Solitons III
(2022-23)

Exercises

January 2, 2023

0 Introduction

Ex 1 Numerical results seen in the lectures suggest that the KdV equation

$$u_t + 6uu_x + u_{xxx} = 0 \tag{0.1}$$

has an exact solution of the form

$$u(x,t) = \frac{2}{\cosh^2(x - vt)} \tag{0.2}$$

for some constant velocity v. Verify this by direct substitution into the KdV equation and determine the value of v.

Ex 2

1. Show that if $u(x,t) = g(x,t)$ solves the KdV equation (0.1), then so does $u(x,t) = Ag(Bx, Ct)$, provided that the constants B and C are related to A in a specific way (which you should determine).
2. Apply this transformation to the basic KdV solution found in exercise 1 to construct a one-parameter family of one-soliton solutions of the KdV equation.
3. Find a formula relating the velocities to the heights for solitons in this one-parameter family. How does the width of a soliton in this family change if its velocity is rescaled by a factor of 4?

Ex 3 Show that if $u(x,t)$ solves the KdV equation and ϵ is a constant, then $v(x,t) := \frac{1}{\epsilon}u(x,t)$ solves the rescaled KdV equation

$$v_t + 6\epsilon vv_x + v_{xxx} = 0\,, \tag{0.3}$$

while $w(x,t) := \epsilon u(x, \epsilon t)$ solves the differently-rescaled KdV equation

$$w_t + 6ww_x + \epsilon w_{xxx} = 0\,. \tag{0.4}$$

Ex 4 Consider a pair of solitons with velocities m and n in the ball and box model, with $m > n$ and the faster soliton to the left of the slower one, with separation $l \geq n$ (*i.e.* there are $l \geq n$ empty boxes between the two solitons). Evolve various such initial conditions forward in time using the ball and box rule, for different values of m, n and l. Prove that the system always evolves into an oppositely-ordered pair of the same two solitons, and find a general formula for the phase shifts[1] of the solitons in terms of m and n.

[Optional:] What can go wrong if $l < n$? **[Hint:** Evolve the system backwards...]

Ex 5 In the two-colour (blue and red) ball and box model, we'll call a row of n consecutive balls a soliton if it keeps its form over time, so that after each time-step its only change is a possible (fixed) translation. There's no need for both colours to be represented, so a row of n blue balls, or a row of n red balls, is also a potential soliton. How many solitons of length n are there? What are their speeds?

Ex 6* The ball and box model can be further generalised to the M-colour ball and box model. The balls now come in M colours, 1, 2, ..., M, and the time-evolution rule is generalised to say that first all balls of colour 1 are moved, then all of colour 2, and so on, with a single time-step being completed once all balls of all colours have been moved. How many solitons of length n are there in this model? Again, there is no need for every colour to be present in a given soliton. You might start by classifying the 'top-speed' solitons of length n, that is, those that move at speed n.

Ex 7* Investigate the scattering of solitons in the two-colour ball and box model. You should find that the lengths of top-speed solitons are preserved under collisions, but their forms can change. Try to formulate a general rule for this behaviour. Can you generalise it to the M-colour model?

[1] The phase shift of a soliton is defined to be the shift of its position, at a time in the far future, relative to the position it would have had at the same time if the other soliton hadn't been there.

1 Waves, dispersion and dissipation

Ex 8

1. Express d'Alembert's general solution of the wave equation $u_{tt} - u_{xx} = 0$ in terms of the initial conditions $u(x, 0) = p(x)$ and $u_t(x, 0) = q(x)$.
2. Find a relation between $p(x)$ and $q(x)$ which produces a single wave travelling to the right.

Ex 9 The wave profile

$$\phi(x,t) = \cos(k_1 x - \omega(k_1)t) + \cos(k_2 x - \omega(k_2)t) \tag{1.1}$$

is a superposition of two plane waves. Rewrite ϕ as a product of cosines, and use this to sketch the wave profile when $|k_1 - k_2| \ll |k_1|$. Find the velocity at which the envelope of the wave profile moves (the **group velocity**), again for $k_1 \approx k_2$; in the limit $k_1 \to k_2$ verify that this reduces to $d\omega/dk$, consistent with the result obtained in lectures.

Ex 10

1. Completing the square, derive the formula

$$\int_{-\infty}^{+\infty} dk \; e^{-A(k-\bar{k})^2} e^{ikB} = \sqrt{\frac{\pi}{A}} \; e^{i\bar{k}B} e^{-B^2/(4A)} \; . \tag{1.2}$$

   (You can quote the result $\int_{-\infty}^{+\infty} dk \; e^{-Ak^2} = \sqrt{\pi/A}$ for $A > 0$.)

2. For the Gaussian wavepacket (where Re denotes the real part)

$$u(x,t) = \mathrm{Re} \int_{-\infty}^{+\infty} dk \; e^{-a^2(k-\bar{k})^2} e^{i(kx - \omega(k)t)} \; , \tag{1.3}$$

   expand $\omega(k)$ to second order in $k - \bar{k}$, and then use the result of part 1 to derive a better approximation for $u(x,t)$ than that obtained in lectures.

3. Given that a function of the form $e^{-(x-x_0)^2/C}$ describes a profile centred at x_0 with width^{-2} equal to the real part of C^{-1}, show that the result of part 2 is a wave profile moving at velocity $\omega'(\bar{k})$, with width^2 increasing with time as $4a^2 + \omega''(\bar{k})^2 t^2/a^2$. (Hence, for $\omega'' \neq 0$, the wave disperses.)

Ex 11 Find the dispersion relation and the phase and group velocities for:

(a) $u_t + u_x + \alpha u_{xxx} = 0\,$;

(b) $u_{tt} - \alpha^2 u_{xx} = \beta^2 u_{ttxx}$.

Ex 12 For which values of n does the equation

$$u_t + u_x + u_{xxx} + \frac{\partial^n u}{\partial x^n} = 0 \tag{1.4}$$

admit "physical" dissipation? (A wave is said to have physical dissipation if the amplitude of plane waves decreases with time.)

2 Travelling waves

Ex 13 Find (if possible) real non-singular travelling wave solutions of the following equations, satisfying the given boundary conditions:

1. Modified KdV (mKdV) equation:

$$\begin{aligned} &u_t + 6u^2 u_x + u_{xxx} = 0 \\ &u \to 0,\ u_x \to 0,\ u_{xx} \to 0 \quad \text{as } x \to \pm\infty\ . \end{aligned} \tag{2.1}$$

2. 'Wrong sign' mKdV equation:

$$\begin{aligned} &u_t - 6u^2 u_x + u_{xxx} = 0 \\ &u \to 0,\ u_x \to 0,\ u_{xx} \to 0 \quad \text{as } x \to \pm\infty\ . \end{aligned} \tag{2.2}$$

3. ϕ^4 theory:

$$\begin{aligned} &u_{tt} - u_{xx} + 2u(u^2 - 1) = 0 \\ &u_t \to 0,\ u_x \to 0,\ u \to -1 \quad \text{as } x \to -\infty \\ &u_t \to 0,\ u_x \to 0,\ u \to +1 \quad \text{as } x \to +\infty\ . \end{aligned} \tag{2.3}$$

4. ϕ^6 theory:

$$\begin{aligned} &u_{tt} - u_{xx} + u(u^2 - 1)(3u^2 - 1) = 0 \\ &u_t \to 0,\ u_x \to 0,\ u \to 0 \quad \text{as } x \to -\infty \\ &u_t \to 0,\ u_x \to 0,\ u \to 1 \quad \text{as } x \to +\infty\ . \end{aligned} \tag{2.4}$$

5. Burgers equation:

$$\begin{aligned} &u_t + uu_x - u_{xx} = 0 \\ &u \to u_0,\ u_x \to 0 \quad \text{as } x \to -\infty \\ &u \to u_1,\ u_x \to 0 \quad \text{as } x \to +\infty\ , \end{aligned} \tag{2.5}$$

   where u_0 and u_1 are real constants with $u_0 > u_1 > 0$.
   [Hint: Start by showing that the boundary conditions relate the velocity v of the travelling wave to the sum of the constants u_0 and u_1.]

