KdV equation

$u_t + 6uu_x + u_{xxx} = 0$

$u_t + 6uu_x + u_{xxx} = L_t + [L,B] = 0$ $\longleftrightarrow$

Lax equation

$L_t + [L,B] = 0$

with $L = D^2 + u$ and $B = -(4D^3 + 6uD + 3u_x)$

If $L_t + [L,B] = 0$ then the time evolution of the associated linear problem $L\psi = \lambda\psi$ is simple: (theorem in chapter 9)

1. The spectrum $\{\lambda\}$ of L is independent of time
2. ψ evolves via $\psi_t = B(u)\psi$

Theorem in chapter 9

Natural question: Are there any other evolution equations for u that can be written in the form $L_t + [L,B] = 0$?

i.e. What is the most general B that solves $L_t = [B,L]$?

Note: $L_t = u_t$ is a multiplicative operator so $L_t = [B,L]$ implies $[B,L]$ is too!

But: L (and generally) B are differential operators in $D \equiv \frac{d}{dx}$. So all Ds must cancel in $[B,L]$!

What is left over will be a polynomial in $u, u_x, u_{xx}, \dots$

... equating this to u_t via $u_t = [B,L]$ gives the evolution equation for u!

> We will making a LOT of use out of the formula (i.e. memorize it!):
>
> $$[D^n, g(x)] = \sum_{m=0}^{n} \binom{n}{m} \frac{d^m g}{dx^m} D^{n-m}$$

Example (i): Does B have to be a differential operator? I.e. $B = \alpha(x)$ for some function α of x

We have $[B,L] = [\alpha(x), D^2 + u]$

$$= \underbrace{[\alpha, D^2]}_{-[D^2,\alpha]} + \cancel{[\alpha, u]}^{\,0} \quad (\alpha \text{ and } u \text{ are numbers})$$

$$= -(2\alpha_x D + \alpha_{xx})$$

$[B,L]$ is a multiplicative operator $\rightarrow$ D's cancel $\rightarrow$ $(\dots)D = 0$

$\rightarrow$ $\alpha_x = 0$ ($\rightarrow \alpha_{xx} = 0$)

$\rightarrow$ $\alpha =$ constant.

What is the evolution equation for u? This is:

$$u_t \overset{!}{=} [B,L] = -\alpha_{xx} = 0$$

$u_t = 0 \quad \longleftrightarrow \quad L_t + [L,B] = 0$ where $B =$ constant.

This is too trivial: u is not evolving with time (i.e. $u_t = 0$) and so neither does L

So whole spectrum is independent of time there is no isospectral flow.

Example (ii): Can we have something non-trivial if B is linear in $D \equiv \frac{d}{dx}$?

The most general form of such a B is :

$$(\dagger) \qquad B = \alpha(x)\, D + \beta(x) \qquad \text{where } \alpha \text{ and } \beta \text{ are some functions of } x$$

Why is $(\dagger)$ the most general form for such a B?

Consider instead $B = \tilde{\alpha}(x) D + \hat{\beta}(x) + D\gamma(x) = \overbrace{(\tilde{\alpha} + \gamma)}^{\alpha(x)} D + \overbrace{(\tilde{\beta} + \gamma_x)}^{\beta(x)}$

Using $[D, \gamma]$ we can bring this into the form $(\dagger)$:

$$[D,\gamma] = D\gamma - \gamma D = \gamma_x \quad \rightarrow \quad D\gamma = \gamma D + \gamma_x$$

We have $[B, L] = [\alpha D + \beta,\, D^2 + u]$

$$= \underbrace{[\alpha D, D^2]}_{-[D^2,\alpha]D} + \underbrace{[\alpha D, u]}_{\alpha[D,u]} + \underbrace{[\beta, D^2]}_{-[D^2,\beta]} + \cancelto{0}{[\beta, u]}$$

$$= -2\alpha_x D^2 - (\alpha_{xx} + 2\beta_x) D - \beta_{xx} + \alpha u_x$$

$[B,L]$ is multiplicative $\Rightarrow$ 1. $(\ldots)D^2 = 0 \rightarrow \alpha_x = 0 \rightarrow \alpha = \text{constant}.$ $(\rightarrow \alpha_{xx} = 0)$

