})$.

Lemma 8.1. *A set of representatives for the left cosets* $(\boldsymbol{P}^n \cap \zeta\boldsymbol{\Gamma}\zeta^{-1}) \setminus \zeta\boldsymbol{\Gamma}$ *is given by*

$$(\lambda,0,0)\gamma, \quad \lambda \in M_{l,n}(\mathfrak{o}), \quad \gamma \in P \cap \zeta\Gamma_0(\mathfrak{b},\mathfrak{c})\zeta^{-1} \setminus \zeta\Gamma_0(\mathfrak{b},\mathfrak{c}).$$

Proof. First note that $\zeta\boldsymbol{\Gamma} = \zeta(H \rtimes \Gamma_0(\mathfrak{b},\mathfrak{c})) = H \rtimes \zeta\Gamma_0(\mathfrak{b},\mathfrak{c})$ and, similarly, $\boldsymbol{P}^n \cap \zeta\boldsymbol{\Gamma}\zeta^{-1} = \boldsymbol{P}^n \cap (H \rtimes \zeta\Gamma_0(\mathfrak{b},\mathfrak{c})\zeta^{-1})$, which is nothing else than the set $(H_0^n \cap H) \rtimes (P \cap \zeta\Gamma_0(\mathfrak{b},\mathfrak{c})\zeta^{-1})$. Now, since

$$(P \cap \zeta\Gamma_0(\mathfrak{b},\mathfrak{c})\zeta^{-1})H = H(P \cap \zeta\Gamma_0(\mathfrak{b},\mathfrak{c})\zeta^{-1}),$$

a set of representatives for the cosets is given by a product of representatives for $(H_0^n \cap H) \setminus H$ and for $(P \cap \zeta\Gamma_0(\mathfrak{b},\mathfrak{c})\zeta^{-1}) \setminus \zeta\Gamma_0(\mathfrak{b},\mathfrak{c})$. This is precisely the statement of the lemma. □

Now recall the expression (11) for an Eisenstein series of Siegel type:

$$\boldsymbol{E}(z,s) = \sum_{\zeta \in Z} N(\mathfrak{a}(\zeta))^{2s} \sum_{\gamma \in Q_\zeta} \chi[\gamma]\delta(z)^{s-k/2}|_{k,S}\gamma,$$

where $Q_\zeta = (P \cap \zeta\Gamma_0(\mathfrak{b},\mathfrak{c})\zeta^{-1}) \setminus \zeta\Gamma_0(\mathfrak{b},\mathfrak{c})$.

We set $\boldsymbol{E}_\zeta(z,s) := \sum_{\gamma \in Q_\zeta} \chi[\gamma]\delta(z)^{s-k/2}|_{k,S}\gamma$. Clearly, the analytic continuation of $\boldsymbol{E}(z,s)$ and its set of possible poles would follow by establishing such a result for all the $\boldsymbol{E}_\zeta(z,s)$, as $\zeta \in Z$.

If we write $\gamma = hg$ and $z = (\tau, w)$, then

$$\boldsymbol{E}_\zeta(z,s) = \sum_{\gamma \in Q_\zeta} \chi[\gamma]\delta(z)^{s-k/2}|_{k,S}\gamma = \sum_{\gamma \in Q_\zeta} \chi[\gamma] J_{k,S}(\gamma,z)^{-1}\delta(g\tau)^{s-k/2}.$$

Further, by Lemma 8.1,

$$\begin{aligned}\boldsymbol{E}_\zeta(z,s) = &\sum_{g \in Q_\zeta} \chi[g] j(g,\tau)^{-k}\delta(g\tau)^{s-k/2}\mathbf{e}_{\mathbf{a}}(-\mathrm{tr}\,(S[w](c_g\tau+d_g)^{-1}c_g)) \\ &\cdot \sum_{\lambda \in M_{l,n}(\mathfrak{o})} \mathbf{e}_{\mathbf{a}}(2\mathrm{tr}\,({}^t\lambda S w(c_g\tau+d_g)^{-1}) + \mathrm{tr}\,(S[\lambda] g\cdot\tau)).\end{aligned}$$

For a lattice L in $M_{l,n}(F)$ we define the Jacobi theta series

$$\Theta_{S,L}(z) = \Theta_{S,L}(\tau,w) := \sum_{\lambda \in L} \mathbf{e}_{\mathbf{a}}(2\mathrm{tr}\,({}^t\lambda S w) + \mathrm{tr}\,(S[\lambda]\tau)).$$

Recall (Lemma 4.2) that the elements ζ may be selected in the form $\mathrm{diag}[1_{n-1}, a_\zeta, 1_{n-1}, a_\zeta^{-1}]$. In particular, for an element $g \in Q_\zeta$ of the form $g = \zeta g_1$,

$$c_g\tau + d_g = (c_\zeta(g_1\tau) + d_\zeta)(c_{g_1}\tau + d_{g_1}) = \begin{pmatrix} 1_{n-1} & \\ & a_\zeta^{-1}\end{pmatrix}(c_{g_1}\tau + d_{g_1})$$

and

$$g\cdot\tau = \zeta g_1 \cdot \tau = \begin{pmatrix} 1_{n-1} & \\ & a_\zeta\end{pmatrix}(g_1\cdot\tau)\begin{pmatrix} 1_{n-1} & \\ & a_\zeta\end{pmatrix}.$$

That is, we may write

$$\sum_{\lambda \in M_{l,n}(\mathfrak{o})} \mathbf{e_a}(2\mathrm{tr}\,({}^t\lambda S w(c_g\tau + d_g)^{-1}) + \mathrm{tr}\,(S[\lambda] g \cdot \tau)) = \Theta_{S,\Lambda_{a_\zeta}}(g_1 \cdot \tau, w(c_{g_1}\tau + d_{g_1})^{-1}),$$

where $\Lambda_{a_\zeta} := M_{l,n}(\mathfrak{o}) \begin{pmatrix} 1_{n-1} & \\ & a_\zeta \end{pmatrix}$ and $g = \zeta g_1$. Moreover, because $c_g = \begin{pmatrix} 1_{n-1} & \\ & a_\zeta^{-1} \end{pmatrix} c_{g_1}$,

$$\mathbf{e_a}(\mathrm{tr}\,(S[w](c_g\tau + d_g)^{-1}c_g)) = \mathbf{e_a}(\mathrm{tr}\,(S[w](c_{g_1}\tau + d_{g_1})^{-1}c_{g_1})).$$

Hence,

$$\boldsymbol{E}_\zeta(z,s) = \sum_{g \in Q_\zeta} \chi[g] j(g,\tau)^{-k}\delta(g\tau)^{s-k/2}\mathbf{e_a}(-\mathrm{tr}\,(S[w](c_{g_1}\tau + d_{g_1})^{-1}c_{g_1}))\Theta_{S,\Lambda_{a_\zeta}}(g_1 z).$$

We now set $\Gamma^\theta := \mathrm{Sp}_n(F) \cap D^\theta$, where $D^\theta := D[\mathfrak{b}^{-1}, \mathfrak{b}]$, if l is even, and $D^\theta := D[\mathfrak{b}^{-1}, \mathfrak{b}] \cap D[2\mathfrak{d}^{-1}, 2\mathfrak{d}]$ if l is odd. For $\gamma \in \Gamma^\theta$, $\tau \in \mathbb{H}^{\mathbf{a}}$ let $j(\gamma,\tau)^{1/2} := h(\gamma,\tau)$, where h is the half-integral factor of automorphy as defined for example in [24, page 180]. Then for l odd and $\gamma \in \Gamma^\theta$ we have

$$j(\gamma,\tau)^{l/2} = h(\gamma,\tau) j(\gamma,\tau)^{[l/2]}.$$

Therefore it makes sense to define

$$\Theta_{S,\Lambda_{a_\zeta}}(z)|_{S,l/2}\gamma := h(\gamma,\tau)^{-1} J_{S,[l/2]}(\gamma,z)^{-1}\Theta_{S,\Lambda_{a_\zeta}}(\gamma z), \quad \gamma \in \Gamma^\theta.$$

In fact, for a sufficiently deep subgroup Γ_{a_ζ} of finite index in $\Gamma_0(\mathfrak{b},\mathfrak{c})) \cap D^\theta$ we have that (see [24])

$$\Theta_{S,\Lambda_{a_\zeta}}(z)|_{S,l/2} g_1 = \psi_S(g_1)\Theta_{S,\Lambda_{a_\zeta}}(z), \quad \text{for all } g_1 \in \Gamma_{a_\zeta},$$

where ψ_S is the Hecke character of F corresponding to the extension $F(\det(2S)^{1/2})/F$ if l is odd, and to the extension $F((-1)^{l/4}\det(2S)^{1/2})/F$ if l is even.