6. * Generalised KdV equation with $n = 1, 2, 3, \ldots$:

$$\begin{aligned} &u_t + (n+1)(n+2)u^n u_x + u_{xxx} = 0 \\ &u \to 0,\ u_x \to 0,\ u_{xx} \to 0 \quad \text{as } x \to \pm\infty\ . \end{aligned} \tag{2.6}$$

Ex 14* Using the analogy with the classical mechanics of a point particle moving in one spatial dimension, determine the qualitative behaviour of travelling wave solutions of the KdV equation on a circle, for which the integration constants A and B are non-zero.

3 Topological lumps and the Bogomol'nyi bound

Ex 15 This exercise involves the infinite chain of identical coupled pendula of section 2.3, whose equations of motion reduce to the sine-Gordon equation in the continuum limit $a \to 0$. We will simplify expression by setting $g = L = \frac{M}{a} = 1$. Let $\theta_n(t)$ be the angle to the vertical of the n-th pendulum ($n \in \mathbb{Z}$), which is hung at the position $x = na$ along the chain, at time t. The configuration of the system at time t is then specified by the collection of angles $\{\theta_n(t)\}_{n\in\mathbb{Z}}$.

1. Starting from the force (note: m is a dummy variable)

$$F_n(\{\theta_m\}) = -a\sin\theta_n + \frac{1}{a}(\theta_{n+1} - \theta_n) + \frac{1}{a}(\theta_{n-1} - \theta_n) \tag{3.1}$$

acting on the n-th pendulum, deduce the potential energy

$$V(\{\theta_m\}) = \sum_{n=-\infty}^{+\infty} (\cdots) \tag{3.2}$$

such that $F_n = -\frac{\partial V}{\partial \theta_n}$ for all $n \in \mathbb{Z}$, and fix the integration constant by requiring that the potential energy be zero when all pendula point down: $V(\{0\}) = 0$.

2. Show that in the continuum limit $a \to 0$, the potential energy computed above becomes

$$V = \int_{-\infty}^{+\infty} dx \, \left[(1 - \cos\theta) + \frac{1}{2}\theta_x^2\right] \ , \tag{3.3}$$

and the kinetic energy

$$T(\{\theta_m\}) = \frac{a}{2} \sum_{n=-\infty}^{+\infty} \dot{\theta}_n^2 \tag{3.4}$$

becomes

$$T = \int_{-\infty}^{+\infty} dx \, \frac{1}{2}\theta_t^2 \ , \tag{3.5}$$

where the function $\theta(x,t)$ is the continuum limit of $\{\theta_n(t)\}_{n\in\mathbb{Z}}$.

[**Hint**: in the continuum limit, $a \sum_{n=-\infty}^{+\infty} \to \int_{-\infty}^{+\infty} dx$.]

Ex 16 A field $u(x,t)$ has kinetic energy T and potential energy V, where

$$\begin{aligned} T &= \int_{-\infty}^{+\infty} dx \; \frac{1}{2} u_t^2 \,, \\ V &= \int_{-\infty}^{+\infty} dx \; \left[\frac{1}{2} u_x^2 + \frac{\lambda}{2} (u^2 - a^2)^2 \right] \,, \end{aligned} \tag{3.6}$$

and a and $\lambda > 0$ are (real) constants. (This is a version of the 'ϕ^4' theory. It's called like that because the scalar potential is quartic, and the field u is usually called ϕ.) The equation of motion for u is

$$u_{tt} - u_{xx} + 2\lambda u(u^2 - a^2) = 0\,. \tag{3.7}$$

1. If u is to have finite energy, what boundary conditions must be imposed on u, u_x and u_t at $x = \pm\infty$?
2. Find the general travelling-wave solution(s) to the equation of motion, consistent with the boundary conditions found in part 1. Compute the total energy $E = T + V$ for these solutions. For which velocity do the solutions have the lowest energy?
3. One of the possible boundary conditions for part 1 implies that u is a kink, with $[u(x)]_{x=-\infty}^{x=+\infty} = 2a$. Use the Bogomol'nyi argument to show that the total energy $E = T + V$ of that configuration is bounded from below by $C\sqrt{\lambda}a^3$, where C is a constant that you should determine, and find the solution u which saturates this bound. Verify that this solution agrees with the lowest-energy solution of part 2.

Ex 17

1. Explain why the Bogomol'nyi argument given in the lectures fails to provide a useful bound on the energy of a two-kink solution of the sine-Gordon equation (a two-kink solution is one with topological charge $n - m$ equal to 2). What is the most that can be said about the energy of a k-kink?
2. For a sine-Gordon field u, generalise the Bogomol'nyi argument to show that

$$\int_A^B dx \; \left[\frac{1}{2} u_t^2 + \frac{1}{2} u_x^2 + (1 - \cos u) \right] \geq \pm 4 \left[\cos \frac{u}{2} \right]_A^B \,. \tag{3.8}$$

3. * Use this result and the intermediate value theorem (look it up if necessary!) to show that if the field u has the boundary conditions of a k-kink, then its energy is at least k times that of a single kink. Can this bound be saturated?

Ex 18 A system on the finite interval $-\pi/2 \leq x \leq \pi/2$ is defined by the following expressions for the kinetic energy T and the potential energy V:

$$\begin{aligned} T &= \int_{-\pi/2}^{\pi/2} dx \; \frac{1}{2} u_t^2 \\ V &= \int_{-\pi/2}^{\pi/2} dx \; \frac{1}{2} \left(u_x^2 + 1 - u^2 \right) \; . \end{aligned} \tag{3.9}$$

The function $u(x,t)$ satisfies the boundary condition $|u(\pm\pi/2, t)| = 1$ and is required to satisfy $|u(x,t)| \leq 1$ everywhere. Show that with "kink" boundary conditions, the total energy E is bounded below by a positive constant, and find a solution for which the bound is saturated.

4 Conservation laws

Ex 19 Check explicitly that the energy

$$E = \int_{-\infty}^{+\infty} dx \, \left[\frac{1}{2} u_t^2 + \frac{1}{2} u_x^2 + \mathbb{V}(u) \right] \tag{4.1}$$

and the momentum

$$P = - \int_{-\infty}^{+\infty} dx \; u_t u_x \tag{4.2}$$

of a relativistic field $u(x,t)$ in 1 space and 1 time dimensions are conserved when the equation of motion

$$u_{tt} - u_{xx} = -\mathbb{V}'(u) \tag{4.3}$$

and the boundary conditions

$$u_t, \; u_x, \; \mathbb{V}(u), \; \mathbb{V}'(u) \underset{x \to \pm\infty}{\longrightarrow} 0 \qquad \forall t \tag{4.4}$$

are satisfied.

