Advanced Quantum Theory IV — Assignment 4

Harmonic oscillators and the complex scalar field

Michaelmas 2021/22

1. *[Optional but might be helpful for understanding]* A system of N decoupled complex simple harmonic oscillators with frequencies $\omega_c,\ c = 1, \ldots, N,$ has action

$$S = \int dt \sum_{c=1}^{N} \left(\dot{q}_c(t)\, \dot{q}_c^\star(t) - \omega_c^2\, q_c(t)\, q_c^\star(t) \right), \tag{1}$$

and Euler-Lagrange equations of motion

$$\ddot{q}_c(t) - \omega_c^2\, q_c(t) = 0, \qquad \ddot{q}_c^\star(t) - \omega_c^2\, q_c^\star(t) = 0, \qquad c = 1, \ldots, N. \tag{2}$$

In the lectures we saw that their general solution can be written as

$$q_c(t) = \frac{1}{\sqrt{2\omega_c}} \left[a_c\, e^{-i\omega_c t} + b_c^\star\, e^{i\omega_c t} \right], \tag{3}$$

$$q_c^\star(t) = \frac{1}{\sqrt{2\omega_c}} \left[b_c\, e^{-i\omega_c t} + a_c^\star\, e^{i\omega_c t} \right]. \tag{4}$$

From the above Lagrangian formulation of the complex Harmonic oscillator, show that the corresponding Hamiltonian takes the form

$$H = \sum_{c=1}^{N} \dot{q}_c(t)\, \dot{q}_c^\star(t) + \omega_c^2\, q_c(t)\, q_c^\star(t). \tag{5}$$

Show that upon canonical quantisation the Hamiltonian takes the following form in terms of $a_c,\ a_c^\dagger$ and $b_c,\ b_c^\dagger$:

$$\hat{H} = \sum_{c=1}^{N} \omega_c \left[\hat{a}_c \hat{a}_c^\dagger + \hat{b}_c^\dagger \hat{b}_c \right] = \sum_{c=1}^{N} \omega_c \left[\hat{a}_c^\dagger \hat{a}_c + \hat{b}_c^\dagger \hat{b}_c + 1 \right]. \tag{6}$$

2. From the lectures we saw that the Hamiltonian of the complex scalar field takes the form

$$H = \int d^3\vec{x}\,\left[\dot{\phi}^\star\dot{\phi} + \vec{\nabla}\phi^\star\cdot\vec{\nabla}\phi + m^2\phi^\star\phi\right], \tag{7}$$

and ϕ admits the decomposition

$$\phi\left(t,\vec{x}\right) = \int \frac{d\vec{k}}{\left(2\pi\right)^3}\frac{1}{\sqrt{2\omega_{\vec{k}}}}\left[a_{\vec{k}}\,e^{i\vec{k}\cdot\vec{x}-i\omega_{\vec{k}}t} + b^\star_{\vec{k}}\,e^{-i\vec{k}\cdot\vec{x}+i\omega_{\vec{k}}t}\right], \tag{8}$$

where $\omega^2_{\vec{k}} = |\vec{k}|^2 + m^2$. Show that:

$$\phi^\star\left(t,\vec{x}\right) = \int \frac{d\vec{k}}{\left(2\pi\right)^3}\frac{1}{\sqrt{2\omega_{\vec{k}}}}\left[b_{\vec{k}}\,e^{i\vec{k}\cdot\vec{x}-i\omega_{\vec{k}}t} + a^\star_{\vec{k}}\,e^{-i\vec{k}\cdot\vec{x}+i\omega_{\vec{k}}t}\right], \tag{9}$$

$$\vec{\nabla}\phi\left(t,\vec{x}\right) = \int \frac{d\vec{k}}{\left(2\pi\right)^3}\frac{1}{\sqrt{2\omega_{\vec{k}}}}\left[i\vec{k}\,a_{\vec{k}}\,e^{i\vec{k}\cdot\vec{x}-i\omega_{\vec{k}}t} - i\vec{k}\,b^\star_{\vec{k}}\,e^{-i\vec{k}\cdot\vec{x}+i\omega_{\vec{k}}t}\right], \tag{10}$$

$$\vec{\nabla}\phi^\star\left(t,\vec{x}\right) = \int \frac{d\vec{k}}{\left(2\pi\right)^3}\frac{1}{\sqrt{2\omega_{\vec{k}}}}\left[i\vec{k}\,b_{\vec{k}}\,e^{i\vec{k}\cdot\vec{x}-i\omega_{\vec{k}}t} - i\vec{k}\,a^\star_{\vec{k}}\,e^{-i\vec{k}\cdot\vec{x}+i\omega_{\vec{k}}t}\right], \tag{11}$$

$$\dot{\phi}\left(t,\vec{x}\right) = \int \frac{d\vec{k}}{\left(2\pi\right)^3}\left(-i\right)\sqrt{\frac{\omega_{\vec{k}}}{2}}\left[a_{\vec{k}}\,e^{i\vec{k}\cdot\vec{x}-i\omega_{\vec{k}}t} - b^\star_{\vec{k}}\,e^{-i\vec{k}\cdot\vec{x}+i\omega_{\vec{k}}t}\right], \tag{12}$$

$$\dot{\phi}^\star\left(t,\vec{x}\right) = \int \frac{d\vec{k}}{\left(2\pi\right)^3}\left(-i\right)\sqrt{\frac{\omega_{\vec{k}}}{2}}\left[b_{\vec{k}}\,e^{i\vec{k}\cdot\vec{x}-i\omega_{\vec{k}}t} - a^\star_{\vec{k}}\,e^{-i\vec{k}\cdot\vec{x}+i\omega_{\vec{k}}t}\right]. \tag{13}$$

Promote the above to operators through the procedure of canonical quantisation. Choosing the order of the operators as they appear in the Hamiltonian (7), show that

$$H = \int \frac{d\vec{k}}{\left(2\pi\right)^3}\,\omega_{\vec{k}}\left[\hat{a}^\dagger_{\vec{k}}\hat{a}_{\vec{k}} + \hat{b}_{\vec{k}}\hat{b}^\dagger_{\vec{k}}\right], \tag{14}$$

Show that

$$\hat{H} = \int \frac{d\vec{k}}{\left(2\pi\right)^3}\,\omega_{\vec{k}}\left[\hat{a}^\dagger_{\vec{k}}\hat{a}_{\vec{k}} + \hat{b}^\dagger_{\vec{k}}\hat{b}_{\vec{k}} + \left(2\pi\right)^3\delta^{(3)}\left(0\right)\right]. \tag{15}$$