Moreover, for every $g \in Q_\zeta$ such that $g = \zeta g_1$, $g_1 \in \Gamma_0(\mathfrak{b},\mathfrak{c})$, we have

$$\begin{aligned}\chi[g] j(g,\tau)^{-k}\delta(g\tau)^{s-k/2}&\mathbf{e_a}(-\mathrm{tr}\,(S[w](c_{g_1}\tau + d_{g_1})^{-1}c_{g_1}))\Theta_{S,\Lambda_{a_\zeta}}(g_1 z) \\ &= N_{F/\mathbb{Q}}(a_\zeta)^{l/2}\psi_S(a_\zeta)\phi[g] j(g,\tau)^{-(k-l/2)}\delta(g\tau)^{s-k/2}(\Theta_{S,\Lambda_{a_\zeta}}(z)|_{S,l/2} g_1),\end{aligned}$$

where $\phi := \chi\psi_S$, and we have used the fact that

$$j(g,\tau) = j(\zeta g_1,\tau) = j(\zeta, g_1 \cdot \tau) j(g_1,\tau) = N_{F/\mathbb{Q}}(a_\zeta)^{-1} j(g_1,\tau).$$

In particular, if we set $Q'_\zeta := \zeta\Gamma_{a_\zeta}$, we obtain

$$\boldsymbol{E}_\zeta(z,s) = N_{F/\mathbb{Q}}(a)^{l/2}\psi_S(a_\zeta) \sum_{\gamma \in \Gamma_{a_\zeta}\backslash\Gamma_0(\mathfrak{b},\mathfrak{c})} \overline{\chi[\gamma]}(E_\zeta(\tau, s - l/4)\Theta_{S,\Lambda_{a_\zeta}}(z))|_{S,k}\gamma,$$

where $E_\zeta(\tau,s) = \sum_{g \in Q'_\zeta} \phi[g] j(g,\tau)^{-(k-l/2)}\delta(g\tau)^{s-k/2+l/4}$ is a symplectic Eisenstein series of Siegel type of weight $k - l/2$. Since the above sum is finite, it follows that the series $\boldsymbol{E}_\zeta(z,s)$ has poles at most at the same places where $E_\zeta(\tau, s - l/4)$ may have.

Hence our focus now moves to detect the poles of the series $E_\zeta(\tau,s)$. Series of this form appear as summands of the classical (i.e. symplectic) Siegel Eisenstein series of some (perhaps half-integral) weight k and character χ, namely

$$E(\tau,\chi,s) := E(\tau,s) = \sum_{\zeta\in Z} N(\mathfrak{a}(\zeta))^{2s} \sum_{\gamma\in R_\zeta} \chi[\gamma]\delta(\tau)^{s-k/2}|_k\gamma,$$

where

$$E_\zeta(\tau,\chi,s) := E_\zeta(\tau,s) := \sum_{\gamma\in R_\zeta} \chi[\gamma]\delta(\tau)^{s-k/2}|_k\gamma.$$

The analytic properties of $E(\tau,s)$ are well known, and thus we may use them to derive similar properties for $E_\zeta(\tau,s)$.

We will use discrete Fourier analysis on the class group $Cl(F)$ of F. Recall that $Cl(F)\cong \mathbb{A}_F^\times/F^\times U$, where $U = F_\infty^\times \prod_v \mathfrak{o}_v^\times$. Moreover, we may pick the representatives $\mathfrak{a}(\zeta)$ for $Cl(F)$ in such a way that the ζ's form the set of representatives for the set Z (see [22, Lemma 3.2]).

Note that for any character χ and any character ψ of $Cl(F)$,

$$E(\tau,\chi\psi,s) = \sum_{\zeta\in Z}\psi(\zeta)N(\mathfrak{a}(\zeta))^{2s}\sum_{\gamma\in R_\zeta}\chi[\gamma]\delta(\tau)^{s-k/2}|_k\gamma = \sum_{\zeta\in Z}\psi(\zeta)N(\mathfrak{a}(\zeta))^{2s}E_\zeta(\tau,s),$$

that is, for every character ψ_i of $Cl(F)$,

$$E(\tau,\chi\psi_i,s) = \sum_{\zeta\in Z}\psi_i(\zeta)N(\mathfrak{a}(\zeta))^{2s}E_\zeta(\tau,s),\;\; i=1,2,\ldots,cl(F),$$

where $cl(F)$ denotes the cardinality of $Cl(F)$. Since the characters ψ_i are linearly independent over the group $Cl(F)$, we can solve the linear system of equations with respect to the unknowns $N(\mathfrak{a}(\zeta))^{2s}E_\zeta(\tau,s)$. In particular, the analytic properties of $E_\zeta(\tau,s)$ can be read off from the ones of $E(\tau,\chi\psi_i,s), i=1,2,\ldots,cl(F)$. Hence, since

$$\boldsymbol{E}_\zeta(z,s) = N_{F/\mathbb{Q}}(a)^{l/2}\sum_{\gamma\in\Gamma_{a_\zeta}\backslash\Gamma_0(\mathfrak{b},\mathfrak{c}))}(E_\zeta(\tau,s-l/4)\Theta_{S,\Lambda_{a_\zeta}}(z))|_{S,k}\gamma,$$

we see that the analytic properties of $\boldsymbol{E}$ can be obtained from those of $E(\tau,\chi\psi_i,s)$ for the various ψ_i's. To do that we will employ the following theorem of Shimura [24] on the analytic properties of symplectic Siegel type Eisenstein series, where

$$\Gamma_n(s) := \pi^{n(n-1)/4}\prod_{j=0}^{n-1}\Gamma(s-j/2).$$

Theorem 8.2 (Shimura, Theorem 16.11 in [24]). *For a weight $k\in\frac{1}{2}\mathbb{Z}^{\mathbf{a}}$ we define*

$$\mathcal{G}_{k,n}(s) := \prod_{v\in\mathbf{a}}\gamma(s,|k_v|),$$

where

$$\gamma(s,h) := \begin{cases} \Gamma\left(s+\frac{h}{2}-\left[\frac{2h+n}{4}\right]\right)\Gamma_n(s+\frac{h}{2}) & \text{if } n/2 \le h \in \mathbb{Z},\ n \in 2\mathbb{Z}, \\ \Gamma_n(s+\frac{h}{2}) & \text{if } n/2 < h \in \mathbb{Z},\ n \in 2\mathbb{Z}+1, \\ \Gamma_{2h+1}(s+\frac{h}{2})\prod_{i=h+1}^{[n/2]}\Gamma(2s-i) & \text{if } 0 \le h < n/2,\ h \in \mathbb{Z}, \\ \Gamma\left(s+\frac{h-1}{2}-\left[\frac{2h+n-2}{4}\right]\right)\Gamma_n(s+h/2) & \text{if } n/2 < h \notin \mathbb{Z},\ n \in 2\mathbb{Z}+1, \\ \Gamma_n(s+h/2) & \text{if } n/2 < h \in \mathbb{Z},\ n \in 2\mathbb{Z}, \\ \Gamma_{2h+1}(s+\frac{h}{2})\prod_{i=[h]+1}^{[(n-1)/2]}\Gamma(2s-i-\frac{1}{2}) & \text{if } 0 < h \le n/2,\ h \notin \mathbb{Z}. \end{cases}$$

We also set $\mathcal{E}(s) := \mathcal{G}(s)\Lambda^n_{k,\mathfrak{c}}(s,\chi)E(z,\chi,s)$, *where*

$$\Lambda^n_{k,\mathfrak{c}}(s,\chi) := \begin{cases} L_{\mathfrak{c}}(2s,\chi)\prod_{i=1}^{[n/2]} L_{\mathfrak{c}}(4s-2i,\chi^2) & \text{if } k \in \mathbb{Z}^{\mathbf{a}}, \\ \prod_{i=1}^{[(n+1)/2]} L_{\mathfrak{c}}(4s-2i+1,\chi^2) & \text{if } k \notin \mathbb{Z}^{\mathbf{a}}. \end{cases}$$

The function $\mathcal{E}(s)$ *has a meromorphic continuation to the whole of* $\mathbb{C}$ *and is entire if* $\chi^2 \neq 1$. *If* $\chi^2 = 1$, *we distinguish two cases:*

(1) *if* $\chi^2 = 1$ *and* $\mathfrak{c} \neq \mathfrak{o}$. *Set* $m := \max_{v\in\mathbf{a}}\{k_v\}$. *Then if* $m > n/2$, *the function* $\mathcal{E}(s)$ *has no poles except for a possible simple pole at* $s = (n+2)/4$, *which occurs only if* $2|k_v| - n \in 4\mathbb{Z}$ *for every* v *such that* $2|k_v| > n$. *If* $m \le n/2$, *then* $\mathcal{E}$ *has possible poles, which are all simple, in the set*

$$S_k^{(1)} := \begin{cases} \{j/2 : j \in \mathbb{Z}, [(n+3)/2] \le j \le n+1-m\} & \text{if } k \in \mathbb{Z}^{\mathbf{a}}, \\ \{(2j+1)/4 : j \in \mathbb{Z}, 1+[n/2] \le j \le n+1/2-m\} & \text{if } k \notin \mathbb{Z}^{\mathbf{a}}. \end{cases}$$

(2) *if* $\chi^2 = 1$, $\mathfrak{c} = \mathfrak{o}$, *and* $k \in \mathbb{Z}^{\mathbf{a}}$. *In this case each pole, which is simple, belongs to the set of poles described in (1) or to*

$$S_k^{(2)} := \{j/2 : j \in \mathbb{Z}, 0 \le j \le [n/2]\},$$

where $j = 0$ *is unnecessary if* $\chi \neq 1$.