Ex 20

1. Compute the conserved topological charge, energy and momentum of a sine-Gordon kink moving with velocity v, and check that the results do not depend on time. [**Hint**: The integral (A.9) might be useful. For the scalar potential term in the energy, write $1 - \cos(u) = 2\sin^2(u/2)$, plug in the kink solution and manipulate the result using trigonometric formulae until (A.9) becomes useful.]
   Confirm that for $|v| \ll 1$ the energy and the momentum take the forms

   $$E = M + \frac{1}{2} M v^2 + \mathcal{O}(v^4) \; , \qquad P = Mv + \mathcal{O}(v^3) \tag{4.5}$$

   where the 'mass' M is the energy of the static kink, which appears in the Bogomol'nyi bound.

2. * If you are fearless and have time on your hands, try also to compute the conserved spin 3 charge

   $$Q_3 = \int_{-\infty}^{+\infty} dx \, \left[u_{++}^2 - \frac{1}{4} u_+^4 + u_+^2 \cos u \right] \tag{4.6}$$

   for the sine-Gordon kink. The integrals are not at all straightforward, but can be evaluated using appropriate changes of variables. (Did I write fearless?)

Ex 21 Find three conserved charges for the mKdV equation (2.1) of Ex 13.1, which involve u, u^2 and u^4 respectively. The boundary conditions on $u(x,t)$ are u, u_x and $u_{xx} \to 0$ as $|x| \to \infty$. Evaluate these quantities for the travelling-wave solution found in Ex 13.1. The list of definite integrals at the end of the problems sheet might help.

Ex 22 Show that u is a conserved density for Burgers' equation (2.5). Why is this result of no use in analysing the travelling wave solution of Ex 13.5?

Ex 23 Consider the KdV equation $u_t + 6uu_x + u_{xxx} = 0$ for the field $u(x,t)$.

1. Show that $\rho_1 \equiv u$, $\rho_2 \equiv u^2$ and $\rho_* \equiv xu - 3tu^2$ are all conserved densities, so that

$$Q_1 = \int_{-\infty}^{+\infty} dx \; u \;, \qquad Q_2 = \int_{-\infty}^{+\infty} dx \; u^2 \;, \qquad Q_* = \int_{-\infty}^{+\infty} dx \; (xu - 3tu^2) \tag{4.7}$$

are all conserved charges.

2. Evaluate the conserved charges Q_1, Q_2 and Q_* for the one-soliton solution centred at x_0 and moving with velocity $v = 4\mu^2$:

$$u_{\mu,\, x_0}(x,t) = 2\mu^2 \text{sech}^2 \left[\mu(x - x_0 - 4\mu^2 t)\right] \;. \tag{4.8}$$

3. According to the KdV equation, the initial condition $u(x,0) = 6\,\text{sech}^2(x)$ is known to evolve into the sum of two well-separated solitons with different velocities $v_1 = 4\mu_1^2$ and $v_2 = 4\mu_2^2$ at late times. Use the conservation of Q_1 and Q_2 to determine v_1 and v_2.

4. A two-soliton solution separates as $t \to -\infty$ into two one-solitons u_{μ_1, x_1} and u_{μ_2, x_2}. As $t \to +\infty$, two one-solitons are again found, with μ_1 and μ_2 unchanged but with x_1, x_2 replaced by y_1, y_2. Use the conservation of Q_* to find a formula relating the *phase shifts* $y_1 - x_1$ and $y_2 - x_2$ of the two solitons.

Ex 24

1. Show that if $u(x,t)$ satisfies the KdV equation $u_t + 6uu_x + u_{xxx} = 0$, and $u = \lambda - v^2 - v_x$ where λ is a constant and $v(x,t)$ some other function, then v satisfies

$$\left(2v + \frac{\partial}{\partial x}\right)\left(v_t + 6\lambda v_x - 6v^2 v_x + v_{xxx}\right) = 0 \,.$$

2. Compute the Gardner transform expansion

$$w(x,t) = \sum_{n=0}^{\infty} w_n(x,t)\varepsilon^n \tag{4.9}$$

up to order ε^4. Use the results to find the conserved charges $\widetilde{Q}_3$ and $\widetilde{Q}_4$, where

$$\widetilde{Q}_n = \int_{-\infty}^{+\infty} dx\ w_n\ . \tag{4.10}$$

Show that $\widetilde{Q}_3$ is the integral of a total x-derivative (and hence is zero), while $\widetilde{Q}_4 = \alpha\, Q_3$, where

$$Q_3 = \int_{-\infty}^{+\infty} dx\ \left(u^3 - \frac{1}{2}u_x^2\right) \tag{4.11}$$

is the third KdV conserved charge (the 'energy') and α a constant that you should determine. * If you're feeling energetic, try to compute $\widetilde{Q}_5$ and $\widetilde{Q}_6$ as well.

Ex 25 This question is also about the KdV equation $u_t + 6uu_x + u_{xxx} = 0$.

1. Evaluate the first three KdV conserved charges

$$Q_1 = \int_{-\infty}^{+\infty} dx\ u\ , \qquad Q_2 = \int_{-\infty}^{+\infty} dx\ u^2\ , \qquad Q_3 = \int_{-\infty}^{+\infty} dx\ \left(u^3 - \frac{1}{2}u_x^2\right) \tag{4.12}$$

for the initial state $u(x,0) = A\,\mathrm{sech}^2(Bx)$, where A and B are constants.

2. The initial state

$$u(x,0) = N(N+1)\,\mathrm{sech}^2(x)\ , \tag{4.13}$$

where N is an integer, is known to evolve at late times into N well-separated solitons, with velocities $4k^2$, $k = 1 \ldots N$. So for $t \to +\infty$, this solution approaches the sum of N single well-separated solitons

$$u(x,t) \approx \sum_{k=1}^{N} 2\mu_k^2\,\mathrm{sech}^2\left[\mu_k(x - x_k - 4\mu_k^2 t)\right]\ , \tag{4.14}$$

where $\mu_1, \ldots, \mu_N$ are N different constants. Since Q_1, Q_2 and Q_3 are conserved, their values at $t = 0$ and $t \to +\infty$ must be equal. Use this fact to deduce formulae for the sums of the first N integers, the first N cubes, and the first N fifth powers.

3. * Use Q_4 and Q_5 and the method just described to find the sum of the first N seventh and ninth powers, $\sum_{k=1}^{N} k^7$ and $\sum_{k=1}^{N} k^9$.

5 The Bäcklund transform

Ex 26 1. Show that the pair of equations

$$\begin{aligned}(u-v)_+ &= \sqrt{2}\, e^{(u+v)/2}\\ (u+v)_- &= \sqrt{2}\, e^{(u-v)/2}\end{aligned} \tag{5.1}$$

provides a Bäcklund transformation linking solutions of $v_{+-} = 0$ (the wave equation in light-cone coordinates) to those of $u_{+-} = e^u$ (the Liouville equation).

2. Starting from d'Alembert's general solution $v = f(x^+) + g(x^-)$ of the wave equation, use the Bäcklund transform (5.1) to obtain the corresponding solutions of the Liouville equation for u. [**Hint**: Set $u(x^+, x^-) = 2U(x^+, x^-) + f(x^+) - g(x^-)$. You might simplify the notation by setting $f(x^+) = \log(F'(x^+))$ and $g(x^-) = -\log(G'(x^-))$, where prime means first derivative.]

Ex 27 Consider the Bäcklund transform

$$v_x + \frac{1}{2} uv = 0 \tag{5.2}$$

$$v_t + \frac{1}{2} u_x v - \frac{1}{4} u^2 v = 0\,. \tag{5.3}$$

1. Show that (5.2) and (5.3) together imply that v satisfies the linear heat equation $v_t = v_{xx}$, while u satisfies Burgers' equation $u_t + uu_x - u_{xx} = 0$.
[**Hint**: for the first, solve (5.2) for u and substitute in (5.3); for the second, start by cross-differentiating.]

