$$\boxed{L(u)\psi = \lambda\psi \quad \text{where} \quad L(u) = D^2 + u(x,t), \quad D \equiv \frac{d}{dx}}$$

"associated linear problem" to the KdV equation

$u(x,t)$ can be determined from info about ψ ("scattering data") $\xrightarrow[\text{(step (6))}]{\text{Goal}}$ We want to know the time evolution of ψ!

To this end we proved the following theorem:

Theorem: If u evolves by KdV:

(i) The set of Eigenvalues $\{\lambda\}$ of $L(u)$ (i.e. the spectrum of L) is independent of time t.

(ii) There is a set of Eigenfunctions $\{\psi\}$ of $L(u)$ which evolve via $\psi_t = \underbrace{B(u)}_{\text{differential operator in } x}\psi$

.... subject to the assumption: $\exists\ B(u)$ such that $\underbrace{L_t = [B, L]}_{\text{Lax equation}}$ i.e. which forms a Lax pair with $L(u)$

Claim: $B(u) = -(4D^3 + 6uD + 3u_x)$ [Notation: $D \equiv \frac{d}{dx}$, $D^2 \equiv \frac{d^2}{dx^2}$, $D^3 \equiv \frac{d^3}{dx^3}$, ..., $D^n \equiv \frac{d^n}{dx^n}$]

Proof: We simply need to show that $L_t = [B, L]$ for the given B!

LHS: $$L_t = \frac{\partial L(u)}{\partial t} = \underbrace{\frac{\partial L(u)}{\partial u}}_{\frac{\partial}{\partial u}[D^2+u] = 1}\frac{\partial u}{\partial t} = u_t$$

RHS: $$[B(u), L(u)] = [-(4D^3 + 6uD + 3u_x), D^2 + u]$$

$$= -4\underbrace{\cancel{[D^3, D^2]}}_{0} - 4[D^3, u]$$
$$- 6[uD, D^2] - 6[uD, u]$$
$$- 3[u_x, D^2] - 3\underbrace{\cancel{[u_x, u]}}_{0}$$

$$= -4[D^3, u]$$
$$+ 6[D^2, u]D$$
$$- 6u[D, u]$$
$$+ 3[D^2, u_x]$$

This was not written down last lecture

Useful properties of $[\cdot,\cdot]$

(i) $[A, B + C] = [A, B] + [A, C]$

(ii) $[A + B, C] = [A, C] + [B, C]$

(iii) $[\lambda A, B] = \lambda[A, B]$

(iv) $[A, \lambda B] = \lambda[A, B]$

(v) $[A, B] = -[B, A]$

(vi) $[AB, C] = A[B, C] + [A, C]B$

Simplifying the commutators

$[D^3, D^2] = \underbrace{D^3D^2}_{D^5} - \underbrace{D^2D^3}_{D^5} = 0$

$[u_x, u] = u_x u - u u_x = 0$

$[uD, u] = u[D, u]$

$[uD, D^2] = [u, D^2]D$

We have reduced the computation of $[B, L]$ to $[D^n, g(x)]$ where "n" is a positive integer where here $n = 1, 2$ or 3 and $g(x) = u$ or u_x. We have the following useful formula

$$[D^n, g(x)] = \sum_{m=1}^{n} \binom{n}{m} \frac{d^m g}{dx^m} D^{n-m}$$

(exercise S1 - problems class 5)

For example:

$n = 1$: $[D, g(x)] = g_x$

$n=2$: $\quad [D^2, g(x)] = 2 g_x D + g_{xx}$

$n=3$: $\quad [D^3, g(x)] = 3 g_x D^2 + 3 g_{xx} D + g_{xxx}$

Coming back to $[B, L]$:

$$[B,L] = -4\left(\cancel{3 u_x D^2} + \cancel{3 u_{xx} D} + u_{xxx}\right)$$
$$+ 6\left(\cancel{2 u_x D} + \cancel{u_{xx}}\right) D$$
$$- 6 u u_x$$
$$+ 3\left(\cancel{2 u_{xx}} D + u_{xxx}\right)$$
$$= - u_{xxx} - 6 u u_x$$

$\uparrow$ u evolves by KdV: $u_t + 6 u u_x + u_{xxx} = 0$

$$= u_t$$
$$= L_t$$

$\square$

Rearrange: $\quad L_t + [L, B] = u_t + 6 u u_x + u_{xxx} = 0$

KdV equation for u $\quad \longleftrightarrow \quad$ $L_t + [L, B] = 0$

Lax equation for Lax pair L and the given B.

Chapter 10 Interlude: The KdV hierarchy and conservation laws

> **Question:** Are there other evolution equations for u such that the spectrum of $L^2 = D + u$ is independent of time?

When the Eigenvalues of an operator L are independent of time the different L at different times t are referred to as "isospectral" and the time evolution is called an "isospectral flow".

10.1: Deriving the KdV equation (+ generalizing it)

Note: When proving the theorem in chapter 9 we did not (at least directly) use the KdV equation. We only used a consequence of it: $L_t = [B, L]$ for some B and the explicit form of B was not relevant!

$\Rightarrow$ The theorem in chapter 9 holds for any evolution equation for u so long as $L_t = [B, L]$ for some B!

But B is not completely arbitrary! We need that:

$$L_t = u_t \stackrel{!}{=} [B, L]$$

In particular, $L_t = u_t$ is a "multiplicative operator" (i.e. its not a differential operator)

$\Rightarrow$ $[B, L]$ is a multiplicative operator!

polynomials in D

But L and (generally) B are differential operators in x i.e. in D
(they are also polynomials in u and its x-derivatives: $u_x, u_{xx}, \dots$)

$\Rightarrow$ All the Ds must cancel in $[B, L]$ and what is left over will be a polynomial in u and $u_x, u_{xx}, \dots$

Equating this polynomial in u and $u_x, u_{xx}, \dots$ with u_t via $L_t = [B, L]$ gives us the evolution equation for u!

This is the game we will play in the next lectures to find other evolution equations for u and also derive the KdV equation.