Lecture 21

Overview of Michaelmas

A soliton is a solution to a non-linear wave equation / partial differential equation in time which

1. Is localizable
2. Keep its localized form over time
3. Is preserved under collisions with other soliton

Only special, integrable, p.d.e.s. admit soliton solutions (e.g. KdV equation and sine - Gordon equation)

↳ infinitely many conservation laws ⇒ dynamics is highly constrained.

Michaelmas term: techniques to find particular solutions to these special p.d.e.s.

(travelling wave ansatz, Backlund transform + Hirota method)

single soliton — multiple solitons.

Epiphany: Can we find the general solution to these p.d.e.s?

Chapter 7: Overview of the inverse scattering method

7.1. initial value problem.

Given a p.d.e. for $u(x,t)$ and "enough" initial data at $t=0$ → find $u(x,t)$ for all $t>0$.

If we have a p.d.e. that is order "n" in time derivatives

unique solution → specify $u(x,t), \frac{\partial u}{\partial t}, \ldots, \frac{\partial^{n-1} u}{\partial t^{n-1}}$ at $t=0$.

$$\boxed{u_t = \frac{\partial u}{\partial t},\quad u_x = \frac{\partial u}{\partial x},\quad u_{tt} = \frac{\partial^2 u}{\partial t^2}}$$

For example

* If p.d.e. is first order in time (e.g. KdV equation) → initial data: $u(x,0)$.

$$u_t + 6uu_x + u_{xxx} = 0 \xrightarrow{\text{rearrange}} u_t = -(6uu_x + u_{xxx})$$

* if p.d.e. is second order in time (e.g. sine - Gordon equation) → initial data $u(x,0),\ u_t(x,0)$.

* etc.

⋮

In practise? Given sufficient initial data, can we construct $u(x, t>0)$?

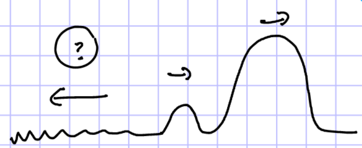
So far in this module: Only when initial data is a snap shot of soliton solutions.

E.g. KdV equation with initial data:

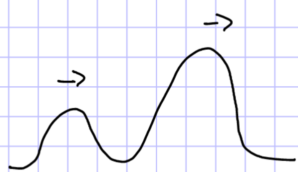
(a) $u(x,0) = 2\,\text{sech}^2 x$ (one soliton, moving to right)

$u(x,t>0) = 2\,\text{sech}^2(x-4t)$

(b) $u(x,0) = 2.1\ \text{sech}^2 x$ (two solitons, one much bigger than other, both moving to right, + complicated junk moving to the left)



(c) $u(x,0) = 6\ \text{sech}^2 x$. (two soliton solution, both moving to right)



This term: 1. Understand complicated scenario's like (b)

2. More complete understanding of when scenario's like (a) and (c) occur.

Recall: N-soliton solution at height $N(N+1)$ of $\text{sech}^2 x$ $N=1,2,\dots$

$\rightsquigarrow$ situation like (a) + (c).

why?

7.2. Linear initial value problems

General solution is a linear transformation of the initial data

Example

1. Heat equation / diffusion equation

$$u_t + u_{xx} = 0 \qquad -\infty < x < \infty, \qquad t>0$$

initial data: $u(x,0) = u_0(x)$. General solution:

$$u(x,t) = \int_{-\infty}^{\infty} \underbrace{\frac{1}{\sqrt{4\pi t}}\, e^{-(x-x')^2/4t}}_{\text{Heat kernel}}\, u_0(x')\, dx'$$

Heat kernel
"Fundamental solution" /
"Greens function" of Heat equation.

2. Klein - Gordon equation.

$$u_{tt} - u_{xx} + u = 0 \,, \qquad -\infty < x < \infty, \qquad t > 0$$

initial data: $u(x,0)$, $u_t(x,0)$

The Fourier transform

Extension of Fourier Series for periodic function (see AMV) to functions defined on the infinite

Consider periodic function $f(x)$, $x \in [-L/2, L/2]$ (period L).

$$f(x) = \sum_{n=-\infty}^{\infty} F_n \, e^{2\pi i n x / L}$$

$$\underline{\underline{F_n}} = \frac{1}{L} \int_{-L/2}^{L/2} f(x) \, e^{-2\pi i n x / L} \, dx$$

For non-periodic functions, send $L \to \infty$

frequency: $\quad n/L \to \frac{k}{2\pi}$

$$L F_n \to \hat{f}(k)$$

$$\frac{1}{L} \sum_{n=-\infty}^{\infty} \to \int_{-\infty}^{\infty} \frac{dk}{2\pi}$$

inverse Fourier transform: $\quad f(x) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} \hat{f}(k) e^{ikx}$

Fourier transform: $\quad \hat{f}(k) = \int_{-\infty}^{\infty} dx \, f(x) \, e^{-ikx}$