2. Find the *general* travelling-wave solution for $v(x,t)$ and, via the Bäcklund transform, re-obtain the travelling-wave for Burgers' equation found in question (2.5).

3. * The linear equation satisfied by $v(x,t)$ allows for the linear superposition of solutions. Use this fact, and your answers to part 2, to construct solutions for v and then u which describe the interaction of *two* travelling waves.

4. * Sketch your solutions functions of x at fixed times both before and after the interaction, and also draw their trajectories in the (x,t) plane, perhaps starting with the help of a computer. Are the travelling waves of Burgers' equation true solitons, in the sense given in lectures?
[**Hints**: Examine the asymptotics of the solution viewed from frames moving at various velocities V (that is, set $X_V = x - Vt$ and consider $t \to \pm\infty$ keeping X_V finite). This should allow you to isolate various travelling waves in these limits, and to decide whether they preserve their form under interactions. For definiteness, consider the case $c_1 > c_2 > 0$, where c_1 and c_2 are the velocities of the two separate travelling waves before they were superimposed. A further hint: as well as the 'expected' special values for V, namely c_1 and c_2, be careful about what happens when $V = c_1 + c_2$.]

Ex 28 1. Show that the two equations

$$\begin{aligned} v_x &= -u - v^2 \\ v_t &= 2u^2 + 2uv^2 + u_{xx} - 2u_x v \end{aligned} \tag{5.4}$$

are a Bäcklund transform relating solutions of the KdV equation

$$u_t + 6uu_x + u_{xxx} = 0 \tag{5.5}$$

and the wrong sign modified KdV (mKdV) equation

$$v_t - 6v^2 v_x + v_{xxx} = 0\,. \tag{5.6}$$

(Note the appearance of the Miura transform in (5.4).)

2. Taking $u = c^2$, where c is a constant, as a seed solution of the KdV equation, find the corresponding solution of the wrong sign mKdV equation.

Ex 29 The 2-soliton solution of the sine-Gordon equation with Bäcklund parameters a_1 and a_2 is

$$u(x,t) = 4\arctan\left(\mu\,\frac{e^{\theta_1} - e^{\theta_2}}{1 + e^{\theta_1+\theta_2}}\right)\,, \qquad \theta_i = \varepsilon_i\gamma_i(x - v_i t - \bar{x}_i) \tag{5.7}$$

where $\mu = (a_2{+}a_1)/(a_2{-}a_1)$, $v_i = (a_i^2{-}1)/(a_i^2{+}1)$, $\gamma_i = 1/\sqrt{1{-}v_i^2}$, $\varepsilon_i = \mathrm{sign}(a_i)$, and $\bar{x}_1$ and $\bar{x}_2$ are constants, as in the lectures. Rewriting u as a function of $X_V \equiv x - Vt$ and t, show that, for $V \neq v_1, v_2$ (and $v_1 \neq v_2$)

$$\lim_{\substack{t\to\infty \\ X_V \ \text{finite}}} u = 2n\pi\ ,$$

where n is an integer. If $v_2 > v_1 > 0$ and $\varepsilon_i = 1$, how does the parity of n (whether it is even or odd) depend on the value of v relative to v_1 and v_2?
[**Hints**: First show that $|\theta_i| \to +\infty$ as $t \to \pm\infty$; then consider each of the four possible options $(\theta_1, \theta_2) \to (+\infty, +\infty)$, $(-\infty, -\infty)$, $(+\infty, -\infty)$, $(-\infty, +\infty)$. Remember that $\arctan(0) = m\pi$ and $\arctan(\pm\infty) = \pm\pi/2 + m\pi$, where the ambiguities of $m\pi$, $m \in \mathbb{Z}$, encode the multivalued nature of the arctan function.]

Ex 30 Find the asymptotics of the 2-soliton sine-Gordon solution defined in equation (5.7), in the case $a_2 > a_1 > 0$, as $t \to \pm\infty$ with $X_{v_2} \equiv x{-}v_2 t$ held finite.

Ex 31 Show by direct analysis (as in the lectures) that taking a_1 and a_2 of opposite signs in (5.7) results in a two-kink, or two-antikink, solution to the sine-Gordon equation.

Ex 32

1. The argument of the arctangent in the sine-Gordon 2-soliton solution (5.7) is a continuous function of x for all $x \in \mathbb{R}$. Show that, in particular, it is never infinite. What does this imply about the range of u? [**Hint**: consider the graph of $\tan u/4$.]
2. By taking the limits of this function as $x \to \pm\infty$ (with $t = \bar{x}_1 = \bar{x}_2 = 0$ for simplicity), show that the topological charge of the two-soliton solution (5.7) is 0 if $\mathrm{sign}(a_1) = \mathrm{sign}(a_2)$, and ± 2 if $\mathrm{sign}(a_1) = -\mathrm{sign}(a_2)$, in units where the topological charge of a kink is 1.

Ex 33 Consider the two-soliton solution of the sine-Gordon equation (5.7) with complex Bäcklund parameters $a_1 = \overline{a_2} := a \in \mathbb{C}$ and with vanishing integration constants, as is appropriate to find the breather solution. Show that

$$\begin{aligned} \mathrm{Re}(\theta_1) &= +\mathrm{Re}(\theta_2) = \gamma(x - vt)\cos\varphi \ , \\ \mathrm{Im}(\theta_1) &= -\mathrm{Im}(\theta_2) = \gamma(vx - t)\sin\varphi \ , \end{aligned} \tag{5.8}$$

where $\varphi = \arg(a)$ and

$$\begin{aligned} v &= \frac{|a|^2 - 1}{|a|^2 + 1} \\ \gamma &= \frac{1}{\sqrt{1 - v^2}} = \frac{1 + |a|^2}{2|a|} \ . \end{aligned} \tag{5.9}$$

Ex 34 The *stationary* breather solution of the sine-Gordon equation (that is the breather solution with $v = 0$) has the form

$$\tan\frac{u}{4} = \frac{\cos\varphi}{\sin\varphi} \cdot \frac{\sin(t\sin\varphi)}{\cosh(x\cos\varphi)} \ . \tag{5.10}$$

Show that in the limit $\varphi \to 0$, in which the kink and antikink that form the breather are very loosely bound, the time period τ of a single oscillation of the breather scales like $\tau \sim |\varphi|^{-1}$, and the spatial size x_{max} of the breather scales like $x_{\mathrm{max}} \sim -\log\varphi$.
[**Hint**: You could define x_{max} as the value of x at which $\tan(u/4) = 1$ when the oscillatory factor in the numerator is at its maximum. Focus only on the parametric dependence on φ, ignoring all numerical factors.]

6 The Hirota method

Ex 35 We have seen in lectures that the KdV equation $u_t + 6uu_x + u_{xxx} = 0$ for the field $u(x,t)$ that describes the profile of a wave translates into the following equation for the new variable $w(x,t) = \int dx\ u$:

$$w_t + 3w_x^2 + w_{xxx} = 0 \ . \tag{6.1}$$

Let $w = 2\frac{\partial}{\partial x}\log f = 2\frac{f_x}{f}$ where $f(x,t)$ is a nowhere vanishing function of x and t, so that $u = 2\frac{\partial^2}{\partial x^2}\log f$. The aim of this exercise is to rewrite (6.1) as an equation for f.

1. Express w_t, w_x, w_{xx} and w_{xxx} in terms of f and its derivatives.
2. Show that the equation (6.1) can be rewritten as

$$f f_{xt} - f_x f_t + 3f_{xx}^2 - 4f_x f_{xxx} + f f_{xxxx} = 0 \ , \tag{6.2}$$

which is known as the *quadratic form* of the KdV equation.

