ASSOCIATED LINEAR PROBLEM

$$\boxed{\begin{aligned} L(u)\psi &= \lambda\psi \\ L(u) &= D^2 + u(x,t) \end{aligned}}$$

... has a simple time evolution if $L_t = [B, L]$. What is the general form of B? Useful basis:

if B is order "m":
$$B = \sum_{j=0}^{m} \beta_j(x) D^j + D^j \beta_j(x) \qquad \text{for real } \beta_j(x)$$
$$= \underbrace{\sum_{j \text{ even}} \beta_j(x) D^j + D^j \beta_j(x)}_{\text{symmetric part } \frac{1}{2}(B+B^+)} + \underbrace{\sum_{j \text{ odd}} \beta_j(x) D^j + D^j \beta_j(x)}_{\text{anti-symmetric part } \frac{1}{2}(B-B^+)}$$

The symmetric part of B, ie $\frac{1}{2}(B+B^+)$, commutes with L $\Rightarrow$ can focus on anti-symmetric part of B since now $u_t = [\frac{1}{2}(B-B^+), L]$! We have:

$$B_n := D^{2n-3} + \sum_{j=1}^{n-2} \beta_j(x) D^{2j-1} + D^{2j-1}\beta_j(x)$$

The remaining $\beta_j(x)$ are constrained by requiring B_n forms Lax pair with L:

$\rightarrow$ all Ds cancel in $[B_n, L]$

$\rightarrow$ $K_n(u, u_x, u_{xx}, \dots) = [B_n, L]$ is a polynomial in $u, u_x, \dots$ of order $2n-3$ in x-derivatives

$\rightarrow$ Evolution equation for u: $\quad u_t = K_n(u, u_x, u_{xx}, \dots)$

First few examples:

$$\left.\begin{aligned}
n=1: &\quad u_t = 0 \\
n=2: &\quad u_t + u_x = 0 \\
n=3: &\quad u_t + 6uu_x + u_{xxx} = 0 \\
n=4: &\quad u_t + 30u^2 u_x + 20 u_x u_{xx} + 10 u u_{xxx} + u_{xxxxx} = 0 \\
&\quad \vdots
\end{aligned}\right\} \text{ KdV hierarchy!}$$

Each equation in the hierarchy leaves invariant the spectrum of L in the associated linear problem!

10.3 Connection with conservation laws

Recall from Michaelmas the KdV equation has an infinite sequence of conserved charges Q_n $n = 1, 2, 3, 4, \dots, \infty$ with $\frac{dQ_n}{dt} = 0$ when the KdV equation is satisfied

$$Q_n = \int_{-\infty}^{\infty} T_n \, dx$$

$\nwarrow$ conserved current $\qquad \frac{\partial T_n}{\partial t} + \frac{\partial X_n}{\partial x} = 0$ with $X_n \to 0$ as $x \to \pm\infty$

where T_n are degree "n" polynomials in u and normalizing such that $T_n = 1 \times u^n + \dots$

first few examples:

$$T_1 = u$$
$$T_2 = u^2$$
$$T_3 = u^3 - \tfrac{1}{2} u_x^2$$

$$T_4 = u^4 - 2uu_x^2 + \frac{1}{5}u_{xx}^2$$

$$\vdots$$

We now have two infinite sequences with a property that is independent of time:

1 KdV conserved charges: $Q_1, Q_2, Q_3, \ldots$ with $\frac{dQ_n}{dt} = 0$

2 KdV hierarchy $u_t = K_n(u, u_x, \ldots)$: $K_1, K_2, K_3, \ldots$ with $\frac{d\lambda}{dt} = 0$ where $L\psi = \lambda\psi$.

Are they related? Yes!

Functional derivative

Suppose we have some function f of u and its x-derivatives. Then

$$F[u] = \int_{-\infty}^{\infty} f(u, u_x, u_{xx}, \ldots)\, dx$$

defines a functional of function $u(x)$, i.e. it takes $u(x)$ and returns a number $F[u]$.
[Note: we are talking t fixed and so it is not integrated over!].