We can now state a theorem regarding the analytic properties of the Eisenstein series $\boldsymbol{E}(z,\chi,s)$, which extends a previous theorem due to Heim [12, Theorem 4.1]. Recall that ψ_S is the Hecke character of F corresponding to the extension $F(\det(2S)^{1/2})/F$ if l is odd, and to the extension $F((-1)^{l/4}\det(2S)^{1/2})/F$ if l is even.

Theorem 8.3. *With notation as above, let*

$$\boldsymbol{\mathcal{E}}(s) := \mathcal{G}_{k-l/2,n}(s-l/4)\Lambda^n_{k-l/2,\mathfrak{c}}(s-l/4,\chi\psi_S)\boldsymbol{E}(z,\chi,s).$$

The function $\boldsymbol{\mathcal{E}}$ *has a meromorphic continuation to the whole of* $\mathbb{C}$, *and its poles are caused by the functions*

$$\frac{\Lambda^n_{k-l/2,\mathfrak{c}}(s-l/4,\chi\psi_S)}{\Lambda^n_{k-l/2,\mathfrak{c}}(s-l/4,\chi\psi_S\psi_i)},\quad i = 1,\dots,cl(F).$$

These poles may appear only when F *has class number larger than one and* $supp(\mathfrak{c}) \neq supp(cond(\chi\psi_S))$. *More precisely:*

(1) *Assume that* $\chi^2\psi_i^2 \neq 1$ *for all* $i = 1,\dots,cl(F)$. *Then* $\boldsymbol{\mathcal{E}}(s)$ *has no extra poles.*

(2) Assume that there exist ψ_i such that $\chi^2\psi_i^2 = 1$. Then we consider the following cases.

(a) $\mathfrak{c} \neq \mathfrak{o}$. Set $m := \max_{v\in\mathbf{a}}\{k_v - l/2\}$. If $m > n/2$, then the function $\boldsymbol{\mathcal{E}}(s)$ has no extra poles except for a possible simple pole at $s = (n+2)/4$, which occurs only if $2|k_v - l/2| - n \in 4\mathbb{Z}$ for every v such that $2|k_v - l/2| > n$. If $m \leq n/2$, then all possible poles of $\mathcal{E}$ are simple and belong to the set $S^{(1)}_{k-l/2}$.

(b) $\mathfrak{c} = \mathfrak{o}$, and $k - l/2 \in \mathbb{Z}^{\mathbf{a}}$. In this case each extra pole is simple and belongs to the set described in (a) or to

$$S^{(2)}_{k-l/2} := \{j/2 : j \in \mathbb{Z}, [0 \leq j \leq [n/2]\},$$

where $j = 0$ is unnecessary if $\chi\psi \neq 1$.

Before we proceed to the proof of the theorem we recall the following fact regarding zeros of Dirichlet series. For a Hecke character ψ of F and an integral ideal $\mathfrak{c}$ we considered the series

$$L_{\mathfrak{c}}(s,\psi) := \prod_{\mathfrak{q}|\mathfrak{c}}(1 - \psi(\mathfrak{q})N(\mathfrak{q})^{-s})L(s,\psi)$$

with functional equation

$$\prod_{v\in\mathbf{a}}\Gamma((s+t_v)/2)L(s,\psi) = W(\psi,s)\prod_{v\in\mathbf{a}}\Gamma((1-s+t_v)/2)L(1-s,\psi),$$

where $W(\psi,s)$ is a non-vanishing holomorphic function, and $t_v \in \{0,1\}$ is the infinite type of the character. It is well known that if $\psi \neq 1$, then $L(s,\psi) \neq 0$ for $\mathrm{Re}(s) \geq 1$, and $\prod_{v\in\mathbf{a}}\Gamma((s+k_v)/2)L(s,\psi)$ is entire. If $\psi = 1$, then this function is meromorphic with simple poles at $s = 0$ and $s = 1$, and $L(s,\psi) \neq 0$ for $\mathrm{Re}(s) > 1$.

The absolute convergence and the functional equation imply that if two non-trivial characters ψ_1 and ψ_2 have the same infinite type, then the zeros of $L(s,\psi_1)$ and $L(s,\psi_2)$ as well as their orders are the same at the integers of the real axis. Namely, for any $0 \leq m \in \mathbb{Z}$, $L(-m,\psi_1) = L(-m,\psi_2) = 0$ if and only if there exists $v \in \mathbf{a}$ such that $\psi_1(x_v) = \psi_2(x_v) = \mathrm{sgn}(x_v)^m$. Moreover, the order of the zero equals precisely the number of places where this is happening. In particular, the function

$$\frac{L_{\mathfrak{c}}(s,\psi_1)}{L_{\mathfrak{c}}(s,\psi_2)} = \left(\prod_{\mathfrak{q}|\mathfrak{c}}\frac{(1-\psi_1(\mathfrak{q})N(\mathfrak{q})^{-s})}{(1-\psi_2(\mathfrak{q})N(\mathfrak{q})^{-s})}\right)\frac{L(s,\psi_1)}{L(s,\psi_2)}$$

may have poles only at the integers where $\prod_{\mathfrak{q}|\mathfrak{c}}\frac{(1-\psi_1(\mathfrak{q})N(\mathfrak{q})^{-s})}{(1-\psi_2(\mathfrak{q})N(\mathfrak{q})^{-s})}$ has poles.

If the characters $\psi_1 = 1$ and ψ_2 have trivial type at infinity, then the same argument as above shows that the function

$$\frac{L_{\mathfrak{c}}(s,\psi_1)}{L_{\mathfrak{c}}(s,\psi_2)}$$

may have poles at the integers where the function $\prod_{\mathfrak{q}|\mathfrak{c}}\frac{(1-\psi_1(\mathfrak{q})N(\mathfrak{q})^{-s})}{(1-\psi_2(\mathfrak{q})N(\mathfrak{q})^{-s})}$ has poles. However, this time there may be an additional zero also at $s = 0$. This is because at this point the order of vanishing of $L(s,\psi_1)$ is smaller by one from the order of vanishing of $L(s,\psi_2)$.

Proof of Theorem 8.3. First note that since ψ_i's are the characters of $Cl(F) \equiv \mathbb{A}_F^\times/F^\times U$, where $U = F_{\mathbf{a}}^\times \prod_v \mathfrak{o}_v^\times$, their signature is trivial, that is, $\psi_{i\infty}(x) = 1$ for all $x \in F_{\mathbf{a}}^\times$. In particular, the characters $\chi\psi_S$ and $\chi\psi_S\psi_i$, $i = 1,\dots,cl(F)$, have the same signature at infinity. The discussion above implies that the functions $\Lambda^n_{k-l/2,\mathfrak{c}}(s-l/2,\chi\psi_S)$ and $\Lambda^n_{k-l/2,\mathfrak{c}}(s-l/2,\chi\psi_S\psi_i)$ have the same zeros on the integers at the real line, and the ratio

$$\frac{\Lambda^n_{k-l/2,\mathfrak{c}}(s-l/4,\chi\psi_S)}{\Lambda^n_{k-l/2,\mathfrak{c}}(s-l/4,\chi\psi_S\psi_i)}$$

may have poles in cases indicated in the theorem. However, then (Theorem 8.2) the series

$$\frac{\Lambda^n_{k-l/2,\mathfrak{c}}(s-l/4,\chi\psi_S)}{\Lambda^n_{k-l/2,\mathfrak{c}}(s-l/2,\chi\psi_S\psi_i)}\mathcal{G}_{k-l/2,n}(s-l/4)\Lambda^n_{k-l/2,\mathfrak{c}}(s-l/4,\chi\psi_S\psi_i)E(\tau,\chi\psi_i\psi_S,s-l/4)$$

does not have any more poles unless $\chi^2\psi_i^2 = 1$ for some i, in which case the poles are as described in the theorem. □

Remark 8.4. The analytic properties of Jacobi Eisenstein series presented in Theorem 8.3 were obtained from the well-studied symplectic Eisenstein series via establishing the link between these two kinds of Eisenstein series. However, perhaps one could also try to use the results of Arakawa in [2] on the Fourier coefficients of Jacobi Eisenstein series.