Ex 36 The Hirota bilinear differential operator $D_t^m D_x^n$ is defined for any pair of natural numbers (m,n) by

$$D_t^m D_x^n(f,g) = \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'}\right)^m \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right)^n f(x,t)g(x',t')\Bigg|_{\substack{x'=x\\ t'=t}} \tag{6.3}$$

and maps a pair of functions $(f(x,t), g(x,t))$ into a single function.

1. Prove that the Hirota operators $B_{m,n} := D_t^m D_x^n$ are bilinear, *i.e.* for all constants a_1, a_2

$$\begin{aligned} B_{m,n}(a_1 f_1 + a_2 f_2, g) &= a_1 B_{m,n}(f_1, g) + a_2 B_{m,n}(f_2, g) \ , \\ B_{m,n}(f, a_1 g_1 + a_2 g_2) &= a_1 B_{m,n}(f, g_1) + a_2 B_{m,n}(f, g_2) \ . \end{aligned} \tag{6.4}$$

2. Prove the symmetry property

$$B_{m,n}(f,g) = (-1)^{m+n} B_{m,n}(g,f) \ . \tag{6.5}$$

3. Compute the Hirota derivatives $D_t^2(f,g)$ and $D_x^4(f,g)$, and verify that your expression for the latter is consistent with the result for $D_x^4(f,f)$ given in lectures.

Ex 37 Define a “non-Hirota" bilinear differential operator $\tilde{D}_t^m \tilde{D}_x^n$ by

$$\tilde{D}_t^m \tilde{D}_x^n(f,g) = \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial t'}\right)^m \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial x'}\right)^n f(x,t)g(x',t')\Bigg|_{\substack{x'=x \\ t'=t}} \tag{6.6}$$

(note the plus signs!).

1. Compute $\tilde{D}_x(f,g)$ and $\tilde{D}_t(f,g)$, verifying that in both cases the answer is given by the corresponding ‘ordinary’ derivative of the product $f(x,t)g(x,t)$.
2. How does this result generalise for arbitrary non-Hirota differential operators (6.6)? Prove your claim.
3. Compare your answer with the Hirota operators defined above.

Ex 38

1. If $\theta_i = a_i x + b_i t + c_i$, prove that

$$D_t D_x(e^{\theta_1}, e^{\theta_2}) = (b_1 - b_2)(a_1 - a_2)e^{\theta_1+\theta_2}.$$

2. Prove the corresponding result for $D_t^m D_x^n(e^{\theta_1}, e^{\theta_2})$, as quoted in lectures.

Ex 39 Prove that

$$D_t^m D_x^n(f,1) = \frac{\partial^m}{\partial t^m}\frac{\partial^n}{\partial x^n} f \ . \tag{6.7}$$

Ex 40 Consider the function f, such that $u = 2\frac{\partial^2}{\partial x^2}\log f$ is the KdV field, which corresponds to a 2-soliton solution:

$$f = 1 + \epsilon f_1 + \epsilon^2 f_2 = 1 + \epsilon\left(e^{\theta_1} + e^{\theta_2}\right) + \epsilon^2 \left(\frac{a_1 - a_2}{a_1 + a_2}\right)^2 e^{\theta_1+\theta_2} \ , \tag{6.8}$$

where $\theta_i = a_i x - a_i^3 t + c_i$, with a_i and c_i constants. Check that $B(f_1, f_2) = 0$ and $B(f_2, f_2) = 0$, where $B = D_x(D_t + D_x^3)$, and show that this implies that the expansion (6.8), which is truncated at order ϵ^2, is a solution of the bilinear form of the KdV equation.

Ex 41* Derive the solution of the bilinear form of the KdV equation $D_x(D_t + D_x^3)(f, f) = 0$ which represents the 3-soliton solution, in the form

$$f = 1 + \epsilon f_1 + \epsilon^2 f_2 + \epsilon^3 f_3 \tag{6.9}$$

where $f_1 = \sum_{i=1}^{3} e^{\theta_i}$. [This includes proving that the higher order terms in the ϵ expansion can be consistently set to zero, as in Ex 40.]

Ex 42 Show that the Boussinesq equation

$$u_{tt} - u_{xx} - 3(u^2)_{xx} - u_{xxxx} = 0 \tag{6.10}$$

can be written in the bilinear form

$$(D_t^2 - D_x^2 - D_x^4)(f, f) = 0 \tag{6.11}$$

where $u = 2\frac{\partial^2}{\partial x^2} \log f$.

Ex 43 Show that the following higher-dimensional version of the KdV equation,

$$(u_t + 6uu_x + u_{xxx})_x + 3\sigma^2 u_{yy} = 0 \tag{6.12}$$

for the field $u(x, y, t)$, also known as the Kadomtsev-Petviashvili (KP) equation, can be written in the bilinear form

$$(D_t D_x + D_x^4 + 3\sigma^2 D_y^2)(f, f) = 0 \tag{6.13}$$

where $u(x, y, t) = 2\frac{\partial^2}{\partial x^2} \log f(x, y, t)$.

7 Overview of the Inverse Scattering Method

In the following exercises Fourier transform will be denoted as $\mathbf{F}[f(x)] = \tilde{f}(k)$.

Ex 44

1. The *convolution* of f and g is defined as

$$(f * g)(x) = \int_{-\infty}^{+\infty} dz\ f(z)\, g(x - z)\ . \tag{7.1}$$

   Prove that $\mathbf{F}[fg] = \frac{1}{2\pi}\tilde{f}(k) * \tilde{g}(k)$ and $\mathbf{F}[f * g] = \tilde{f}(k)\tilde{g}(k)$.

2. The *cross-correlation* of f and g is defined as

$$(f \otimes g)(x) = \int_{-\infty}^{+\infty} dz\ f^*(z)\, g(x + z)\ . \tag{7.2}$$

   Prove the *Weiner-Kinchin theorem*, that $\mathbf{F}[f \otimes g] = \tilde{f}^*(k)\tilde{g}(k)$.

3. The *auto-correlation* of $f(x)$ is defined as

$$a(x) = (f \otimes f)(x). \tag{7.3}$$

   Using the answer to 2, verify that $\mathbf{F}[a] = |\tilde{f}(k)|^2$. This is called the *energy spectrum* of f.

4. Using the above result obtain the FT version of *Parseval's theorem*, which you may have already seen for Fourier series:

$$\int_{-\infty}^{+\infty} dx\ |f(x)|^2 = \int_{-\infty}^{+\infty} \frac{dk}{2\pi}\ |\tilde{f}(k)|^2\ . \tag{7.4}$$

Ex 45

1. Show that $e^{-x^2/2}$ is (up to a factor $\sqrt{2\pi}$) its own FT.

2. Find the FT of

$$f(x) = \begin{cases} 1/(2\varepsilon) & |x| \le \varepsilon \\ 0 & |x| > \varepsilon \end{cases} \tag{7.5}$$

   and discuss the $\varepsilon \to 0$ limit.

3. Find the FT of

$$f(x) = \begin{cases} 1 - x^2 & |x| < 1 \\ 0 & |x| > 1 \end{cases}. \tag{7.6}$$

Ex 46

1. Find the general solution of the heat equation $u_t = u_{xx}$ in the form

$$u(x,t) = \int_{-\infty}^{+\infty} dk\ \tilde{u}(k,0) f(k,x,t)\ , \tag{7.7}$$

   where $\tilde{u}(k,0)$ is the Fourier transform of the initial condition $u(x,0)$ and $f(k,x,t)$ is a function of k, x and t that you should determine.