2. $(\ldots)D = 0 \rightarrow \cancelto{0}{\alpha_{xx}} + 2\beta_x = 0 \rightarrow \beta = \text{some other constant}.$ $(\beta_{xx} = 0)$

Evolution equation for u? This is :

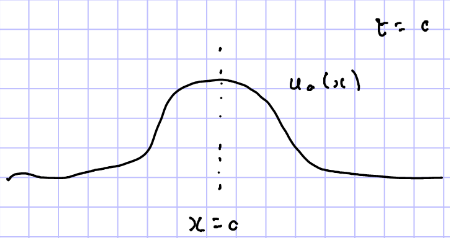
$$u_t \doteq [B, L] = -\cancelto{0}{\beta_{xx}} + \alpha u_x = \alpha u_x \qquad \text{Advection equation}$$

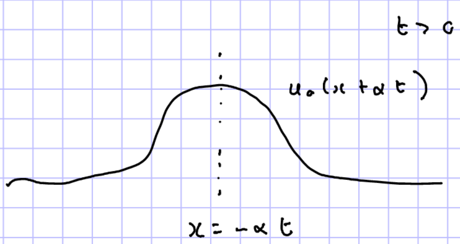
$$u_t = \alpha u_x \quad \longleftrightarrow \quad L_t + [L, B] = 0 \quad \text{where } B = \alpha D + \beta$$

w. α and β constants.

Meaning? This is still trivial.

Initial data : $u(x,0) = u_0(x) \longrightarrow u(x, t>0) = u_0(x + \alpha t)$





Example (iii): Deriving the KdV. Consider :

$$B = \alpha(x) D^3 + \beta(x) D + \gamma(x) \quad \longleftarrow$$

We have :
$$[B, L] = -2\alpha_x D^4 - \alpha_{xx} D^3 - (2\beta_x - 3\alpha u_x) D^2$$
$$- (\beta_{xx} + 2\gamma_x - 3\alpha u_{xx}) D$$
$$- \gamma_{xx} + \alpha u_{xxx} + \beta u_x$$

1. $(\ldots)D^4 = 0 \quad \rightarrow \quad \alpha_x = 0 \quad \rightarrow \quad \alpha = \text{constant}.$

2. $(\ldots)D^3 = 0 \quad \rightarrow \quad \alpha_{xx} = 0$ (automatic)

3. $(\ldots)D^2 = 0 \quad \rightarrow \quad 2\beta_x - 3\alpha u_x = 0 \quad \rightarrow \quad \beta = \frac{3}{2}\alpha u + k_1$

4. $(\ldots)D = 0 \quad \rightarrow \quad \beta_{xx} + 2\gamma_x - 3\alpha u_{xx} = 0$

arbitrary constant

$$\beta_{xx} = \frac{3}{2}\alpha u_{xx} \quad \rightarrow \quad \gamma_x = \frac{3}{4}\alpha u_{xx}$$

$$\rightarrow \quad \gamma = \frac{3}{4}\alpha u_x + k_2$$

arbitrary constant.

Evolution equation for u ?

$$u_t \doteq [B, L] = -\frac{3}{4}\alpha u_{xxx} + \alpha u_{xxx} + \left(\frac{3}{2}\alpha u + k_1\right) u_x$$

$$= \frac{1}{4}\alpha u_{xxx} + \frac{3}{2}\alpha u u_x + k_1 u_x$$

The KdV equation corresponds to $\alpha = -4$ and $k_1 = 0$ $\quad (u_t + 6uu_x + u_{xxx} = 0)$

For other values of α and k_1, we define (we make a field redefinition):

$$\tilde{u}(x,t) = u\left(x + \frac{4k_1}{\alpha} t, -\frac{4}{\alpha} t\right)$$

$\tilde{u}$ solves the KdV equation! $\quad$ Note: $\tilde{u}_t = \frac{4k_1}{\alpha} u_x - \frac{4}{\alpha} u_t$

(exercise S6) $\quad \tilde{u}_x = u_x$

$\tilde{u}_{xxx} = u_{xxx}$

Show: $\tilde{u}_{xxx} + 6\tilde{u}\tilde{u}_x + \tilde{u}_t = 0$