9. Analytic continuation of the standard L-function

We are now ready to establish two main theorems regarding the analytic properties of the standard L-function and the Klingen-type Jacobi Eisenstein series. The approach taken here can be regarded as an extension from the symplectic to the Jacobi setting of the method utilized in [22].

We keep the notation introduced at the beginning of section 7 and additionally we define groups

$$\boldsymbol{D}' := \{(\lambda,\mu,\kappa)x \in C[\mathfrak{o},\mathfrak{b}^{-1},\mathfrak{b}^{-1}]D[\mathfrak{b}^{-1}\mathfrak{c},\mathfrak{b}\mathfrak{e}] : (a_x - 1_n)_v \in M_n(\mathfrak{e}_v) \text{ for every } v|\mathfrak{e}\},$$

$$\boldsymbol{\Gamma}' := \boldsymbol{G}^n(F) \cap \boldsymbol{D}'$$

and

$$R(\mathfrak{e},\mathfrak{c}) := \{\mathrm{diag}[\tilde{q},q] : q \in Q(\mathfrak{e}), q_v \in M_n(\mathfrak{c}_v) \text{ for every } v|\mathfrak{e}^{-1}\mathfrak{c}\}.$$

For $\mathrm{diag}[\tilde{q},q] \in R(\mathfrak{e},\mathfrak{c})$ and $f \in M^n_{k,S}(\boldsymbol{\Gamma},\psi)$, in a manner similar to $f|T_{r,\psi}$, we define

$$f|U_{q,\psi} := \sum_{\boldsymbol{\beta}\in\boldsymbol{B}} \psi_{\mathfrak{c}}(\det(a_\beta)_{\mathfrak{c}})^{-1} f|_{k,S}\boldsymbol{\beta}, \tag{27}$$

where $\boldsymbol{B} \subset \boldsymbol{G}^n(F)$ is such that $\boldsymbol{G}^n(F) \cap \boldsymbol{D}\mathrm{diag}[\tilde{q},q]\boldsymbol{D}' = \coprod_{\boldsymbol{\beta}\in\boldsymbol{B}} \boldsymbol{\Gamma}\boldsymbol{\beta}$. As in section 7, if we write $\mathbf{f}|U_{q,\psi}$ for the adelic Jacobi form associated to $f|U_{q,\psi}$ (with $\boldsymbol{g} = 1$) and

$\boldsymbol{D}\mathrm{diag}[\tilde{q}, q]\boldsymbol{D}' = \coprod_{\boldsymbol{\beta}\in\boldsymbol{B}} \boldsymbol{D}\boldsymbol{\beta}$ with $\boldsymbol{B} \subset \boldsymbol{G}_{\mathbf{h}}$, then

$$(\mathbf{f}|U_{q,\psi})(x) = \sum_{\boldsymbol{\beta}\in\boldsymbol{B}} \psi_{\mathfrak{c}}(\det(a_\beta)_{\mathfrak{c}})^{-1}\mathbf{f}(x\boldsymbol{\beta}^{-1}), \qquad x \in \boldsymbol{G}^n(\mathbb{A}).$$

For the rest of this section we assume that $0 \neq f \in S^n_{k,S}(\boldsymbol{\Gamma}, \psi)$ is an eigenfunction of $T_\psi(\mathfrak{a})$ for every $\mathfrak{a}$ with eigenvalues $\lambda(\mathfrak{a})$. Note that $T_\psi(\mathfrak{a}) \neq 0$ only if $\mathfrak{a}$ is coprime to $\mathfrak{e}$.

We start with a version of [22, Lemma 6.2] for Hecke operators in our Jacobi setting.

Lemma 9.1. *Let h be an element of $\mathbb{A}^\times_{\mathbf{h}}$ such that its corresponding ideal is $\mathfrak{e}^{-1}\mathfrak{c}$ and $h_v = 1$ for $v \nmid \mathfrak{e}^{-1}\mathfrak{c}$. Then $U_{hr,\psi} = T_{r,\psi}U_{h1_n,\psi}$ for every $r \in Q(\mathfrak{e})$. Moreover, for $f \in M^n_{k,S}(\boldsymbol{\Gamma}, \psi)$ we have $f|T_{h1_n,\psi} \neq 0$ only if $f|U_{h1_n,\psi} \neq 0$.*

Proof. To prove the first statement it suffices to show that

$$\boldsymbol{D}\begin{pmatrix} h^{-1}\tilde{r} & \\ & hr\end{pmatrix}\boldsymbol{D}' = \boldsymbol{D}\begin{pmatrix} \tilde{r} & \\ & r\end{pmatrix}\boldsymbol{D}\cdot\boldsymbol{D}\begin{pmatrix} h^{-1}1_n & \\ & h1_n\end{pmatrix}\boldsymbol{D}'.$$

This may be done place by place. As we established in (25),

$$\boldsymbol{D}_v\begin{pmatrix}\tilde{r}_v & \\ & r_v\end{pmatrix}\boldsymbol{D}_v = \coprod_{d,b,\mu}\boldsymbol{D}_v(0,\mu,0)\begin{pmatrix}\tilde{d} & \tilde{d}b\\ & d\end{pmatrix}$$

at each place $v|\mathfrak{c}$, where $d \in \mathrm{GL}_n(\mathfrak{o}_v)\backslash\mathrm{GL}_n(\mathfrak{o}_v)r_v\mathrm{GL}_n(\mathfrak{o}_v)$, $b \in Sym_n(\mathfrak{b}_v^{-1})/{}^t dSym_n(\mathfrak{b}_v^{-1})d$ and $\mu \in M_{l,n}(\mathfrak{b}_v^{-1})d^{-1}/M_{l,n}(\mathfrak{b}_v^{-1})$. Using the same argument and a double coset decomposition for symplectic groups, we get

$$\boldsymbol{D}_v\begin{pmatrix}h_v^{-1}\tilde{r}_v & \\ & h_vr_v\end{pmatrix}\boldsymbol{D}'_v = \coprod_{d_1,b_1,\upsilon_1}\boldsymbol{D}_v(0,\upsilon_1,0)\begin{pmatrix}\tilde{d}_1 & \tilde{d}_1b_1\\ & d_1\end{pmatrix},$$

where $d_1 \in \mathrm{GL}_n(\mathfrak{o}_v)\backslash\mathrm{GL}_n(\mathfrak{o}_v)h_vr_v\mathrm{GL}_n(\mathfrak{o}_v)$, $b_1 \in Sym_n(\mathfrak{b}_v^{-1}\mathfrak{c}_v)/{}^t d_1Sym_n(\mathfrak{b}_v^{-1})d_1$ and $\upsilon_1 \in M_{l,n}(\mathfrak{b}_v^{-1})d_1^{-1}/M_{l,n}(\mathfrak{b}_v^{-1})$. In particular, if we take $r = 1_n$ and a coset decomposition over d_2, b_2, υ_2, then we can take $d_2 = h_v1_n$ and it is easy to see that the set

$$\begin{aligned}\{(0,\mu,0)&\begin{pmatrix}\tilde{d} & \tilde{d}b\\ & d\end{pmatrix}(0,\upsilon_2,0)\begin{pmatrix}h_v^{-1}1_n & h_v^{-1}b_2\\ & h_v1_n\end{pmatrix} : \mu,\upsilon_2,b,b_2,d\}\\ &= \{(0,\mu+\upsilon_2d^{-1},0)\begin{pmatrix}h_v^{-1}\tilde{d} & h_v^{-1}\tilde{d}(b_2+h_v^2b)\\ & h_vd\end{pmatrix} : \mu,\upsilon_2,b,b_2,d\}\end{aligned}$$

represents $\boldsymbol{D}_v\backslash(\boldsymbol{D}\begin{pmatrix}h^{-1}\tilde{r} & \\ & hr\end{pmatrix}\boldsymbol{D}')_v$ for each $v|\mathfrak{c}$.