2. Evaluate the previous integral over k in the case where the initial condition is $u(x,0) = \delta(x)$, to obtain the corresponding solution $u(x,t)$ for $t > 0$ explicitly. [**Hint**: look at the definite integrals in the last page of the problem sheet and read the note below.]

3. Finally, use this last result to rewrite the general solution as in equation (7.2) in the lecture notes.

Ex 47 Find the general solution of the linearised KdV equation $u_t + u_{xxx} = 0$. Your answer should be in the form of an integral involving $\tilde{u}(k,0)$, the Fourier transform of the initial condition $u(x,0)$.

Ex 48 Try to solve the full (non-linear) KdV equation using the same method, Fourier transform. [Do not try too hard as it is impossible! Just convince yourself that it is impossible and understand what goes wrong/why the Fourier transform doesn't work in the non-linear case.]

8 The KdV-Schrödinger connection

Ex 49 Show that if $u(x,t)$ satisfies the KdV equation $u_t + 6uu_x + u_{xxx} = 0$, and $u = \lambda - v^2 - v_x$ where λ is a constant and $v(x,t)$ some other function, then v satisfies

$$\big(2v + \frac{\partial}{\partial x}\big)\big(v_t + 6(\lambda - v^2)v_x + v_{xxx}\big) = 0\,.$$

(You might recognise this problem from last term!)

9 Time evolution of the scattering data

Ex 50 If λ is an eigenvalue of $\frac{d^2}{dx^2}\psi(x) + u(x)\psi(x) = \lambda\psi(x)$, where we require $\int_{-\infty}^{+\infty}|\psi(x)|^2 dx < \infty$, and $u(x)$ is real, prove that λ must also be real. (Hint: start by multiplying by $\psi(x)^*$ and integrating.)

Ex 51 Let $D = d/dx$ and let $g(x)$ be a general function of x.

1. Show that, as differential operators,

$$Dg = g_x + gD \ , \qquad D^2 g = g_{xx} + 2g_x D + gD^2 \ .$$

2. Show more generally that

$$D^n g = \sum_{m=0}^{n} \binom{n}{m} \frac{d^m g}{dx^m} D^{n-m} \ .$$

[**Hint**: to show that two differential operators are equal, you just have to show that they have the same effect on any function $f(x)$. For part (b), either try induction or think about the formula for the differentiation of a product.]

Ex 52 Let $D = \partial/\partial x$, and

$$L(u) = D^2 + u(x,t) \ , \qquad B(u) = -(4D^3 + 6uD + 3u_x) \ .$$

Check that

$$L(u)_t + [L(u), B(u)] = u_t + 6uu_x + u_{xxx} \ .$$

10 Interlude: the KdV hierarchy and conservation laws

Ex 53 Let $L(u) = D^2 + u(x,t)$ and $B(u) = \alpha D$ for some constant α.

1. Check that
$$L(u)_t = [B(u), L(u)] \quad \Longleftrightarrow \quad u_t = \alpha u_x \ .$$

2. Let $\psi(x,0)$ be an eigenfunction of $L(u)$ at $t = 0$ with eigenvalue λ, so that
$$(D^2 + u(x,0))\psi(x,0) = \lambda\psi(x,0) \ .$$
If $u(x,t)$ evolves according to the equation of part 1, find an eigenfunction $\psi(x,t)$ for each later time t, with the same eigenvalue λ, so that
$$(D^2 + u(x,t))\psi(x,t) = \lambda\psi(x,t) \ .$$
Verify that $\psi(x,t)$ can be arranged to satisfy $\psi_t = B(u)\psi$. (You can assume that the eigenfunction is non-degenerate, namely that there is a single eigenfunction with that eigenvalue. This is the case both for bound state solutions and for scattering solutions.)

Ex 54

1. Show that the differential operator $D = \partial/\partial x$ is anti-symmetric with respect to the inner product
$$(\psi_1, \psi_2) := \int_{-\infty}^{+\infty} dx \ \psi_1(x)^* \psi_2(x)$$
on the space $L^2(\mathbb{R})$ of square integrable functions, that is $(\psi_1, D\psi_2) = -(D\psi_1, \psi_2)$ for all ψ_1, $\psi_2 \in L^2(\mathbb{R})$.

2. Show that $L(u) = D^2 + u(x,t)$ is self-adjoint, given that u is real.

3. Given a Lax pair $L(u)$, $B(u)$, show that the symmetric part of $B(u)$ commutes with $L(u)$ and therefore drops out of the Lax equation $L(u)_t = [B(u), L(u)]$.

4. Now assume that $B(u)$ is anti-symmetric. Show that (ψ_1, ψ_2) is independent of time t if $\psi_i(x;t)$ evolves according to the equation $(\psi_i)_t = B(u)\psi_i$.

Ex 55

1. Show that the differential operator of order $2m - 1$
$$B(u) = \sum_{j=1}^{m} \big(\beta_j(x) D^{2j-1} + D^{2j-1}\beta_j(x)\big)$$
is anti-symmetric if the functions $\beta_j(x)$ are real.

2. If $L(u) = D^2 + u(x,t)$, compute the leading term of $[L(u), B(u)]$ in the form $\gamma(x)D^{2m}$. If $[L(u), B(u)]$ is to be purely multiplicative (forcing $\gamma(x)$ to be zero), deduce that $\beta_m(x)$ must be a constant.

Ex 56 Consider the $m = 2$ case of the equation from Ex 55 part 1. Given the result of that question, you can assume that β_2 is a constant. Fix a normalization by imposing $\beta_2 = 1/2$, and find the most general form of β_1 which allows $[L(u), B(u)]$ to be multiplicative. Show that the Lax equation $L(u)_t + [L(u), B(u)] = 0$ is equivalent to the following alternative version of the KdV equation

$$u_t = \frac{1}{4}u_{xxx} + \frac{3}{2}uu_x + 2ku_x \ , \tag{*}$$

where k is an integration constant. Finally, check that the redefined field

$$\tilde{u}(x,t) = u(x + 8kt, -4t)$$

solves the standard KdV equation $\tilde{u}_t + 6\tilde{u}\tilde{u}_x + \tilde{u}_{xxx} = 0$.

Ex 57 Consider the $m = 3$ case of the equation from Ex 55 part 1. Given the result of that question, you can assume that $L(u)_t + [L(u), B(u)] = 0$ forces β_3 to be a constant. Complete the calculation to find the most general form of β_2 and β_1 which allow $[L(u), B(u)]$ to be multiplicative. Deduce from a special case of your result that a function $u(x,t)$ evolving according to the fifth-order KdV equation

$$u_t + 30u^2u_x + 20u_xu_{xx} + 10uu_{xxx} + u_{xxxxx} = 0$$

leaves the eigenvalues of $L(u) = D^2 + u$ invariant.

