* Last time * Lecture 22

Michaelmas term: techniques to find particular solutions to special, integrable, wave equations

(travelling wave ansatz, Backlund transform + Hirota method)

single solitons — Multiple solitons

Epiphany term: Can we find the general solution to these wave equations? ←

7.1 Initial value problems

Wave equation for $u(x,t)$ order "n" in time: unique solution → specify $u(x,0)$, $\frac{\partial u}{\partial t}, \dots, \frac{\partial^{n-1} u}{\partial t^{n-1}}$ at $t=0$

But how do we construct this solution in practise? Toy model: Linearized wave equations ←

$u(x,t)$ is a linear transformation of initial data!

Useful tool: The Fourier transform

Consider $f(x)$ where $-\infty < x < \infty$.

frequency k

Inverse Fourier transform: $f(x) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} \hat{f}(k) e^{ikx}$

Fourier transform: $\hat{f}(k) = \int_{-\infty}^{\infty} dx\, f(x) e^{-ikx}$

Useful property for solving wave equations:

derivative w.r.t. to x $\xrightarrow{\text{F.T.}}$ factor ik

Consider $\partial_x f(x) = \partial_x \left[\int_{-\infty}^{\infty} \frac{dk}{2\pi} \hat{f}(k) e^{ikx} \right] = \int_{-\infty}^{\infty} \frac{dk}{2\pi} \hat{f}(k) \underbrace{\partial_x e^{ikx}}_{ik\, e^{ikx}} = \int_{-\infty}^{\infty} \frac{dk}{2\pi} ik \hat{f}(k) e^{ikx}$

i.e. the Fourier transform of $\partial_x f(x)$ is $ik \hat{f}(k)$!

→ The Fourier transform converts wave equations into ordinary differential equations in time!

For example: Klein-Gordon equation

$$u_{tt} - u_{xx} + u = 0, \qquad -\infty < x < \infty \quad \text{and} \quad t > 0$$

initial data: $u(x,0)$ and $u_t(x,0)$ ←

Approach: Use Fourier transform to replace $u(x,t)$ with $\hat{u}(k,t)$

$$u(x,t) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} \hat{u}(k,t) e^{ikx}$$

$$u_{tt}(x,t) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} \hat{u}_{tt}(k,t) e^{ikx}$$

$$u_{xx}(x,t) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} \underbrace{(ik)^2}_{-k^2} \hat{u}(k,t) e^{ikx}$$

The evolution equation for $\hat{u}$ is therefore:

$$\hat{u}_{tt} + \underbrace{(k^2+1)}_{\omega^2} \hat{u} = 0$$

i.e. very simple o.d.e.

~ the transformed field $\hat{u}$ is simpler than the original field u

$$\rightarrow \quad \hat{u}_{tt} = -\omega^2 \hat{u} \quad \leftarrow$$

general solution: $\hat{u}(k,t) = A(k)\, e^{+i\omega t} + B(k)\, e^{-i\omega t}$, $\hat{u}_t(k,t) = i\omega A(k) e^{i\omega t} - i\omega B(k) e^{-i\omega t}$

$$\hat{u}(k,0) = A(k) + B(k)$$

$$\hat{u}_t(k,0) = i\omega A(k) - i\omega B(k)$$

$$\underbrace{\begin{pmatrix} 1 & 1 \\ i\omega & -i\omega \end{pmatrix}}_{M} \begin{pmatrix} A(k) \\ B(k) \end{pmatrix} = \begin{pmatrix} \hat{u}(k,0) \\ \hat{u}_t(k,0) \end{pmatrix} \rightarrow \begin{pmatrix} A(k) \\ B(k) \end{pmatrix} = M^{-1} \begin{pmatrix} \hat{u}(k,0) \\ \hat{u}_t(k,0) \end{pmatrix}$$

$$= \frac{1}{2} \begin{pmatrix} \hat{u}(k,0) + \frac{1}{i\omega}\hat{u}_t(k,0) \\ \hat{u}(k,0) - \frac{1}{i\omega}\hat{u}_t(k,0) \end{pmatrix}$$