To prove the second statement we use Proposition 3.4. We recall that the Siegel-Jacobi modular form f and its adelic counterpart are related by $\mathbf{f}(\boldsymbol{y}) = J_{k,S}(\boldsymbol{y},\boldsymbol{i}_0)^{-1}f(\boldsymbol{y}\cdot\boldsymbol{i}_0)$, for every $\boldsymbol{y} \in \boldsymbol{G}_{\mathbf{a}}$. Moreover, recall that the symmetric space $\mathcal{H}_{n,l}$ is contained in $\{\boldsymbol{y}\cdot\boldsymbol{i}_0 : \boldsymbol{y} \in \boldsymbol{G}_{\mathbf{a}}$ of the form $(\lambda,\mu,0)\begin{pmatrix}q & \sigma\tilde{q}\\ & \tilde{q}\end{pmatrix}\}$.

For an $\boldsymbol{\alpha}$ of the form $(0,\nu,0)\begin{pmatrix}h^{-1}1_n & h^{-1}b\\ & h1_n\end{pmatrix}$, with $\nu_{\mathbf{a}} = 0$, $b_{\mathbf{a}} = 0$, and $\boldsymbol{y} \in \boldsymbol{G}(\mathbb{A})$ such that $\boldsymbol{y}_{\mathbf{h}} = (0,0,0)1_{2n}$ and $\boldsymbol{y}_{\mathbf{a}}$ as above, we have

$$\begin{aligned}\boldsymbol{y}\boldsymbol{\alpha}^{-1} &= (\lambda,\mu,0)(0,-h\nu\,{}^tq,0)\begin{pmatrix}hq & h^{-1}(-qb+\sigma\tilde{q})\\ & h^{-1}\tilde{q}\end{pmatrix}\\ &= (\lambda,\mu-h\nu\,{}^tq,-h\lambda q\,{}^t\nu-h\nu\,{}^tq\,{}^t\lambda)\begin{pmatrix}hq & (-qb\,{}^tq+\sigma)h^{-1}\tilde{q}\\ & h^{-1}\tilde{q}\end{pmatrix},\end{aligned}$$

and thus by the expansion (6),

$$\mathbf{f}(\boldsymbol{y}\boldsymbol{\alpha}^{-1}) = \sum_{t,r} c(t,r;hq,\lambda)\mathbf{e}_{\mathbb{A}}(\mathrm{tr}\,(t\sigma - tqb\,{}^tq))\mathbf{e}_{\mathbb{A}}(\mathrm{tr}\,({}^tr(\lambda\sigma - \lambda qb\,{}^tq + \mu - h\nu\,{}^tq))) =$$

$$\sum_{t,r} c(t,r;hq,\lambda)\left(\prod_{v|\mathfrak{c}} \mathbf{e}_v(\mathrm{tr}\,(-tq_vb_v\,{}^tq_v))\right)\left(\prod_{v|\mathfrak{c}} \mathbf{e}_v(\mathrm{tr}\,({}^tr(-h_v\nu_v\,{}^tq_v)))\right)\mathbf{e}_{\mathbf{a}}(\mathrm{tr}\,(t\sigma + {}^tr(\lambda\sigma+\mu))).$$

Hence,

$$\mathbf{f}|T_{h1_n,\psi}(\boldsymbol{y}) = \sum_{b,\nu}\psi_{\mathfrak{c}}(h_{\mathfrak{c}})^n \sum_{t,r} c(t,r;hq,\lambda)\left(\prod_{v|\mathfrak{c}} \mathbf{e}_v(\mathrm{tr}\,(-tq_vb_v\,{}^tq_v + {}^tr(-h_v\nu_v\,{}^tq_v)))\right)$$
$$\cdot\,\mathbf{e}_{\mathbf{a}}(\mathrm{tr}\,(t\sigma + {}^tr(\lambda\sigma+\mu))),$$

where $b \in \prod_{v|\mathfrak{c}} Sym_n(\mathfrak{b}_v^{-1})/h_v^2 Sym_n(\mathfrak{b}_v^{-1})$, and $\nu \in \prod_{v|\mathfrak{c}} M_{l,n}(\mathfrak{b}_v^{-1})h_v^{-1}/M_{l,n}(\mathfrak{b}_v^{-1})$. That is, if we write $c(\mathbf{f}|T_{h1_n,\psi};t,r;q,\lambda)$ for the (t,r)-coefficient of $\mathbf{f}|T_{h1_n,\psi}$, we have

$$c(\mathbf{f}|T_{h1_n,\psi};t,r;q,\lambda) = \psi_{\mathfrak{c}}(h_{\mathfrak{c}})^n \sum_{b,\nu}\left(\prod_{v|\mathfrak{c}} \mathbf{e}_v(\mathrm{tr}\,(-tq_vb_v\,{}^tq_v + {}^tr(-h_v\nu_v\,{}^tq_v)))\right).$$

Therefore, if

$$\mathbf{e}_{\mathbf{h}}(\mathrm{tr}\,({}^tqtqh^{-2}Sym_n(\mathfrak{b}^{-1}))) = 1 \quad \text{and} \quad \mathbf{e}_{\mathbf{h}}(\mathrm{tr}\,({}^tq\,{}^trM_{l,n}(\mathfrak{b}^{-1}))) = 1,$$

then

$$c(\mathbf{f}|T_{h1_n,\psi};t,r;q,\lambda) = N(\mathfrak{e}^{-1}\mathfrak{c})^{n(l+n+1)}\psi_{\mathfrak{c}}(h_{\mathfrak{c}})^n c(t,r;hq,\lambda),$$

otherwise $c(\mathbf{f}|T_{h1_n,\psi};t,r;q,\lambda) = 0$.

Arguing exactly in the same way we can also conclude that if both

$$\mathbf{e}_{\mathbf{h}}(\mathrm{tr}\,({}^tqtqh^{-2}Sym_n(\mathfrak{b}^{-1}\mathfrak{c}))) = 1 \quad \text{and} \quad \mathbf{e}_{\mathbf{h}}(\mathrm{tr}\,({}^tq\,{}^trM_{l,n}(\mathfrak{b}^{-1}))) = 1,$$

then

$$c(\mathbf{f}|U_{h1_n,\psi};t,r;q,\lambda) = N(\mathfrak{e}^{-1}\mathfrak{c})^{nl+n(n+1)/2}\psi_{\mathfrak{c}}(h_{\mathfrak{c}})^n c(t,r;hq,\lambda),$$

otherwise $c(\mathbf{f}|U_{h1_n,\psi};t,r;q,\lambda) = 0$, where we write $c(\mathbf{f}|U_{h1_n,\psi};t,r;q,\lambda)$ for the (t,r)-coefficient of $\mathbf{f}|U_{h1_n,\psi}$.

Hence, if $\mathbf{f}|U_{h1_n,\psi} = 0$, then $c(\mathbf{f}|U_{h1_n,\psi};t,r;q,\lambda) = 0$ for all t,r. In particular, if for a pair t,r both $\mathbf{e}_{\mathbf{h}}(\mathrm{tr}\,({}^tqtqh^{-2}Sym_n(\mathfrak{b}^{-1}\mathfrak{c}))) = 1$ and $\mathbf{e}_{\mathbf{h}}(\mathrm{tr}\,({}^tq\,{}^trM_{l,n}(\mathfrak{b}^{-1}))) = 1$, then $c(t,r;hq,\lambda) = 0$ and hence also $c(\mathbf{f}|T_{h1_n,\psi};t,r;q,\lambda) = 0$. If on the other hand for a pair t,r either $\mathbf{e}_{\mathbf{h}}(\mathrm{tr}\,({}^tqtqh^{-2}Sym_n(\mathfrak{b}^{-1}\mathfrak{c}))) \neq 1$ or $\mathbf{e}_{\mathbf{h}}(\mathrm{tr}\,({}^tq\,{}^trM_{l,n}(\mathfrak{b}^{-1}))) \neq 1$, then also either $\mathbf{e}_{\mathbf{h}}(\mathrm{tr}\,({}^tqtqh^{-2}Sym_n(\mathfrak{b}^{-1}))) \neq 1$ (since $Sym_n(\mathfrak{b}^{-1}\mathfrak{c}) \subset Sym_n(\mathfrak{b}^{-1})$) or $\mathbf{e}_{\mathbf{h}}(\mathrm{tr}\,({}^tq\,{}^trM_{l,n}(\mathfrak{b}^{-1}))) \neq 1$, which also implies that $c(\mathbf{f}|T_{h1_n,\psi};t,r;q,\lambda) = 0$. Therefore $\mathbf{f}|T_{h1_n,\psi} = 0$. $\square$