Ex 58 The functional derivative $\delta F/\delta u$ of $F[u]$ is defined by the equation

$$F[u + \delta u] = F[u] + \int_{-\infty}^{+\infty} dx\ \frac{\delta F[u]}{\delta u(x)}\delta u(x) + \mathcal{O}((\delta u)^2) \ ,$$

where the infinitesimal variation $\delta u(x)$ is small everywhere and goes to zero at the boundaries of the integration range (the same applies to its derivatives δu_x, δu_{xx}, ...). If

$$F[u] = \int_{-\infty}^{+\infty} dx\ f(u, u_x, u_{xx}, u_{xxx}, \dots) \ ,$$

show that

$$\frac{\delta F[u]}{\delta u} = \frac{\partial f}{\partial u} - \frac{\partial}{\partial x}\frac{\partial f}{\partial u_x} + \frac{\partial^2}{\partial x^2}\frac{\partial f}{\partial u_{xx}} - \frac{\partial^3}{\partial x^3}\frac{\partial f}{\partial u_{xxx}} + \dots$$

Ex 59

1. Find a function $f(u, u_x, u_{xx})$ and a functional
$$F[u] = \int_{-\infty}^{+\infty} dx\ f(u, u_x, u_{xx})$$
such that the equation
$$u_t = \frac{\partial}{\partial x}\frac{\delta F}{\delta u}$$
is the same as the fifth-order KdV equation from exercise 57.
2. Show that your $F[u]$ is a conserved quantity if u evolves according to the standard third order KdV equation.
3. Show that $\int_{-\infty}^{+\infty} dx\ u$ is a conserved quantity if u evolves according to the fifth-order KdV equation (γ).

11 Basics of scattering theory

Ex 60 The *Wronskian* $W[f,g](x)$ of two differentiable functions $f(x)$ and $g(x)$ is defined as

$$W[f,g](x) = f'(x)g(x) - f(x)g'(x) \ . \tag{11.1}$$

If the functions f and g are linearly dependent, then their Wronskian vanishes identically: $W[f,g](x) = 0$. (Equivalently, if $W[f,g](x) \neq 0$, the functions f and g are linearly independent.) Conversely, if the Wronskian vanishes identically for two *analytic* functions f and g, then f and g are linearly dependent.

1. Write down the Wronskian $W[\overline{\psi}_1, \psi_2](x)$ of two eigenfunctions $\psi_{1,2}(x)$ of the time-independent Schrödinger equation with the same potential $V(x)$ and possibly different eigenvalues k_i^2:

$$\psi_i''(x) - V(x)\psi_i(x) = -k_i^2\psi_i(x) \qquad (i = 1,2) \ . \tag{11.2}$$

   (This is just preparation for what follows, no computation is needed.)

2. Show that the Wronskian is constant if the two eigenfunctions correspond to the same eigenvalue.
3. Show that two eigenfunctions with different eigenvalues are orthogonal with respect to the hermitian inner product

$$\langle \psi_1, \psi_2 \rangle := \int_{-\infty}^{+\infty} dx \ \overline{\psi}_1(x)\psi_2(x) \tag{11.3}$$

   if at least one of the two eigenfunctions describes a bound state.

4. Show that the Wronskian vanishes for two eigenfunctions with the same eigenvalue in the discrete spectrum. (This implies the linear dependence of the two eigenfunctions, provided that they are sufficiently differentiable.) [**Hint**: consider the limit $x \to \pm\infty$.]
5. The $x \to \pm\infty$ asymptotics of a scattering solution $\psi(x)$ with eigenvalue $k^2 > 0$ is

$$\psi(x) \approx \begin{cases} e^{ikx} + R(k) \ e^{-ikx} \ , & x \to -\infty \\ T(k) \ e^{ikx} \ , & x \to +\infty \end{cases} \tag{11.4}$$

   By evaluating the Wronskian $W[\overline{\psi}, \psi]$ at $x \to \pm\infty$, show that the reflection and transmission coefficients $R(k)$ and $T(k)$ satisfy

$$|R(k)|^2 + |T(k)|^2 = 1 \ . \tag{11.5}$$

Ex 61 Consider the time independent Schrödinger equation

$$\left(-\frac{d^2}{dx^2} + V(x)\right)\psi(x) = k^2\psi(x) \tag{11.6}$$

with energy $E = k^2$ for the square barrier/well potential

$$V(x) = \begin{cases} 0\ , & x < 0 \\ V_0\ , & 0 < x < a \\ 0\ , & x > a \end{cases} \tag{11.7}$$

where $a > 0$ and V_0 are constants.

1. Show that the matching conditions to be imposed at $x = 0$ and a, where the potential (11.7) is discontinuous (but finite), are that $\psi(x)$ and $\psi'(x)$ are continuous.
2. Solve the Schrödinger equation (11.6) in the three regions (11.7) and impose the matching conditions to find the scattering solutions associated to energy eigenvalues $k^2 > 0$ in the continuous spectrum, and determine the reflection and transmission coefficients $R(k)$ and $T(k)$ in terms of a and $l = \sqrt{k^2 - V_0}$.
3. For which values of the wavenumber k is the square potential (11.7) transparent, that is $R(k) = 0$?
4. Write down the bound state solutions corresponding to the discrete spectrum $k^2 = -\mu^2 < 0$. Find the equations that determine implicitly the allowed values of μ in terms of a and l (or V_0).
5. Do bound state solutions exist for $V_0 > 0$? And for $V_0 < 0$? In the latter case, use a graphical argument to show that a new bound state solution appears every time that $\sqrt{-V_0}$ crosses a non-negative integer multiple of π/a.
6. Show that in the limit $a \to 0$, $V_0 \to +\infty$ with $b = aV_0$ fixed, the reflection and transmission coefficients reduce to those of the delta-function potential $V(x) = b\delta(x)$.

Ex 62 Consider the time independent Schrödinger equation

$$-\psi''(x) + V(x)\psi(x) = k^2\psi(x)\ , \tag{11.8}$$

where the potential $V(x)$ is the sum of two delta functions:

$$V(x) = -a\delta(x) - b\delta(x-r)\ . \tag{11.9}$$

Taking $r > 0$, the solution $\psi(x)$ can be split into three pieces, $\psi_1(x)$, $\psi_2(x)$ and $\psi_3(x)$, defined on $(-\infty, 0)$, $(0, r)$, and $(r, +\infty)$ respectively.

1. Write down the four matching conditions relating ψ_1, ψ_2 and ψ_3, and their derivatives, at $x = 0$ and $x = r$.

2. For a scattering solution describing waves incident from the left, ψ_1 and ψ_3 are given by
$$\psi_1(x) = e^{ikx} + R(k)e^{-ikx}, \quad \psi_3(x) = T(k)e^{ikx}. \tag{11.10}$$
Write down the general form of ψ_2, and then use the matching conditions found in part 1 to eliminate the unknowns and determine $R(k)$ and $T(k)$.
3. Show from the answer to part 2 that, for there to be a bound state pole at $k = i\mu$, μ must satisfy
$$e^{-2\mu r} = (1 - 2\mu/a)(1 - 2\mu/b)\ . \tag{11.11}$$
4. The solutions to (11.11) can be analysed using a graphical method. Show that:
   (a) if both a and b are negative, then there are no bound states;
   (b) if a and b have opposite signs, then there is at most one bound state, occurring when $a + b > rab$ (note: since a and b have opposite signs, rab is negative);
   (c) if a and b are positive, then the number of bound states is one if $rab \leq a + b$, and two otherwise.

   Sketch on the ab-plane the regions which correspond to zero, one and two bound states, and indicate the form of $\psi(x)$ for each of the two bound states found when $ab/(a + b) > r^{-1}$.

Ex 63 The time independent Schrödinger equation
$$-\psi''(x) + V(x)\psi(x) = k^2\psi(x) \tag{11.12}$$
is conjectured to have solutions in the form
$$\psi(x) = e^{ikx}(2k + iw(x))\ , \tag{11.13}$$
where $w(x)$ is real, non-singular for all x, independent of k, and has finite limits as $x \to \pm\infty$. Substituting in, deduce the equation
$$w'(x) + \frac{1}{2}w^2(x) = 2\mu^2\ , \tag{11.14}$$
where μ is an integration constant. [**Hint**: take real and imaginary parts of an intermediate equation.] Solve this via the substitution $w(x) = 2f'(x)/f(x)$, and deduce that $V(x)$ must have the form
$$V(x) = -2\mu^2\mathrm{sech}^2(\mu(x - x_0))\ . \tag{11.15}$$
Show also that $u = -V$ is a solution of the KdV equation provided that x_0 depends on t in a certain way that you should determine.

