Advanced Quantum Theory IV

Michaelmas Term 2021

Study Sheet: *Special Relativity, Index Notation and Summation Convention*

Note: Any blue text is a hyperlink.

1. **Lorentz Boosts.** Consider two inertial frames S and S' with space-time co-ordinates $(ct, \vec{x})$ and $(ct', \vec{x}')$ respectively, where $\vec{x} = \left(x^1, x^2, x^3\right)$. In this exercise we are going to walk through how S and S' are related when S' has speed v in the positive x^1-direction. Assume that at $t = 0$ the origin $\vec{x} = 0$ of S coincides with the origin $\vec{x}' = 0$ of S'. This transformation is a Lorentz boost along the x^1 axis.

   An inertial frame is a frame of reference which is not undergoing acceleration. A particle with no net force acting upon it will therefore travel at a constant velocity in an inertial frame, meaning that its trajectory is a straight line. Since S and S' are inertial frames, the map $(ct, \vec{x}) \mapsto (ct', \vec{x}')$ between them must therefore send straight lines to straight lines. This means that such a map is a linear transformation and, since the origins of S and S' coincide at $t = 0$, can be written in matrix form:

$$\begin{pmatrix} ct' \\ (x')^1 \\ (x')^2 \\ (x')^3 \end{pmatrix} = \begin{pmatrix} a_{00} & a_{01} & a_{02} & a_{03} \\ a_{10} & a_{11} & a_{12} & a_{13} \\ a_{20} & a_{21} & a_{22} & a_{23} \\ a_{30} & a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} ct \\ x^1 \\ x^2 \\ x^3 \end{pmatrix}, \tag{1}$$

   for some constants a_{ij}, $i, j = 0, 1, 2, 3$, that we are going to determine.

   Now use that S' is moving at speed v in the positive x^1-direction, meaning that its origin $\vec{x}' = 0$ follows the trajectory $\vec{x} = (vt, 0, 0)$. Based on this information, show that:

$$\left(x'\right)^1 = \gamma\left(x^1 - vt\right), \qquad \left(x'\right)^2 = \alpha x^2, \qquad \left(x'\right)^3 = \beta x^3, \tag{2}$$

   for some constant coefficients γ, α, β. Conversely, using that S has speed v in the negative $(x')^1$-direction from the perspective of S', show:

$$x^1 = \gamma\left(\left(x'\right)^1 + vt'\right), \qquad x^2 = \alpha\left(x'\right)^2, \qquad x^3 = \beta\left(x'\right)^3, \tag{3}$$

   for the *same* constants γ, α, β as in (2). [Hint: You should convince yourself that one can obtain (2) from (3) by sending $v \to -v$ and interchanging S and S']. Conclude that $\alpha = 1$ and $\beta = 1$.

   The second postulate of Special Relativity states that the speed of light c in the vacuum is the same in all reference frames. Using this, show that:

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}. \tag{4}$$

   [Hint: In S a light ray that passes through the origin at $t = 0$ and is travelling in the x^1 direction has trajectory $x^1 = ct$. Using the second postulate of special relativity, convince yourself that this has trajectory $\left(x'\right)^1 = ct'$ in S']. Finally, show that:

$$t' = \gamma\left(t - \frac{v}{c^2}x^1\right). \tag{5}$$

   The final part of this exercise continues on the next page.

Using the results you obtained above, deduce that a Lorentz boost in the (positive) x^1-direction is given by the following 4×4 matrix Λ:

$$\begin{pmatrix} ct' \\ (x')^1 \\ (x')^2 \\ (x')^3 \end{pmatrix} = \underbrace{\begin{pmatrix} \gamma & -\frac{v\gamma}{c} & 0 & 0 \\ -\frac{v\gamma}{c} & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}}_{\Lambda} \begin{pmatrix} ct \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} . \tag{6}$$

Extension exercise: A similar argument gives the matrix form for Lorentz Boosts along the x^2- and x^3-axes. Is there a quicker way to obtain them given the knowledge of the matrix form (6) of the Lorentz Boost along the x^1-axis? [Hint: the x^2- and x^3-axes can be reached from the x^1 axis through a rotation.] In the first assignment we shall see yet another derivation which follows from a group-theoretical approach to Lorentz transformations.

