ASSOCIATED LINEAR PROBLEM

$$\boxed{\begin{aligned} L(u)\psi &= \lambda \psi \\ L(u) &= D^2 + u(x,t) \end{aligned}}$$

The associated linear problem has a simple time evolution if the evolution equation for u can be written as a Lax equation $u_t = [B, L]$ for a Lax pair L and B. The KdV equation is an example!

Any other examples? These are found by finding B such that $[B, L]$ is a multiplicative operator.

Simple examples

1. Does B have to be a differential operator? I.e. can $B = \alpha(x)$ for some function α of x?
   $[B, L]$ is multiplicative $\to$ α = constant and $u_t = [B, L] = 0$! TOO TRIVIAL!

2. What about $B = \alpha(x) D + \beta(x)$? In this case we need α and β constant.
   This gives: $u_t = [B, L] = \alpha u_x$ THE ADVECTION EQUATION (still trivial time evolution)

3. Now try: $B = \alpha(x) D^3 + \beta(x) D + \gamma(x)$! In this case $[B, L]$ multiplicative imposes:
   * $\alpha(x) = \text{constant}$
   * $\beta(x) = \frac{3}{2} \alpha u(x,t) + k_1$ ← constant
   * $\gamma(x) = \frac{3}{4} \alpha u_x + k_2$ ← constant.

Evolution equation for u:

$$u_t = [B, L] = \frac{1}{4} \alpha u_{xxx} + \frac{3}{2} \alpha u u_x + k_1 u_x$$

For $\alpha = -4$ and $k_1 = 0$ this is the KdV equation! For all other values of α and k_1, the re-defined field:

$$\tilde{u}(x,t) = u(x + 4k_1/\alpha \, t, \, -4/\alpha \, t) \quad \text{solves KdV!}$$

10.2 Hints for the general case

Claim: The self adjoint (symmetric) part of B commutes with L

M is self adjoint (symmetric) if $M^+ = M$

M is skew (anti-symmetric) if $M^+ = -M$.

B can be uniquely decomposed into symmetric and anti-symmetric parts

$$B = \underbrace{\tfrac{1}{2}(B + B^+)}_{\text{symmetric part}} + \underbrace{\tfrac{1}{2}(B - B^+)}_{\text{anti-symmetric part.}}$$

Proof: We want to show that $[\frac{1}{2}(B + B^+), L] = 0$.

If B forms a Lax pair with L then $([B, L])^+ = [B, L]$.

$$\Downarrow$$

$L_t = [B, L]$ and $L^+ = L$

$\Rightarrow L^+_t = L_t$

LHS : $\left([B,L]\right)^\dagger = (BL - LB)^\dagger = L^\dagger B^\dagger - B^\dagger L^\dagger \underset{\substack{\uparrow \\ L^\dagger = L}}{=} LB^\dagger - B^\dagger L$

RHS : $[B,L] = BL - LB$

$\Rightarrow LB^\dagger - B^\dagger L = BL - LB$

$\Rightarrow \frac{1}{2}(B+B^\dagger)L - L\frac{1}{2}(B+B^\dagger) = 0$

$\Rightarrow \left[\frac{1}{2}(B+B^\dagger), L\right] = 0$ $\square$

$\Rightarrow$ We can focus now on the anti-symmetric part of B since the symmetric part doesn't contribute to the evolution equation for u : $u_t = [B,L] = \left[\frac{1}{2}(B+B^\dagger) + \frac{1}{2}(B-B^\dagger), L\right]$

$$= \underbrace{\left[\frac{1}{2}(B-B^\dagger), L\right]}_{\text{only anti-symmetric part contributes!}}$$

What is the anti-symmetric part of B? If B is order "m" in x-derivatives :

$$B = \sum_{j=0}^{m} \alpha_j(x) D^j \qquad \text{for some real function } \alpha_j(x) \text{ of } x.$$

using $[D^n, g(x)]$ this can be equivalently written as :

$$B = \sum_{j=0}^{m} \beta_j(x) D^j + D^j \beta_j(x) \qquad \text{for some real function } \beta_j(x) \text{ of } x$$

We need to determine $B^\dagger$!

\# 1 $D^\dagger = -D$

$$\langle D^\dagger \phi, \chi \rangle = \langle \phi, D\chi \rangle = \int_{-\infty}^{\infty} \phi^* D \chi \, dx = \int_{-\infty}^{\infty} (-D\phi^*)\chi \, dx = \langle -D\phi, \chi \rangle$$

\# 2 $(\lambda(x) D)^\dagger = D^\dagger \lambda^*(x)$ $\qquad \left((MN)^\dagger = N^\dagger M^\dagger \right)$

$\qquad\qquad\qquad\qquad\qquad\qquad \lambda(x)\mathbb{I} \quad N = D$

$(D\lambda(x))^\dagger = \lambda^*(x) D^\dagger$

\# 3 $(D^{2j})^\dagger = (D^\dagger)^{2j} = D^{2j}$

\# 4 $(D^{2j+1})^\dagger = -D^{2j+1}$

All in all :

$$B^\dagger = \left(\sum_{j \text{ even}} D^j \beta_j + \beta_j D^j \right) - \sum_{j \text{ odd}} D^j \beta_j + \beta_j D^j \quad \leftarrow$$

$\Rightarrow$ Symmetric part : $\frac{1}{2}(B + B^\dagger) = \sum_{j \text{ even}} D^j \beta_j + \beta_j D^j$

anti-symmetric part : $\frac{1}{2}(B - B^\dagger) = \sum_{j \text{ odd}} D^j \beta_j + \beta_j D^j$

Since the symmetric part does not contribute to the equation for u we can redefine B, i.e. the object that must form a Lax pair with L, to be:

$$B := \sum_{j \text{ odd}} D^j \beta_j(x) + \beta_j(x) D^j$$

Other properties of B?

Exercise 55 (assignment 6): The coefficient of the highest power of D in B is constant

i.e. If B is order $2n-3$ in D, then:

$$B_n = \text{constant} \times D^{2n-3} + \sum_{j=1}^{n-2} \beta_j(x) D^{2j-1} + D^{2j-1} \beta_j(x)$$

tells us the order in D

Without loss of generality we can take constant $= 1$. This is because:

$L_t = [B, L]$ is unchanged after sending: $B_n \to \frac{1}{\text{constant}} B_n$, $t \to \text{constant} \times t$

All in all:

$$B_n := D^{2n-3} + \sum_{j=1}^{n-2} D^{2j-1} \beta_j(x) + \beta_j(x) D^{2j-1}$$

As before, the remaining functions $\beta_j(x)$ are constrained by the requirement that B_n forms a Lax pair with L

$\rightarrow$ All the Ds cancel in $[B_n, L]$.

Upon ensuring this, one finds that

$K_n(u, u_x, u_{xx}, \ldots) := [B_n, L]$ polynomial in $u, u_x, u_{xx}, \ldots, \frac{d^{2n-3} u}{dx^{2n-3}}$

i.e. the polynomial K_n is order $2n-3$ in x-derivatives

Evolution equation for u? $\quad u_t \stackrel{!}{=} K_n(u, u_x, \ldots)$

$$L_t + [L, B_n] = 0 \quad \Longleftrightarrow \quad u_t = K_n(u, u_x, u_{xx}, \ldots)$$