Lecture 25

$$\boxed{\psi_{xx} + u\psi = \lambda\psi}$$

"associated linear problem" to KdV.

$u(x,t)$ can be determined from info about ψ ("scattering data") $\longrightarrow$ We want to know the time evolution of ψ! (step (b))

Recast as Eigenvalue problem:

$$L(u)\psi = \lambda\psi$$

$$L(u) = \frac{d^2}{dx^2} + u(x,t)$$

($\int_{-\infty}^{\infty} |\psi|^2\, dx < \infty$ "square integrable")

Theorem: If u evolves by KdV:

(i) The set of Eigenvalues $\{\lambda\}$ of $L(u)$ (i.e. the spectrum of L) is independent of time t.

(ii) There is a set of Eigenfunctions $\{\psi\}$ of $L(u)$ which evolve via $\psi_t = B(u)\psi$

$B(u)$: differential operator in x

Proof: For now, assume that: $L(u)_t = B(u)L(u) - B(u)L(u) = \underbrace{[B(u), L(u)]}_{\text{commutator of } B \text{ and } L}$

later this will be shown to be a fact!

Tool box

1. Hermitian inner product on space of complex square integrable functions:

$$(\phi, \chi) := \int_{-\infty}^{\infty} \phi^* \chi\, dx$$

2. Adjoint M^+ of an operator M:

$$(\phi, M\chi) = (M^+\phi, \chi)$$

(i) We need to show that $\lambda_t = 0$!

$$L(u)\psi = \lambda\psi \quad \rightarrow \quad \frac{\partial}{\partial t}\left[L(u)\psi = \lambda\psi\right]$$

$$\rightarrow \quad \lambda_t \underbrace{(\psi, \psi)}_{0 < \quad < \infty} = (\psi, (L-\lambda)(\psi_t - B\psi))$$

$$= ((L-\lambda)^+\psi, \psi_t - B\psi) \qquad (*)$$

[we know that $(L-\lambda)\psi = \lambda\psi - \lambda\psi = 0$].

Claim: L is a self adjoint operator $L^+ = L$ (this is the operator analogue of a Hermitian matrix)

Proof: We need to show that $(\phi, L\chi) = (L\phi, \chi)$

Start from LHS: $(\phi, L\chi) = \int_{-\infty}^{\infty} \phi^* L\chi\, dx$

$$= \int_{-\infty}^{\infty} \phi^* \frac{d^2\chi}{dx^2} + u\phi^*\chi\, dx$$

$$= \int_{-\infty}^{\infty} -\frac{d\phi^*}{dx}\frac{d\chi}{dx} + u\phi^*\chi\, dx + \cancel{\underbrace{\int_{-\infty}^{\infty} \frac{d}{dx}\left(\phi^* \frac{d\chi}{dx}\right) dx}_{\left[\phi^* \frac{d\chi}{dx}\right]_{x=-\infty}^{x=\infty}}} \rightarrow 0$$

integrate by parts #1

But ϕ & χ are square integrable $\Rightarrow$ $\phi, \chi \rightarrow 0$ as $x \rightarrow \pm\infty$.

$$= \int_{-\infty}^{\infty} \frac{d^2\phi^*}{dx^2}\chi + u\phi^*\chi \, dx + \int_{-\infty}^{\infty} \cancel{\frac{d}{dx}\left(-\frac{d\phi^*}{dx}\chi\right)}^{\,0} dx$$

IBP # 2

$$= \int_{-\infty}^{\infty} \Big(\big(\underbrace{\tfrac{d^2}{dx^2} + u}_{L}\big)\phi\Big)^* \chi \, dx$$

u and $\frac{d}{dx}$ are real

$$= (L\phi, \chi) \quad \square$$

Corollary (assignment 6): The Eigenvalues λ of L are real [Recall that Eigenvalues of Hermitian matrix are real!]

Coming back to (*): $L^\dagger = L$ and λ is real $\rightarrow$ $(L-\lambda)^\dagger = (L-\lambda)$

$$\lambda_t \underbrace{(\psi,\psi)}_{0<\ \ <\infty} = \Big(\underbrace{(L-\lambda)\psi}_{0},\ \psi_t - B\psi\Big) = 0$$

$$\rightarrow \quad \lambda_t = 0 \quad \square.$$

Proof of part (ii) We want to show that if $\psi_t = B\psi$ then $\underbrace{\frac{\partial}{\partial t}}\underbrace{\left[(L-\lambda)\psi\right]} = 0$

if the RHS is not zero then ψ does not stay an Eigenvalue as time evolves.

$$\frac{\partial}{\partial t}\left[(L-\lambda)\psi\right] = L_t\psi + L\psi_t - \cancel{\lambda_t\psi}^{\,0} - \lambda\psi_t$$

$$= BL\psi - LB\psi + L\psi_t - \lambda\psi_t$$

$L_t = [B, L]$

$$= -\cancel{(L-\lambda)B\psi} + \cancel{(L-\lambda)}\underbrace{\psi_t}_{B\psi}$$

$$= 0$$

$\rightarrow$ If $\psi_t = B\psi$ and at $t=0$ ψ is an Eigenfunction with Eigenvalue λ then it stays that way for all times.

We're not done! Recall: We assumed the existence of B such that $\underbrace{L_t} = \underbrace{[B, L]}$

L and B are Lax pair for KdV!

9.2 The Lax pair for KdV

We know that $L = \frac{d^2}{dx^2} + u(x,t) = D^2 + u$

Claim: B is given by $B(u) = -(4D^3 + 6uD + 3u_x)$ [Notation: $D \equiv \frac{d}{dx}$, $D^2 \equiv \frac{d^2}{dx^2}$, $D^3 \equiv \frac{d^3}{dx^3}$, ..., $D^n = \frac{d^n}{dx^n}$].

Proof: We simply need to show that $L_t = [B, L]$ for the given B.

First look at LHS: $L(u)_t = \frac{\partial L(u)}{\partial t} = \underbrace{\frac{\partial L(u)}{\partial u}}\frac{\partial u}{\partial t} = u_t$

$$\frac{\partial}{\partial u}[D^2 + u] = 1$$

Consider now the RHS: $[B(u), L(u)] = [-(4D^3 + 6uD + 3u_x), D^2 + u]$

$$= -4\cancel{[D^3, D^2]}^{0} - 4[D^3, u] - 6[uD, D^2] - 6[uD, u] - 3[u_x, D^2] - 3\cancel{[u_x, u]}^{0}.$$

Useful properties of $[\cdot, \cdot]$

(i) $[A, B + C] = [A, B] + [A, C]$

(ii) $[A + B, C] = [A, C] + [B, C]$

(iii) $[\lambda A, B] = \lambda [A, B]$

(iv) $[A, \lambda B] = \lambda [A, B]$.

(v) $[A, B] = -[B, A]$

(vi) $[AB, C] = A[B, C] + [A, C]B$.

Simplifying the commutators

$[D^3, D^2] = \underbrace{D^3 D^2}_{D^5} - \underbrace{D^2 D^3}_{D^5} = 0$

$[u_x, u] = u_x u - u u_x = 0$

$[uD, u] = uDu - uuD$

$= u(Du - uD)$

$= u[D, u]$.

$[\underset{A}{u}\underset{B}{D}, \underset{C}{D^2}] = u\cancel{[D, D^2]}^{0} + [u, D^2]D$