Ex 64 Using the results of the last question, show that $V(x) = -2\mu^2 \mathrm{sech}^2(\mu(x - x_0))$ is an example of a reflectionless potential, for which $R(k) = 0$. By adjusting the normalisation of the wavefunction $\psi(x)$ correctly, find out what the reflection coefficient $T(k)$ is for this potential. Verify that $|T(k)|^2 = 1$, consistently with the idea that for such a potential an incident particle must certainly be transmitted.

Ex 65 Show by induction or otherwise that the general solution to the differential equation

$$\psi_n''(x) = (-k^2 - n(n+1)\mathrm{sech}^2 x)\psi_n(x) \qquad (n = 0, 1, 2, \dots) \tag{11.16}$$

is given by $\psi_n(x) = \mathcal{O}_n \mathcal{O}_{n-1} \dots \mathcal{O}_1 \psi_0(x)$, where

$$\psi_0(x) = A(k)e^{ikx} + B(k)e^{-ikx} \ , \tag{11.17}$$

$A(k)$ and $B(k)$ are constants (with respect to x), and $\mathcal{O}_l$ is the differential operator

$$\mathcal{O}_l = \frac{d}{dx} - l \tanh x \ . \tag{11.18}$$

Find the asymptotic behaviour of this solution as $x \to \pm\infty$ and hence find the eigenvalues k^2 for the bound states of the potential $V(x) = -n(n+1)\mathrm{sech}^2 x$.

12 The Marchenko equation

13 Integrable systems in classical mechanics

Background The following exercise applies the idea of a Lax pair to a system with a finite number of degrees of freedom in classical mechanics.[2] The degrees of freedom are the *coordinates* $q_i(t)$ and the *momenta* $p_i(t)$, with $i = 1, \ldots, n$. Their time evolution is governed by Hamilton's equations

$$\dot{q}_i = \frac{\partial H}{\partial p_i} , \qquad \dot{p}_i = -\frac{\partial H}{\partial q_i} , \tag{13.1}$$

where a dot denotes a time derivative and the function $H(q, p)$ is called the Hamiltonian, which is constant under time evolution and equals the total energy of the system. If the Hamiltonian takes the form

$$H(q, p) = T(p) + V(q) = \sum_i \frac{p_i^2}{2} + V(q) , \tag{13.2}$$

then Hamilton's equations are equivalent to $p_i = \dot{q}_i$ and Newton's equations of motion $\ddot{q}_i = -\frac{\partial V}{\partial q_i}$. For a classical *integrable system*, the equations of motion are equivalent to the Lax equation $\dot{L} + [L, M] = 0$, where L and M are $n \times n$ matrices.

Ex 64 Let $n = 3$, with $q_{i+3} = q_i$ and $p_{i+3} = p_i$. A Lax pair of matrices L and M are given by

$$L = \begin{bmatrix} p_1 & b_1 & b_3 \\ b_1 & p_2 & b_2 \\ b_3 & b_2 & p_3 \end{bmatrix} , \quad M = \begin{bmatrix} 0 & b_1 & -b_3 \\ -b_1 & 0 & b_2 \\ b_3 & -b_2 & 0 \end{bmatrix} , \tag{13.3}$$

where $p_i = \dot{q}_i$ and $b_i = \exp[c(q_i - q_{i+1})]$ for some constant c. Use the Lax equation $\dot{L} + [L, M] = 0$ to find the constant c and to obtain equations of motion in the form $\ddot{q}_i = f_i(q)$, for some functions $f_i(q)$ that you should determine.

[2] This is to be contrasted with a field $u(x, t)$, which has infinitely many degrees of freedom, namely the values of u at all the different positions x at a fixed time t.

14 Exam-style problem

Ex 44 The complex field $u(x,t)$ obeys the equation

$$iu_t + \frac{1}{2}u_{xx} + |u|^2 u = 0 \; , \tag{14.1}$$

where $i = \sqrt{-1}$, and the boundary conditions

$$u, u_x, u_{xx} \to 0 \qquad \text{as} \quad x \to \pm\infty \; . \tag{14.2}$$

1. Show that the quantities

$$\begin{aligned} Q_1 &= \int_{-\infty}^{+\infty} dx \; |u|^2 \\ Q_2 &= \int_{-\infty}^{+\infty} dx \; \mathrm{Im}(\bar{u}u_x) \\ Q_3 &= \int_{-\infty}^{+\infty} dx \; \left(\frac{1}{2}|u_x|^2 + C|u|^4\right) \end{aligned} \tag{14.3}$$

   are conserved provided that the constant C takes a value that you should find. (Here Im denotes the imaginary part and a bar denotes complex conjugation.)

2. Show that given a 'seed' solution $u(x,t)$ of equation (14.1),

$$u^{(v)}(x,t) := u(x - vt, t)e^{i(Ax+Bt)} \tag{14.4}$$

   is also a solution for all $v \in \mathbb{R}$, provided that the constants A and B depend on v in a way that you should find.

3. Determine the functional dependence of the conserved charges Q_1, Q_2, Q_3 in (14.3) on the parameter v that labels the one-parameter family of solution (14.4).

4. Find all solutions of the form

$$u(x,t) = \rho(x)e^{i\varphi(t)} \tag{14.5}$$

   of equation (14.1) with boundary conditions (14.2), where ρ are φ are real and $u(x,0)$ is a real even function of x. [You can use the integrals at the end of the problem sheet.] Apply the method of part 2 to this seed solution to find the associated one-parameter family of solutions $u^{(v)}(x,t)$.

A Useful integrals

You can freely quote the following formulae, athough deriving them may be instructive:

- **Indefinite integrals:** [**Note:** the integration constant is in principle complex]

$$\int \frac{dx}{x\sqrt{1-x}} = -2\text{arcsech}(\sqrt{x}) \tag{A.1}$$

$$\int \frac{dx}{x\sqrt{1-x^2}} = -\text{arcsech}(x) \tag{A.2}$$

$$\int \frac{dx}{x\sqrt{1+x^2}} = -\text{arccosech}(x) \tag{A.3}$$

$$\int \frac{dx}{\sin(x/2)} = 2\log\tan(x/4) \tag{A.4}$$

$$\int \frac{dx}{\cosh(x)} = 2\arctan(e^x) \tag{A.5}$$

$$\int \frac{dx}{1-x^2} = \text{arctanh}(x) \tag{A.6}$$

$$\int dx\ \sqrt{1-x^2} = \frac{1}{2}\left[x\sqrt{1-x^2} + \arcsin(x)\right] \tag{A.7}$$

$$\int \frac{dx}{\cos^2(x)} = \tan(x) \tag{A.8}$$

$$\int \frac{dx}{\cosh^2(x)} = \tanh(x) \tag{A.9}$$

- **Definite integrals:**

$$\int_{-\infty}^{+\infty} dx\ e^{-Ax^2} = \sqrt{\frac{\pi}{A}} \qquad (A > 0) \tag{A.10}$$

$$\int_{-\infty}^{+\infty} dx\ \text{sech}^{2n}(x) = \frac{2^{2n-1}((n-1)!)^2}{(2n-1)!} \tag{A.11}$$

Note: the result of the Gaussian integral (A.10) does not change if the integration variable x is shifted by a finite imaginary amount c, namely if you replace $x \to x + ic$.