We now fix uniformizers $\pi_v \in \mathfrak{o}_v$ for every finite place v in the support of $\mathfrak{e}$. Then for a fractional ideal $\mathfrak{t}$ we pick $t \in \mathbb{A}_{\mathbf{h}}^{\times}$, such that $\mathfrak{t}$ is the ideal corresponding to the idele t, and at every place $v|\mathfrak{e}$ we have $t_v = \pi_v^{\mathrm{ord}_v(\mathfrak{t})}$, where $\mathrm{ord}_v(\cdot)$ is the usual valuation at the place v. Further, we set $\boldsymbol{\tau} := 1_H\mathrm{diag}[t^{-1}1_n, t1_n]$ and define an isomorphism

$$I_{\mathfrak{t}}\colon \mathcal{M}_{k,S}^n(\boldsymbol{D},\psi) \to \mathcal{M}_{k,S}^n(\boldsymbol{\tau}^{-1}\boldsymbol{D}\boldsymbol{\tau},\psi), \quad \mathbf{f}|I_{\mathfrak{t}}(x) := \psi(t^n)\mathbf{f}(x\boldsymbol{\tau}^{-1}) \quad (x \in \boldsymbol{G}^n(\mathbb{A})).$$

Lemma 9.2. *The map $I_{\mathfrak{t}}$ has the following properties:*

1. *it is independent of the choice of $\mathfrak{t}$,*
2. *it commutes with the operators $T_{r,\psi}$ and $U_{q,\psi}$,*
3. *$(f|I_{\mathfrak{t}})^c = f^c|I_{\mathfrak{t}}$, where f is the Siegel-Jacobi form corresponding to $\mathbf{f}$.*

Proof. (1) If $t' \in \mathbb{A}_{\mathbf{h}}^{\times}$ is another idele that corresponds to the ideal $\mathfrak{t}$, then $t = t'l$ for some $l \in \prod_{v\in\mathbf{h}} \mathfrak{o}_v^{\times}$.

$$\begin{aligned}\psi(t^n)\mathbf{f}(x\boldsymbol{\tau}^{-1}) &= \psi((lt')^n)\mathbf{f}(x\mathrm{diag}[t'1_n, t'^{-1}1_n]\mathrm{diag}[l1_n, l^{-1}1_n]1_H)\\ &= \psi(t'^n)\mathbf{f}(x\mathrm{diag}[t'1_n, t'^{-1}1_n]1_H),\end{aligned}$$

where we have used the fact that $\mathrm{diag}[l1_n, l^{-1}1_n] \in \boldsymbol{D}$ since $l_v = 1$ if $v|\mathfrak{e}$.

(2) This follows from direct computation, e.g. in case of $T_{r,\psi}$:

$$\boldsymbol{\tau}^{-1}\boldsymbol{D}\mathrm{diag}[\tilde{r}, r]\boldsymbol{D} = \boldsymbol{D}_{\mathfrak{t}}\mathrm{diag}[\tilde{r}, r]\boldsymbol{D}_{\mathfrak{t}}\boldsymbol{\tau}^{-1},$$

where

$$\boldsymbol{D}_{\mathfrak{t}} := \{(\lambda, \mu, \kappa)x \in C[\mathfrak{t}^{-1}, \mathfrak{t}\mathfrak{b}^{-1}, \mathfrak{t}\mathfrak{b}^{-1}]D[\mathfrak{b}^{-1}\mathfrak{e}\mathfrak{t}^2, \mathfrak{b}\mathfrak{c}\mathfrak{t}^{-2}] \colon (a_x - 1_n)_v \in M_n(\mathfrak{e}_v) \text{ for } v|\mathfrak{e}\}.$$

(3) By strong approximation we may write $\tau = \boldsymbol{\gamma}\boldsymbol{d}$ for some $\boldsymbol{\gamma} \in \boldsymbol{G}(F)$ and $\boldsymbol{d} \in \boldsymbol{D}$. We moreover notice that since $\boldsymbol{\tau}$ has no Heisenberg part we may take $\boldsymbol{\gamma} = \gamma \in G(F) \hookrightarrow \boldsymbol{G}(F)$, and $\boldsymbol{d} \in D \hookrightarrow \boldsymbol{D}$. Furthermore, for $\epsilon := \mathrm{diag}[1_n, -1_n]$, $\epsilon\boldsymbol{\tau}\epsilon^{-1} = \epsilon\gamma\epsilon^{-1}\epsilon\boldsymbol{d}\epsilon^{-1}$ as elements of $G(F)$. Note that $\epsilon\boldsymbol{d}\epsilon^{-1} \in \boldsymbol{D}$ and $\epsilon\boldsymbol{\tau}\epsilon^{-1} = \boldsymbol{\tau}$. Clearly, without loss of generality we may assume that $\psi = 1$. Then $(f|I_{\mathfrak{t}})^c = (f|_{k,S}\boldsymbol{\gamma})^c = f^c|_{k,S}\epsilon\gamma\epsilon^{-1} = f^c|I_{\mathfrak{t}}$, where for the second equality we have used Proposition 7.9.

□

Let χ be a Hecke character as in subsection 4.1 and assume that $\chi = \psi$ on $\prod_{v\nmid\mathfrak{c}} \mathfrak{o}_v^{\times}$. Then $\mathcal{S}_{k,S}^n(\boldsymbol{D}, \psi) = \mathcal{S}_{k,S}^n(\boldsymbol{D}, \chi)$ since the nebentype depends only on the finite places that divide $\mathfrak{c}$ and is trivial on places that divide $\mathfrak{e}$ ($\det(a_g) \equiv 1 \bmod \mathfrak{e}_v$ for $hg \in \boldsymbol{D}$). Moreover, the Hecke operators are related via:

$$(\chi/\psi)^*(\mathfrak{a})\psi^*(\mathfrak{a}')T_{\psi}(\mathfrak{a}) = \chi^*(\mathfrak{a}')T_{\chi}(\mathfrak{a}), \quad (\chi/\psi)^*(\mathfrak{e}^{-1}\mathfrak{c})^n U_{h1_n,\psi} = U_{h1_n,\chi},$$

where $\mathfrak{a}' := \prod_{v\nmid\mathfrak{c}} \mathfrak{a}_v$. Put $\boldsymbol{\tau} := 1_H\mathrm{diag}[\theta^{-1}1_n, \theta 1_n]$ with θ as in Lemma 5.3. Then the set $\boldsymbol{Y}_v$ is equal to the set $(\boldsymbol{\tau}^{-1}\boldsymbol{D}R(\mathfrak{e},\mathfrak{c})\boldsymbol{D}'\boldsymbol{\tau})_v$ at every place v. Put

$$\Delta(q) := \left(\boldsymbol{G}^n(F) \cap \boldsymbol{\tau}^{-1}\boldsymbol{D}\boldsymbol{\tau}\right) \backslash \left(\boldsymbol{G}^n(F) \cap \boldsymbol{G}_{\mathbf{a}}^n \prod_{v\in\mathbf{h}} (\boldsymbol{\tau}^{-1}\boldsymbol{D}\left(\begin{smallmatrix}\tilde{q} & \\ & q\end{smallmatrix}\right)\boldsymbol{D}'\boldsymbol{\tau})_v\right).$$

For $f \in S_{k,S}^n(\boldsymbol{\Gamma}, \psi)$ such that $f|T_{\psi}(\mathfrak{a}) = \lambda(\mathfrak{a})f$ and for $\mathcal{D}$ defined as in (20) we have:

$$\begin{aligned}\mathcal{D}(z,s,f|I_{\mathfrak{b}}) &= \sum_{\boldsymbol{\xi}\in\boldsymbol{Y}} \ell'(\xi)^{-s}\chi^*(\ell_1'(\xi))\chi_{\mathfrak{c}}(\det(a_{\xi}))^{-1}(f|I_{\mathfrak{b}})|_{k,S}\boldsymbol{\xi}(z)\\ &= \sum_{q\in R(\mathfrak{e},\mathfrak{c})}\sum_{\boldsymbol{\beta}\in\Delta(q)} N(\det(q)\mathfrak{o})^{-s}\chi^*(\prod_{v\nmid\mathfrak{c}}(\det(q)\mathfrak{o})_v)\chi_{\mathfrak{c}}(\det(a_{\beta}))^{-1}(f|I_{\mathfrak{b}})|_{k,S}\boldsymbol{\beta}(z)\\ &\overset{Lemma\,9.1}{=} N(\mathfrak{e}^{-1}\mathfrak{c})^{-ns}\sum_{\mathfrak{a}} N(\mathfrak{a})^{-s}\chi^*(\mathfrak{a}')(f|I_{\mathfrak{b}})|T_{\chi}(\mathfrak{a})U_{h1_n,\chi}(z)\end{aligned}$$

$$= N(\mathfrak{e}^{-1}\mathfrak{c})^{-ns}\sum_{\mathfrak{a}} N(\mathfrak{a})^{-s}(\chi/\psi)^*(\mathfrak{a})\psi^*(\mathfrak{a}')\lambda(\mathfrak{a})f|U_{h1_n,\chi}I_{\mathfrak{b}}(z).$$