2. Lorentz transformations and the Minkowski metric.

Consider a bilinear form on a 4-dimensional vector space,

$$B(v,u) = v^{\mathsf{T}} B\, u, \tag{7}$$

where B is the 4×4 matrix of the bilinear form and vectors v, u,

$$v = \begin{pmatrix} v^0 \\ v^1 \\ v^2 \\ v^3 \end{pmatrix}, \qquad u = \begin{pmatrix} u^0 \\ u^1 \\ u^2 \\ u^3 \end{pmatrix}. \tag{8}$$

Let's transform them under the Lorentz Boost (6),

$$v' = \Lambda v, \qquad u' = \Lambda u. \tag{9}$$

Suppose that $B = \mathbb{I}_{4\times4}$, the identity matrix:

$$\mathbb{I}_{4\times4} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{10}$$

The bi-linear form (7) is then the usual dot product:

$$B(v,u) = v^0u^0 + v^1u^1 + v^2u^2 + v^3u^3. \tag{11}$$

Show that this is not invariant under the Lorentz Boost (6).

Suppose instead $B = \eta$, the Minkowski metric:

$$\eta = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{12}$$

Show that in this case we have:

$$B(v,u) = -v^0u^0 + v^1u^1 + v^2u^2 + v^3u^3. \tag{13}$$

Show that this is invariant under the Lorentz Boost (6).

In Special Relativity, our usual 3-dimensional Euclidean space with metric given by the dot product is extended to a four-dimensional space-time endowed with the Minkowski metric (12), known as Minkowski space. Lorentz transformations are defined as the group of transformations which preserve the Minkowski metric. This is in analogy to the group of orthogonal transformations (rotations and reflections) which preserve the dot product of Euclidean space.

3. Contravariant and covariant vectors, index notation and Einstein summation convention.

Note: For the AQT course it is not necessary to know what is a dual space. It is touched upon in this exercise purely for a wider understanding of contravariant and covariant vectors.

Every bilinear form on a vector space V defines a map to the dual space $V^\star$,

$$B : V \to V^\star. \tag{14}$$

The components of vectors $v \in V$ are labelled by upper indices v^μ, while the components of dual vectors $\omega \in V^\star$ are labelled by lower indices ω_μ.

Take V to be a four-dimensional vector space and B to be the Minkowski metric (12). Consider a (contravariant) vector $v \in V$ with components v^μ, $\mu = 0, 1, 2, 3$. Show that the components v_μ of the dual (covariant) vector are given by:

$$v_0 = -v^0, \quad v_1 = v^1, \quad v_2 = v^2, \quad v_3 = v^3. \tag{15}$$

In index notation this map reads:

$$v_\mu = (\eta v)_\mu = \sum_{\nu=0}^{3} \eta_{\mu\nu} v^\nu, \tag{16}$$

where $\eta_{\mu\nu}$, $\mu, \nu = 0, 1, 2, 3$, label the elements of the Minkowski metric (12). The expression on the right hand side can be made more concise by adopting a convenient notation known as Einstein Summation convention. In this notation, when an index appears twice in a given term we are instructed to sum the term over all the values that the index takes. In Einstein summation convention the map (16) therefore reads:

$$v_\mu = (\eta v)_\mu = \eta_{\mu\nu} v^\nu. \tag{17}$$

This notation is much less cumbersome since we do not need to include the sum over the index ν explicitly – it is implied. In the following we will work through further examples using Einstein summation convention.

The inverse map to (14) is simply given by the inverse matrix η^{-1}:

$$\eta^{-1}\eta = \mathbb{I}_4. \tag{18}$$

Write (18) in index notation and show that in Einstein summation convention it reads:

$$\left(\eta^{-1}\right)^{\mu\nu} \eta_{\nu\sigma} = \delta^\mu{}_\sigma, \tag{19}$$

where the Kronecker delta $\delta^\mu{}_\nu$ encodes the elements of the identity matrix,

$$\delta^\mu{}_\nu = \begin{cases} 1 & \text{if} \quad \mu = \nu \\ 0 & \phantom{\text{if}} \quad \mu \neq \nu \end{cases}. \tag{20}$$

