Goal: Initial value problem for KdV equation $\rightarrow$ Toy model: Klein-Gordon equation (Lecture 23)

Key steps for Klein-Gordon equation:

$$
\begin{array}{lccc}
\text{Original problem:} & \boxed{\text{initial data: } u(x,0),\ u_t(x,0)} & \xrightarrow[u_{tt} - u_{xx} + u = 0]{\text{Evolved by Klein-Gordon}} & \boxed{\text{unique } u(x,t>0)} \\
 & \Big\downarrow \text{Fourier Transform} & & \Big\uparrow \text{inverse Fourier Transform} \\
\text{Simpler problem:} & \boxed{\text{initial data: } \hat{u}(k,0),\ \hat{u}_t(k,0)} & \xrightarrow[\hat{u}_{tt} + (k^2+1)\hat{u} = 0]{\text{Evolved by simpler o.d.e.}} & \boxed{\text{unique } \hat{u}(k,t>0)}
\end{array}
$$

This is the correct big picture for KdV, though it is more complicated as the KdV equation is non-linear.

Big idea for KdV:

$$
\begin{array}{lccc}
\text{Original problem:} & \boxed{\text{initial data: } u(x,0)} & \xrightarrow{\text{(d) Evolve by KdV equation}} & \boxed{\text{unique } u(x,t>0)} \\
 & \text{(a)} \Big\downarrow \text{"disassembly"} & & \text{(c)} \Big\uparrow \text{"reassembly"} \\
\text{Simpler problem:} & \boxed{\text{data "equivalent" to } u(x,0)} & \xrightarrow{\text{(b) Simpler evolution problem}} & \boxed{\text{data "equivalent" to } u(x,t>0)}
\end{array}
$$

Achieve (d) by following steps (a) $\rightarrow$ (b) $\rightarrow$ (c)!

Chapter 8: The KdV - Schrödinger connection

Follow approach of $\underbrace{\text{Gardner, Greene, Kruskal and Miura}}_{\text{GGKM}}$ (late 1960s)

Aim: Solve the KdV equation

$$u_t + 6uu_x + u_{xxx} = 0$$

for $t>0$ and $-\infty < x < \infty$ with initial data: $u(x,0)$

First step: Consider the Miura transform (Michaelmas term)

If $v(x,t)$ solves $v_t + 6(\lambda - v^2)v_x + v_{xxx} = 0$ then

$$u = \lambda - v^2 - v_x \qquad (8.2b)$$

solves the KdV equation.

Second step: Take now u known and solve for v! The trick is to write

$$v = \frac{\psi_x}{\psi} \qquad \text{for some function } \psi \text{ and find } \psi.$$

(8.2b) becomes:
$$u = \lambda - \cancel{\left(\frac{\psi_x}{\psi}\right)^2} - \underbrace{\partial_x\left(\frac{\psi_x}{\psi}\right)}_{\frac{\psi_{xx}}{\psi} - \cancel{\frac{\psi_x^2}{\psi^2}}} = \lambda - \frac{\psi_{xx}}{\psi}$$

i.e.
$$\psi_{xx} + \underset{\substack{\uparrow \\ u(x,t)}}{u}\,\psi = \lambda\psi \qquad (8.3)$$

This is a simple linear o.d.e. for ψ known as the time independent Schrödinger equation.
This is the quantum mechanical equation for a particle moving in potential $V = -u$.
[But the quantum mechanical interpretation won't be too important!].

Remarks

* To the KdV field u we associate a field ψ governed by (8.3). This is known as an associated linear problem.

* The time t is just a parameter in (8.3) i.e. no time derivatives.

* The problem of finding u is translated into the problem of finding the potential V given some information about ψ.

* At any time t the potential and hence $u(x,t)$ can be reconstructed from the "scattering data" (limited information about ψ for different values of λ)

* If $u(x,t)$ evolves by the KdV equation then the scattering data evolves in a particularly simple way.

KdV	$\longleftrightarrow$	Klein - Gordon equation
"scattering data" at $t=0$		$\hat{u}(k,0)$, $\hat{u}_t(k,0)$
λ		k

Finding the scattering data at $t=0$ would complete step (a), which we postpone to later lecture.

Chapter 9: $\underbrace{\text{Time evolution of the scattering data}}_{(b)}$

The time evolution of scattering data follows from time evolution of ψ
we focus on this for now!

9.1 The idea of the Lax pair (Peter Lax, 1968)

For fixed time t we recast the differential equation for ψ

$$\psi_{xx} + u\psi = \lambda\psi$$

as an Eigenvalue problem:

Eigenfunction

$$L(u)\psi = \lambda\psi \qquad (9.1)$$

Eigenvalue

where $L(u)$ is the following differential operator :

$$L(u) = \frac{d^2}{dx^2} + u(x,t)$$

To be absolutely careful, since t is fixed, we might write

$$L(u) = \frac{\partial^2}{\partial x^2} + u(x,t)$$

Comments

* We think of differential operators (e.g. $L(u)$, $\frac{d}{dx}$, $\frac{d^2}{dx^2}$, $\frac{d}{dt}$, ...) as acting on everything to their right

i.e. $L(u)\,fg = L(u)(fg)$

$(\neq (L(u)f)\,g$
$\neq (L(u)g)\,f\;)$