Joining the above formula for $\mathcal{D}(z,s,f|I_{\mathfrak{b}})$ together with (21), after setting $f^c|I_{\mathfrak{b}}$ for f there, we obtain

$$N(\mathfrak{b}\mathfrak{e}^{-1}\mathfrak{c})^{2ns}\chi_{\mathbf{h}}(\theta)^{-n}(-1)^{n(s-k/2)}vol(A) < (E|_{k,S}\boldsymbol{\rho})(\mathrm{diag}[z_1,z_2],s),(f^c|I_{\mathfrak{b}})(z_2) >$$
$$= \nu_{\mathfrak{e}}c_{S,k}(s-k/2)E(z_1,s;(f|U_{h1_n,\chi}I_{\mathfrak{b}})|_{k,S}\boldsymbol{\eta}_n^{-1})\sum_{\mathfrak{a}} N(\mathfrak{a})^{-2s}(\chi/\psi)^*(\mathfrak{a})\psi^*(\mathfrak{a}')\lambda(\mathfrak{a}),$$

where we have used the fact that $(f^c|I_{\mathfrak{b}})^c = f|I_{\mathfrak{b}}$.

After multiplying both sides of the above equation with $\mathcal{G}_{k-l/2,n+m}(s-l/4)\Lambda^{n+m}_{k-l/2,\mathfrak{c}}(s-l/4,\chi\psi_S)$ with notation as in Theorem 8.3 and setting $\mathcal{E}(z,s) := \mathcal{G}_{k-l/2,n+m}(s-l/4)\Lambda^{n+m}_{k-l/2,\mathfrak{c}}(s-l/4,\chi\psi_S)E(z,s)$, we obtain

$$N(\mathfrak{b}\mathfrak{e}^{-1}\mathfrak{c})^{2ns}\chi_{\mathbf{h}}(\theta)^{-n}(-1)^{n(s-k/2)}vol(A) < (\mathcal{E}|_{k,S}\boldsymbol{\rho})(\mathrm{diag}[z_1,z_2],s),(f^c|I_{\mathfrak{b}})(z_2) >$$
$$= \nu_{\mathfrak{e}}c_{S,k}(s-k/2)\mathcal{G}_{k-l/2,n+m}(s-l/4)E(z_1,s;(f|U_{h1_n,\chi}I_{\mathfrak{b}})|_{k,S}\boldsymbol{\eta}_n^{-1})$$
$$\cdot\Lambda^{n+m}_{k-l/2,\mathfrak{c}}(s-l/4,\chi\psi_S)\sum_{\mathfrak{a}} N(\mathfrak{a})^{-2s}(\chi/\psi)^*(\mathfrak{a})\psi^*(\mathfrak{a}')\lambda(\mathfrak{a}),$$

where we recall that

$$\Lambda^{n+m}_{k-l/2,\mathfrak{c}}(s,\chi\psi_S) := \begin{cases} L_{\mathfrak{c}}(2s-l/2,\chi\psi_S)\prod_{i=1}^{[(n+m)/2]} L_{\mathfrak{c}}(4s-l-2i,\chi^2) & \text{if } l\in 2\mathbb{Z},\\ \prod_{i=1}^{[(n+m+1)/2]} L_{\mathfrak{c}}(4s-l-2i+1,\chi^2) & \text{if } l\notin 2\mathbb{Z}.\end{cases}$$

By the discussion in subsection 7.3, we have that

$$\mathfrak{L}_{\psi}(\chi\psi^{-1},2s-n-l/2)\sum_{\mathfrak{a}} N(\mathfrak{a})^{-2s}(\chi/\psi)^*(\mathfrak{a})\psi^*(\mathfrak{a}')\lambda(\mathfrak{a}) = L_{\psi}(2s-n-l/2,\mathbf{f},\chi\psi^{-1})$$

with $\mathfrak{L}_{\psi}(\chi\psi^{-1},2s-n-l/2) = \prod_{(\mathfrak{p},\mathfrak{c})=1}\mathfrak{L}_{\mathfrak{p}}(\chi,2s-n-l/2)$, where

$$\mathfrak{L}_{\mathfrak{p}}(\chi,2s) := G_{\mathfrak{p}}(\chi,2s-n-l/2)\begin{cases}\prod_{i=1}^{n} L_{\mathfrak{p}}(4s-l-2i,\chi^2) & \text{if } l\in 2\mathbb{Z},\\ \prod_{i=1}^{n} L_{\mathfrak{p}}(4s-l-2i+1,\chi^2) & \text{if } l\notin 2\mathbb{Z}.\end{cases}$$

That is, we obtain

$$\begin{aligned} &N(\mathfrak{b}\mathfrak{e}^{-1}\mathfrak{c})^{2ns}\chi_{\mathbf{h}}(\theta)^{-n}(-1)^{n(s-k/2)}vol(A) < (\mathcal{E}|_{k,S}\boldsymbol{\rho})(\mathrm{diag}[z_1,z_2],s),(f^c|I_{\mathfrak{b}})(z_2) >\\ &= \nu_{\mathfrak{e}}c_{S,k}(s-k/2)\mathcal{G}_{k-l/2,n+m}(s-l/4)E(z_1,s;(f|U_{h1_n,\chi}I_{\mathfrak{b}})|_{k,S}\boldsymbol{\eta}_n^{-1})\\ &\cdot G(\chi,2s-n-l/2)^{-1}L_{\psi}(2s-n-l/2,\mathbf{f},\chi\psi^{-1})\\ &\cdot\begin{cases} L_{\mathfrak{c}}(2s-l/2,\chi\psi_S)\prod_{i=n+1}^{[(n+m)/2]} L_{\mathfrak{c}}(4s-l-2i,\chi^2) & \text{if } l\in 2\mathbb{Z},\\ \prod_{i=n+1}^{[(n+m+1)/2]} L_{\mathfrak{c}}(4s-l-2i+1,\chi^2) & \text{if } l\notin 2\mathbb{Z},\end{cases}\end{aligned} \tag{28}$$

where we have set

$$G(\chi,2s-n-l/2) = \prod_{(\mathfrak{p},\mathfrak{c})=1} G_{\mathfrak{p}}(\chi,2s-n-l/2). \tag{29}$$

In particular, if $m=n$, we obtain

$$N(\mathfrak{b}\mathfrak{e}^{-1}\mathfrak{c})^{2ns}\chi_{\mathbf{h}}(\theta)^{-n}(-1)^{n(s-k/2)}vol(A) < (\mathcal{E}|_{k,S}\boldsymbol{\rho})(\mathrm{diag}[z_1,z_2],s),(f^c|I_{\mathfrak{b}})(z_2) >$$

$$(30)\qquad = \nu_{\mathfrak{e}} c_{S,k}(s-k/2)\mathcal{G}_{k-l/2,2n}(s-l/4)(f|U_{h1_n,\chi}I_{\mathfrak{b}})|_{k,S}\boldsymbol{\eta}_n^{-1}G(\chi,2s-n-l/2)^{-1}$$
$$\cdot L_{\psi}(2s-n-l/2,\mathbf{f},\chi\psi^{-1})\begin{cases} L_{\mathfrak{c}}(2s-l/2,\chi\psi_S), & \text{if } l\in 2\mathbb{Z},\\ 1, & \text{if } l\notin 2\mathbb{Z}.\end{cases}$$

We are now ready to prove our first main theorem regarding the analytic properties of the function $L_{\psi}(s,\mathbf{f},\chi)$, which should be seen as an extension of the Theorem 6.1 in [22] to the Siegel-Jacobi setting.