By explicit computation, show that:

$$\eta^{-1} = \eta. \tag{21}$$

In index notation the above equation reads [*exercise continues on next page*]:

$$\left(\eta^{-1}\right)^{\mu\nu} = \eta^{\mu\nu}, \tag{22}$$

where now the elements of the Minkowski metric are labelled with upper indices $\eta^{\mu\nu}$, since we are mapping from the dual space V^* to V. Show that:

$$v^\mu = \eta^{\mu\nu} v_\nu. \tag{23}$$

In summary, indices are *raised* with $\eta^{\mu\nu}$ and *lowered* with $\eta_{\mu\nu}$:

$$v_\mu = \eta_{\mu\nu} v^\nu, \qquad v^\mu = \eta^{\mu\nu} v_\nu. \tag{24}$$

Show that using Einstein summation convention (7) can be written as:

$$B(v, u) = v_\mu u^\mu = v^\mu u_\mu. \tag{25}$$

In Euclidean space, where the metric is $B = \mathbb{I}$, the distinction between upper and lower indices is not so important since $v_\mu = v^\mu$ for all μ (verify this!). In Minkowski space however, where the metric is $B = \eta$, this is no longer true (see equation (15)) and the distinction between upper and lower indices is *very* important. In particular, an object like

$$v^\mu u^\mu = v^0 u^0 + v^1 u^1 + v^2 u^2 + v^3 u^3, \tag{26}$$

doesn't make much sense in Minkowski space since it is not invariant under Lorentz transformations (see exercise 2). *We therefore only sum over pairs of repeated indices when one of them is raised and the other is lowered.*

4. More exercises on index notation and Einstein summation convention.

a. Show that using Einstein summation convention the Lorentz transformation (9) reads

$$\left(v'\right)^{\mu} = \Lambda^{\mu}{}_{\nu} v^{\nu}, \tag{27}$$

where $\Lambda^{\mu}{}_{\nu}$ are the elements of the 4×4 matrix Λ. Show also that:

$$\left(v'\right)_{\mu} = \Lambda_{\mu}{}^{\nu} v_{\nu}, \tag{28}$$

where

$$\Lambda_{\mu}{}^{\nu} = \eta_{\mu\rho} \Lambda^{\rho}{}_{\sigma} \eta^{\sigma\nu}. \tag{29}$$

Recalling that $v^{\mu} v_{\mu}$ is invariant under Lorentz transformations, show that:

$$\Lambda_{\mu}{}^{\nu} = \left(\Lambda^{-1}\right)^{\nu}{}_{\mu}. \tag{30}$$

b. Consider the gradient vector:

$$\frac{\partial}{\partial x^{\mu}} = \left(\frac{\partial}{\partial x^{0}}, \frac{\partial}{\partial x^{1}}, \frac{\partial}{\partial x^{2}}, \frac{\partial}{\partial x^{3}}\right). \tag{31}$$

Show that

$$\frac{\partial x^{\nu}}{\partial x^{\mu}} = \delta_{\mu}{}^{\nu}, \tag{32}$$

where $\delta_{\mu}{}^{\nu}$ is the Kronecker delta (20). The gradient vector (31) is therefore a covariant vector with *lower* index μ. In the AQT course we shall often abbreviate (31) as ∂^{x}_{μ} or, when there is no risk of ambiguity, ∂_{μ}. If $x = (ct, \vec{x})$, i.e. so that $x^{0} = ct$, show that:

$$\partial_{0} = \frac{1}{c}\frac{\partial}{\partial t} \qquad \text{and} \qquad \partial^{0} = -\frac{1}{c}\frac{\partial}{\partial t}. \tag{33}$$

Show that:

$$\partial^{\mu}\partial_{\mu} = -\frac{1}{c^{2}}\frac{\partial^{2}}{\partial t^{2}} + \frac{\partial^{2}}{\partial\left(x^{1}\right)^{2}} + \frac{\partial^{2}}{\partial\left(x^{2}\right)^{2}} + \frac{\partial^{2}}{\partial\left(x^{3}\right)^{2}}. \tag{34}$$

[*Note: We will often make use of the formulas* (33) *and* (34) *in the AQT course. A common mistake is to forget the minus signs – you have been warned!*]

c. If p_{ν} is a constant vector, show that

$$\frac{\partial\left(p_{\nu} x^{\nu}\right)}{\partial x^{\mu}} = p_{\mu}. \tag{35}$$