| KdV equation | Miura transform | Modified KdV equation |
|---|---|---|
| $u_t + 6uu_x + u_{xxx} = 0$ | $u = \lambda - v^2 - v_x$ (8.2b) | $v_t + 6(\lambda - v^2)v_x + v_{xxx} = 0$ |

$$\Big\downarrow \; v = \frac{\psi_x}{\psi}$$

$$\boxed{\psi_{xx} + u(x,t)\psi = \lambda\psi} \qquad (8.3)$$

"Time independent Schrödinger equation"
for a particle moving in a potential $V = -u$.

$\Rightarrow$ **The problem of solving for the KdV field $u(x,t)$ is transformed to the problem of finding the potential $V$ at time $t$ given some information about $\psi$!**

Key points: * The potential $V$ can be reconstructed at any time $t$ by knowing how particles scatter from the potential for different values of $\lambda$. I.e. the "scattering data"

* If $u$ evolves by the KdV equation, the scattering data has a simple time evolution.

Big picture so far:

$$\begin{array}{ccc}
\boxed{\begin{array}{c}\text{initial data}\\ u(x,0)\end{array}} & \xrightarrow[\text{(d)}]{\text{Evolve by KdV equation}} & \boxed{\text{unique } u(x,t>0)} \\
\text{(a)} \big\downarrow \text{"disassembly"} & & \text{(c)} \big\uparrow \text{"reassembly"} \\
\boxed{\begin{array}{c}\text{"scattering data"}\\ \text{at } t=0\end{array}} & \xrightarrow[\text{(b)}]{\text{Simple time evolution}} & \boxed{\begin{array}{c}\text{"scattering data"}\\ \text{at time } t\end{array}}
\end{array}$$

We postpone defining "scattering data" to a later lecture. For now let's focus on step (b)

## Chapter 9: Time evolution of the scattering data

Time evolution of scattering data follows from time evolution of $\psi$ focus on this for now!

### 9.1 The idea of the Lax pair (Peter Lax, 1968)

For fixed time $t$, recast $\psi_{xx} + u\psi = \lambda\psi$ as an Eigenvalue problem:

$$L(u)\underbrace{\psi}_{\text{Eigenfunction}} = \underbrace{\lambda}_{\text{Eigenvalue}}\psi \qquad (9.1)$$

where $L(u)$ is the differential operator:

$$L(u) = \frac{d^2}{dx^2} + u(x,t)$$

comments :

* We think of differential operators (e.g. $L$, $\frac{d}{dx}$, $\frac{d}{dt}$, ...) as acting on everything to their right :

$$\text{i.e. } L f g = L(fg)$$

* (9.1) picks out special values of $\lambda$ ("Eigenvalues") provided $\psi$ is square integrable :

$$\int_{-\infty}^{\infty} \underbrace{|\psi|^2}_{\psi^* \psi} \, dx < \infty$$

[ This is normalizability of the wave function $\psi$ in quantum mechanics ]

This implies that $\psi \to 0$ as $|x| \to +\infty$

[ later on we just require that $\psi$ is bounded ].

[ we will also assume (like in Michaelmas) that $u(x,t) \to 0$ as $|x| \to \infty$ ].

* $L$ depends on $u$ and $u$ depends on time $t$, so $L$ evolves with time

$\rightarrow$ Eigenfunctions $\psi$ and (in principle) the Eigenvalues $\lambda$ depend on time $t$ as well.

Theorem

(i) If $u$ evolves by KdV then the set of Eigenvalues $\{\lambda\}$ of $L(u)$ (i.e. the spectrum of $L$) is independent of time $t$. [ Spectra of $L(u(x,t))$ and $L(u(x,0))$ coincide even if $u(x,t)$ and $u(x,0)$ are very different ].

(ii) If $u$ evolves by KdV there is a set of Eigenfunctions $\{\psi\}$ of $L$ which evolve in time $t$ simply as follows :

$$\psi_t = B(u) \psi$$

where $B$ is some differential operator in $x$

$L(u)$ and $B(u)$ are a "Lax pair" for the KdV equation.

Proof

For now, assume that : $\underbrace{L(u)_t}_{\frac{\partial L}{\partial t}} = B(u) L(u) - L(u) B(u)$

$$\frac{\partial L}{\partial t} = [B(u), L(u)]$$

later on this will be a fact

[ Since $L$ & $B$ are operators, this is not necessarily zero

e.g. $\frac{d}{dx} f(x) \neq f(x) \frac{d}{dx}$

$\frac{d}{dx} f(x) - f(x) \frac{d}{dx} = f_x$ ]

where $[B, L] := BL - LB$ is the commutator of $B$ and $L$

(i) We want to show that $\lambda_t = 0$

We have: $$L(u)\psi = \lambda\psi$$

$$\frac{\partial}{\partial t}\left[ L(u)\psi = \lambda\psi \right]$$

$$\rightarrow \quad L(u)_t\psi + L(u)\psi_t = \lambda_t\psi + \lambda\psi_t$$

rearrange: $$\lambda_t\psi = L(u)_t\psi + (L(u)-\lambda)\psi_t$$

$$\underset{L_t = [B,L]}{=} BL\psi - LB\psi + (L(u)-\lambda)\psi_t$$

$$= \underbrace{B\lambda\psi}_{\lambda B\psi} - LB\psi + (L(u)-\lambda)\psi_t$$

$$= (L-\lambda)(\psi_t - B\psi)$$

Multiply both sides by $\psi^*$ and integrate:

$$(*) \qquad \lambda_t \underbrace{\int_{-\infty}^{\infty} |\psi|^2\, dx}_{\substack{>0 \\ <\infty}} = \int_{-\infty}^{\infty} \psi^* (L-\lambda)(\psi_t - B\psi)\, dx$$

To proceed, let's introduce some tools!

Tool Box

Consider space of square integrable complex functions

1. Define Hermitian inner product of functions $\phi$, $\chi$

$$(\phi, \chi) := \int_{-\infty}^{\infty} \phi^* \chi \, dx$$

Useful defining properties:

(i) $(\phi_1 + \phi_2, \chi) = (\phi_1, \chi) + (\phi_2, \chi)$

(ii) $(\phi, \chi_1 + \chi_2) = (\phi, \chi_1) + (\phi, \chi_2)$

(iii) $(\lambda\phi, \chi) = \lambda^*(\phi, \chi)$

(iv) $(\phi, \lambda\chi) = \lambda(\phi, \chi)$

anti-linearity in LHS / linearity on RHS.

(v) $(\phi, \chi)^* = (\chi, \phi)$ (conjugate symmetry)

(vi) $(\phi, \phi) > 0$ if $\phi \neq 0$ positive definiteness.

2. The adjoint of an operator

Consider differential operator $M$. The adjoint $M^\dagger$ is defined such that:

$$(\phi, M x) = (M^{\dagger} \phi, x)$$

useful properties:

(i) $(M^{\dagger})^{\dagger} = M$

(ii) $(MN)^{\dagger} = N^{\dagger} M^{\dagger}$

(iii) $(M + N)^{\dagger} = M^{\dagger} + N^{\dagger}$

(iv) $(\lambda M)^{\dagger} = \lambda^{*} M^{\dagger}$

Coming back to $(*)$:

$$\lambda_t (\psi, \psi) = (\psi, (L - \lambda)(\psi_t - B\psi))$$

$$= ((L - \lambda)^{\dagger} \psi, \psi_t - B\psi)$$