Theorem 9.3. *Let $\mathbf{f}\in\mathcal{S}^n_{k,S}(\boldsymbol{D},\psi)$ be a Hecke eigenform of index S which satisfies the $M^+_{\mathfrak{p}}$ condition for every prime $\mathfrak{p}\nmid\mathfrak{c}$. Moreover, let ϕ be a Hecke character of F of conductor $\mathfrak{f}_\phi$ such that $\phi_{\mathbf{a}}(x)=\operatorname{sgn}(x_{\mathbf{a}})^k$. Write $\mathfrak{x}$ for the product of all primes ideals $\mathfrak{p}$ in the support of $\mathfrak{e}^{-1}\mathfrak{c}$ such that $\mathbf{f}|T_{\pi_{\mathfrak{p}}1_n,\psi}=0$. Then the function*

$$\Lambda_{\psi,\mathfrak{x}}(s,\mathbf{f},\phi) := L_{\mathbf{a}}(s,k)L_{\psi,\mathfrak{x}}(s,\mathbf{f},\phi)\cdot\begin{cases} L_{\mathfrak{c}}(s+n,\phi\psi\psi_S), & \text{if } l\in 2\mathbb{Z},\\ 1, & \text{if } l\notin 2\mathbb{Z},\end{cases}$$

where

$$L_{\mathbf{a}}(s,k) := c_{S,k}((s+n-k)/2+l/4)\mathcal{G}_{k-l/2,2n}((s+n)/2)$$

has a meromorphic continuation to the whole complex plane. More precisely, the poles are exactly the poles of the Eisenstein series $\mathcal{E}((s+n+l/2)/2)$ as described in Theorem 8.3 plus the poles of the function $G(\chi,s+n)$.

Proof. The theorem follows now from equation (30) and Theorem 8.3 arguing similarly to the proof of [22, Theorem 6.1]. Assume first that $\mathfrak{f}_\phi|\mathfrak{e}$, which is equivalent to $\phi_v(\mathfrak{o}_v^\times)=1$ (i.e. ϕ_v is unramified) for all v that do not divide $\mathfrak{e}$ and that $\mathfrak{f}_\phi|\mathfrak{c}$. Then we can use the equation (30) with $\chi:=\phi\psi$. We obtain the statement of the theorem by observing that the function $L_{\psi,\mathfrak{x}}(s,\mathbf{f},\phi)$ may be obtained by changing $\mathfrak{e}$ to $\mathfrak{e}\prod_{v|\mathfrak{x}}\mathfrak{c}_v$ and employing Lemma 9.1. This guarantees that the equation (30) is not trivial (0=0) and hence we can read off the analytic properties of $L_{\psi,\mathfrak{x}}(s,\mathbf{f},\phi)$ from those of $\mathcal{E}$.
We also give the proof of the general case by repeating the idea which was used to show [22, Theorem 6.1]. Set $\mathfrak{c}^0:=\mathfrak{c}\cap\mathfrak{f}_\phi$ and decompose $\mathfrak{c}^0=\mathfrak{e}^0\mathfrak{e}^1$ with $(\mathfrak{e}^0,\mathfrak{e}^1)=1$, such that $\mathfrak{e}^0_v=\mathfrak{c}^0_v$ for every $v|\mathfrak{e}\mathfrak{x}\mathfrak{f}_\phi$, and $\mathfrak{e}^0_v=\mathfrak{o}_v$ otherwise. Then if $\boldsymbol{D}^0$ denotes the group $\boldsymbol{D}$ with $\mathfrak{c}^0,\mathfrak{e}^0$ in place of $\mathfrak{c}$ and $\mathfrak{e}$, $\mathbf{f}\in\mathcal{S}^n_{k,S}(\boldsymbol{D}^0,\psi)=\mathcal{S}^n_{k,S}(\boldsymbol{D}^0,\chi)$. In particular, we can apply the argument of the previous paragraph with $\chi:=\phi\psi$ and the group $\boldsymbol{D}^0$ to conclude the proof. ☐

Remark 9.4. The proof above indicates the significance of considering in the whole paper the case of a non-trivial ideal $\mathfrak{e}$. Indeed, let us consider a cusp form $\mathbf{f}\in S^n_{k,S}(\boldsymbol{D}[\mathfrak{b}^{-1},\mathfrak{b}\mathfrak{c}],\psi)$, that is with $\mathfrak{e}=\mathfrak{o}$, and assume for simplicity that $\mathfrak{x}$ is trivial. Moreover, consider a Hecke character ϕ whose conductor $\mathfrak{f}_\phi$ - again, for simplicity - is prime to $\mathfrak{c}$. Then $\mathfrak{c}^0=\mathfrak{c}\mathfrak{f}_\phi$ and $\mathfrak{e}^0=\mathfrak{f}_\phi$, and thus we need to consider non-trivial $\mathfrak{e}$ even if we start with a form of trivial one.

Now we can also prove a theorem regarding the analytic continuation of the Klingen-type Jacobi Eisenstein series attached to a form f in the case of $\mathfrak{e}=\mathfrak{c}$.

Theorem 9.5. *Let $f\in S^n_{k,S}(\boldsymbol{\Gamma})$ be a Hecke eigenform with $\boldsymbol{\Gamma}=\boldsymbol{D}\cap\boldsymbol{G}$ where we take $\mathfrak{e}=\mathfrak{c}$ (i.e., in particular $\psi=1$) and let χ be a Hecke character of F such that*

$\chi_{\boldsymbol{a}}(x) = \mathrm{sgn}_{\boldsymbol{a}}(x)^k$. *Then the Klingen-type Eisenstein series*

$$\mathcal{E}(z,s;f,\chi) := c_{S,k}(s-k/2)\mathcal{G}_{k-l/2,n+m}(s-l/4)\boldsymbol{\Lambda}(s,f,\chi)E(z,s;f,\chi),$$

where

$$\boldsymbol{\Lambda}(s,f,\chi) := L(2s-n-l/2,\mathbf{f},\chi)\begin{cases} L_{\mathfrak{c}}(2s-l/2,\chi\psi_S)\prod_{i=n+1}^{[(n+m)/2]} L_{\mathfrak{c}}(4s-l-2i,\chi^2), & l\in 2\mathbb{Z},\\ \prod_{i=n+1}^{[(n+m+1)/2]} L_{\mathfrak{c}}(4s-l-2i+1,\chi^2), & l\notin 2\mathbb{Z},\end{cases}$$

has a meromorphic continuation to the entire complex plane.

Proof. We need to rewrite the equation (28). First note that since $\mathfrak{e}=\mathfrak{c}$, we have $U_{h1_n,\chi}=1$. Now we extend an argument in [23, page 569] to the Siegel-Jacobi case. Observe that for every finite place v we have $\mathbf{Y}_v = \boldsymbol{\eta}_n\boldsymbol{D}_v R_v(\mathfrak{c})\boldsymbol{D}_v\boldsymbol{\eta}_n^{-1}$. Further, consider the isomorphism

$$S^n_{k,S}(\boldsymbol{D}) \cong S^n_{k,S}(\widetilde{\boldsymbol{D}}), \quad \mathbf{f}\mapsto \mathbf{f}|_{k,S}\boldsymbol{\eta}_n,$$

where $\widetilde{\boldsymbol{D}} := C[\mathfrak{b}^{-1},\mathfrak{o},\mathfrak{b}^{-1}]D[\mathfrak{b}\mathfrak{c},\mathfrak{b}^{-1}\mathfrak{c}]$. Note that since $\mathfrak{e}=\mathfrak{c}$ we do not have any nebentype (i.e. $\psi=1$). Now note that for any $g\in R(\mathfrak{c})$

$$\boldsymbol{\eta}_n\widetilde{\boldsymbol{D}}g\widetilde{\boldsymbol{D}}\boldsymbol{\eta}_n^{-1} = \boldsymbol{D}g\boldsymbol{D},$$

and hence we can conclude that $(f|T_g)|_{k,S}\boldsymbol{\eta}_n = (f|_{k,S}\boldsymbol{\eta}_n)|\widetilde{T_g}$, where $\widetilde{T_g}$ denotes the Hecke operator defined with respect to the group $\widetilde{\boldsymbol{D}}$. Putting all these observations together we see that the equation (28) can be also written as

$$\begin{aligned} &G(\chi,2s-n-l/2)N(\mathfrak{b}\mathfrak{e}^{-1}\mathfrak{c})^{2ns}\chi_{\mathbf{h}}(\theta)^{-n}(-1)^{n(s-k/2)}vol(A)\\ &\quad\cdot < (\mathcal{E}|_{k,S}\boldsymbol{\rho})(\mathrm{diag}[z_1,z_2],s),(f_{k,S}|\boldsymbol{\eta}_n)^c(z_2) >\\ &\quad= \nu_{\mathfrak{e}}c_{S,k}(s-k/2)\mathcal{G}_{k-l/2,n+m}(s-l/4)\boldsymbol{\Lambda}(s,f,\chi)E(z_1,s;f),\end{aligned} \tag{31}$$

where, recall, $G(\chi,2s-n-l/2)$ is meromorphic on $\mathbb{C}$. In particular, we can extend the Klingen-type Eisenstein series to the whole of $\mathbb{C}$ with respect to variable s by using the analytic properties of the Siegel-type Eisenstein series. Moreover, we can read off the various poles from this expression. □

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