Advanced Quantum Theory IV **Assignment 3**

Noether's theorem in field theory Michaelmas 2021/22

Part 1. Consider the following action for two real scalar fields ϕ_1 and ϕ_2,

$$S = \int d^4x \left(-\frac{1}{2}\partial_\mu\phi_1\partial^\mu\phi_1 - \frac{1}{2}\partial_\mu\phi_2\partial^\mu\phi_2 - \frac{1}{2}m^2\phi_1^2 - \frac{1}{2}m^2\phi_2^2 - \lambda\left(\phi_1^2+\phi_2^2\right)^2 \right). \tag{1}$$

(a) Derive the Euler-Lagrange equations of motion for the fields ϕ_1 and ϕ_2.

(b) Show that the action (1) is invariant under the continuous transformation

$$\phi_1 \to \phi_1' = (\cos\alpha)\,\phi_1 - (\sin\alpha)\,\phi_2, \tag{2}$$
$$\phi_2 \to \phi_2' = (\sin\alpha)\,\phi_1 + (\cos\alpha)\,\phi_2, \tag{3}$$

where α is a constant continuous parameter.

(c) Determine the current and charge associated to the symmetry given in part (b).

(d) Show that the current in part (c) is indeed conserved if the Euler-Lagrange equations of motion are satisfied.

Part 2: Another way to solve the same problem is to use the fact that a theory of two real scalar fields with the same mass can be recast as a theory of a complex scalar field ϕ, which you will show in the following.

(a). We can assemble two real scalar fields ϕ_1 and ϕ_2, with the same mass m, into a single complex scalar field $\phi = (\phi_1 + i\phi_2)/\sqrt{2}$. Show that the action (1) in terms of the complex scalar field ϕ reads

$$S = \int d^4x \left(-\partial_\mu\phi\partial^\mu\phi^* - m^2\phi\phi^* - 4\lambda\left(\phi\phi^*\right)^2 \right). \tag{4}$$

(b) Derive the Euler-Lagrange equations of motion for the field ϕ and its complex conjugate ϕ^*.

(c) Show that the action (4) is invariant under the continuous transformation

$$\phi \to \phi' = e^{i\alpha}\phi, \qquad \phi^* \to \left(\phi'\right)^* = e^{-i\alpha}\phi^*. \tag{5}$$

where α is a constant continuous parameter. *[This question continues on the next page.]*

(d) Determine the current and charge associated to the symmetry given in part 2 (c).

(e) Show that the current you determined in part 2 (d) is equivalent to the current you obtained in part 1 (c).