$$M^{-1} = \frac{1}{\det M} \begin{pmatrix} -i\omega & -1 \\ -i\omega & 1 \end{pmatrix}$$

$$\det M = -2i\omega$$

$$\rightarrow \quad A(k) = \frac{1}{2}\left(\hat{u}(k,0) + \frac{1}{i\omega}\hat{u}_t(k,0)\right)$$

$$B(k) = \frac{1}{2}\left(\hat{u}(k,0) - \frac{1}{i\omega}\hat{u}_t(k,0)\right)$$

$$\rightarrow \quad \hat{u}(k,t) = \cos(\omega t)\, \underline{\hat{u}(k,0)} + \frac{1}{\omega}\sin(\omega t)\, \underline{\underline{\hat{u}_t(k,0)}}$$

$$= \int_{-\infty}^{\infty} dx' \left[\cos(\omega t)\, u(x',0) + \frac{1}{\omega}\sin(\omega t)\, u_t(x',0)\right] e^{-ikx'}$$

$$\cos(\omega t) = \frac{1}{2}\left(e^{i\omega t} + e^{-i\omega t}\right)$$

$$\sin(\omega t) = \frac{1}{2i}\left(e^{i\omega t} - e^{-i\omega t}\right)$$

$$\hat{u}(k,0) = \int_{-\infty}^{\infty} dx'\, u(x',0)\, e^{-ikx'}$$

$$\hat{u}_t(k,0) = \int_{-\infty}^{\infty} dx'\, u_t(x',0)\, e^{-ikx'}$$

inverse Fourier transform:

$$u(x,t) = \int_{-\infty}^{\infty} \frac{dk}{2\pi}\, \hat{u}(k,t)\, e^{+ikx}$$

$$= \int_{-\infty}^{\infty} \frac{dk}{2\pi}\, dx'\, e^{ik(x-x')} \left[\cos(\omega t)\, u(x',0) + \frac{1}{\omega}\sin(\omega t)\, u_t(x',0)\right]$$

Remarks

1. This is a linear function of the initial data [this will not carry over to KdV]

2. Key feature: the transformed data $\hat{u}$ for each value of k evolved separately and in particular in a simpler way by the transformed equation.
[Quite remarkably, an analogue of this will hold for KdV]

Recap

$$
\begin{tikzcd}[column sep=huge, row sep=large]
\boxed{\begin{array}{c}\text{initial data}\\ u(x,0) \text{ and } u_t(x,0)\end{array}} \arrow[r, "\text{evolves by}", "u_{tt}-u_{xx}+u=0"'] \arrow[d, "\text{Fourier transform}"] & \boxed{\text{unique } u(x,t>0)} \\
\boxed{\begin{array}{c}\text{transformed initial data}\\ \hat{u}(k,0) \text{ and } \hat{u}_t(k,0)\end{array}} \arrow[r, "\text{Evolves by}", "\begin{array}{c}\hat{u}_{tt}+\omega^2\hat{u}=0\\ \text{v. simple o.d.e.}\end{array}"'] & \boxed{\text{Explicit unique } \hat{u}(k,t>0)} \arrow[u, "\text{inverse Fourier transform.}"']
\end{tikzcd}
$$

This is the correct big idea for KdV equation, though its complicated since KdV is non-linear

General strategy for KdV equation

$$
\begin{tikzcd}[column sep=huge, row sep=large]
\boxed{\begin{array}{c}\text{initial data}\\ u(x,0)\end{array}} \arrow[r, "\begin{array}{c}(d)\\ \text{Evolve by KdV}\end{array}"] \arrow[d, "(a)"', "\text{“disassembly”}"] & \boxed{\text{unique } u(x,t>0)} \\
\boxed{\begin{array}{c}\text{data “equivalent” to}\\ u(x,0)\end{array}} \arrow[r, "\begin{array}{c}(b)\\ \text{Simpler evolution}\\ \text{problem}\end{array}"] & \boxed{\begin{array}{c}\text{data “equivalent” to}\\ u(x,t>0)\end{array}} \arrow[u, "(c)", "\text{“reassembly”}"']
\end{tikzcd}
$$

Achieve step (d) by following steps (a) → (b) → (c).