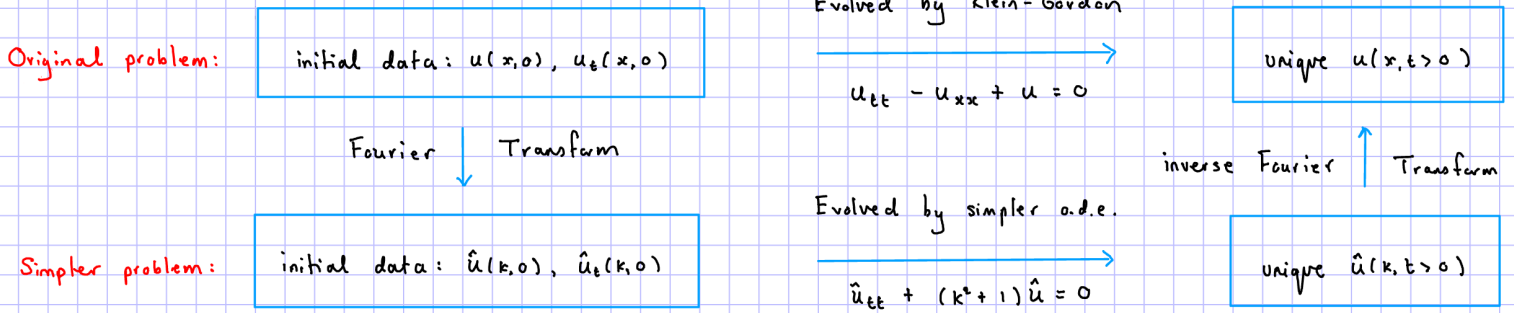
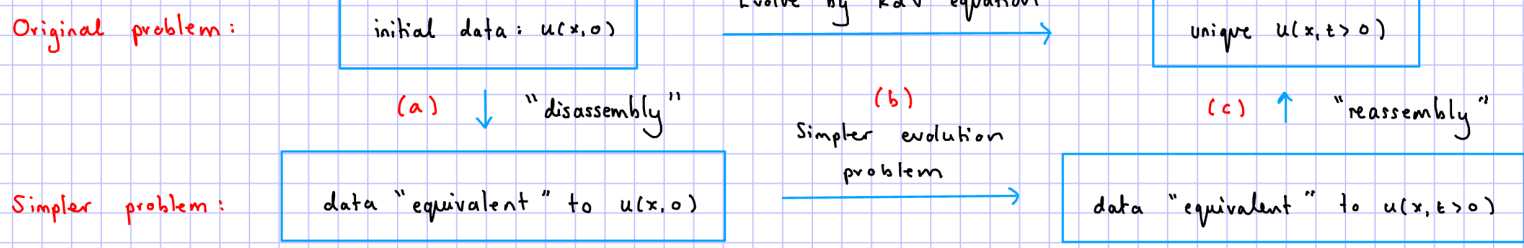


Key steps for Klein-Gordon equation:



This is the correct big picture for KdV, though it is more complicated as the KdV equation is non-linear.

Big idea for KdV:



Achieve (d) by following steps (a) → (b) → (c)!

Chapter 8: The KdV-Schrödinger connection

Follow approach of Gardner, Greene, Kruskal and Miura (late 1960s)

GGKM

Aim: Solve the KdV equation

$$u_t + 6uu_x + u_{xxx} = 0$$

for $t > 0$ and $-\infty < x < \infty$ with initial data: $u(x,0)$

First step: Consider the Miura transform (Miura's term)

If $v(x,t)$ solves $v_t + 6(1-v^2)v_x + v_{xxx} = 0$ then

$$u = 1 - v^2 - v_x \tag{8.2b}$$

solves the KdV equation.

Second step: Take now u known and solve for v ! The trick is to write

$$v = \frac{\psi_x}{\psi} \quad \text{for some function } \psi \text{ and find } \psi.$$

$$(8.2b) \text{ becomes: } u = 1 - \left(\frac{\psi_x}{\psi}\right)^2 - \partial_x \left(\frac{\psi_x}{\psi}\right) = 1 - \frac{\psi_{xx}}{\psi}$$

$$\frac{\psi_{xx}}{\psi} - \frac{\psi_x^2}{\psi^2}$$

i.e. $\psi_{xx} + u\psi = \lambda\psi \tag{8.3}$

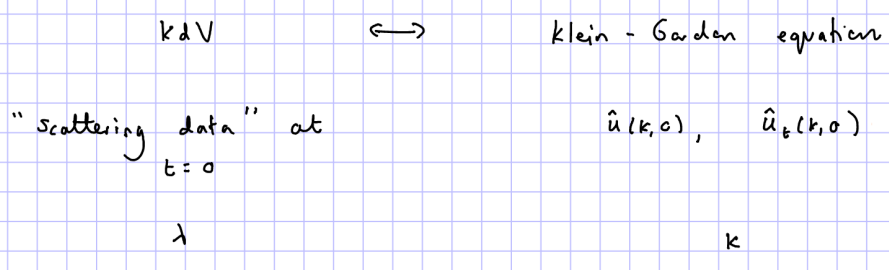
page 2 This is a simple linear o.d.e. for ψ known as the time independent Schrödinger equation.

This is the quantum mechanical equation for a particle moving in potential $V = -u$.

[But the quantum mechanical interpretation won't be too important!]

Remarks

- * To the kdv field u we associate a field ψ governed by (8.3). This is known as an associated linear problem.
- * The time t is just a parameter in (8.3) i.e. no time derivatives.
- * The problem of finding u is translated into the problem of finding the potential V given some information about ψ .
- * At any time t the potential and hence $u(x,t)$ can be reconstructed from the "scattering data" (limited information about ψ for different values of λ)
- * If $u(x,t)$ evolves by the kdv equation then the scattering data evolves in a particularly simple way.



Finding the scattering data at $t=0$ would complete step (a), which we postpone to later lecture.

Chapter 9: Time evolution of the scattering data
(b)

The time evolution of scattering data follows from time evolution of ψ
we focus on this for now!

9.1 The idea of the Lax pair (Peter Lax, 1968)

For fixed time t we recast the differential equation for ψ

$$\psi_{xx} + u\psi = \lambda\psi$$

as an Eigenvalue problem:

$$L(u)\psi = \lambda\psi \tag{9.1}$$

↙ Eigenfunction
↖ Eigenvalue

where $L(u)$ is the following differential operator :

$$L(u) = \frac{d^2}{dx^2} + u(x, t)$$

To be absolutely careful,
since t is fixed, we might write

$$L(u) = \frac{\partial^2}{\partial x^2} + u(x, t)$$

Comments

* We think of differential operators (e.g. $L(u)$, $\frac{d}{dx}$, $\frac{d^2}{dx^2}$, $\frac{d}{dt}$, ...) as acting on everything to their right

$$\begin{aligned} \text{i.e. } L(u)fg &= L(u)(fg) \\ &(\neq (L(u)f)g \\ &(\neq (L(u)g)f)) \end{aligned}$$