To define the functional derivative we need to understand how $F[u]$ varies w.r.t. u!
Consider small variation $\delta u(x)$ of $u(x)$ i.e. $u(x) \to u(x) + \delta u(x)$ with $\delta u \to 0$ as $x \to \pm\infty$:

$$F[u + \delta u] = \int_{-\infty}^{\infty} f(u + \delta u, u_x + \delta u_x, u_{xx} + \delta u_{xx}, \ldots)\, dx$$

Consider Taylor expansion of f around $u, u_x, u_{xx}, \ldots$:

$$f(u+\delta u, u_x + \delta u_x, u_{xx} + \delta u_{xx}, \ldots) = f(u, u_x, u_{xx}, \ldots) + \frac{\partial f}{\partial u}\delta u + \frac{\partial f}{\partial u_x}\underbrace{\delta u_x}_{\frac{\partial}{\partial x}\delta u} + \frac{\partial f}{\partial u_{xx}}\underbrace{\delta u_{xx}}_{\frac{\partial^2}{\partial x^2}\delta u} + \ldots + O(\delta u^2)$$

$$= F[u] + \int_{-\infty}^{\infty}\left(\frac{\partial f}{\partial u}\delta u + \frac{\partial f}{\partial u_x}\frac{\partial}{\partial x}\delta u + \frac{\partial f}{\partial u_{xx}}\frac{\partial^2}{\partial x^2}\delta u + \ldots\right) dx + O(\delta u^2)$$

$$= F[u]$$

integrate by parts and $\delta u \to 0$ as $x \to \pm\infty$

$$+ \int_{-\infty}^{\infty}\underbrace{\left(\frac{\partial f}{\partial u} - \frac{\partial}{\partial x}\frac{\partial f}{\partial u_x} + \frac{\partial^2}{\partial x^2}\frac{\partial f}{\partial u_{xx}} + \ldots\right)}_{\frac{\delta F}{\delta u(x)}}\delta u\, dx$$

$+ \; o(\delta u^2)$

Consider g of x : $g(x+\delta x) = g(x) + \delta x \underbrace{\frac{dg}{dx}} + o(\delta x^2)$

derivative is defined as coefficient of δx :

$$\frac{dg}{dx} = \lim_{\delta x \to 0} \frac{g(x+\delta x) - g(x)}{\delta x}$$

The functional derivative $\frac{\delta F}{\delta u(x)}$ of $F[u]$ is the coefficient of δu :

$$F[u+\delta u] = F[u] + \int_{-\infty}^{\infty} \frac{\delta F}{\delta u(x)} \delta u(x) \, dx + o(\delta u^2)$$

$$\frac{\delta F}{\delta u(x)} = \frac{\partial f}{\partial u} - \frac{\partial}{\partial x} \frac{\partial f}{\partial u_x} + \frac{\partial^2}{\partial x^2} \frac{\partial f}{\partial u_{xx}} + \cdots$$

Examples : Take $F[u] = \int_{-\infty}^{\infty} f(u, u_x, u_{xx}, \dots) \, dx$

(a) $f(u) = u$, $\frac{\partial f}{\partial u} = 1$, $\frac{\partial f}{\partial u_x} = 0$ ... not zero

$$\rightarrow \frac{\delta F}{\delta u} = \frac{\partial f}{\partial u} = 1$$

(b) $f(u) = u^2$, $\frac{\partial f}{\partial u} = 2u$, $\frac{\partial f}{\partial u_x} = 0$, $\frac{\partial f}{\partial u_{xx}} = 0$, ... not zero

$$\rightarrow \frac{\delta F}{\delta u} = 2u$$

(c) $f(u) = u_x^2$, $\frac{\partial f}{\partial u} = 0$, $\frac{\partial f}{\partial u_x} = 2u_x$, not vanishing

$$\rightarrow \frac{\delta F}{\delta u} = -\frac{\partial}{\partial x}(2u_x) = -2u_{xx}.$$

Consider now the functional derivative of KdV charges $Q_n = \int_{-\infty}^{\infty} T_n \, dx$

$n=1$: $\frac{\delta Q_1}{\delta u} = \frac{\partial T_1}{\partial u} = 1$

$n=2$: $\frac{\delta Q_2}{\delta u} = \frac{\partial T_2}{\partial u} = 2u$

$n=3$: $\frac{\delta Q_3}{\delta u} = \frac{\partial T_3}{\partial u} - \frac{\partial}{\partial x} \frac{\partial T_3}{\partial u_x} = 3u^2 + u_{xx}$

$\vdots$

Key observation : now take x-derivative :

$n=1$: $\frac{\partial}{\partial x} \frac{\delta Q_1}{\delta u} = 0 = K_1(u)$

$n=2$: $\frac{\partial}{\partial x} \frac{\delta Q_2}{\delta u} = 2u_x = -2K_2(u)$

$n=3$ $$\frac{\partial}{\partial x}\frac{\delta Q_3}{\delta u} = 6uu_x + u_{xxx} = -K_3(u)$$

$$\vdots$$

More generally: $$\frac{\partial}{\partial x}\frac{\delta Q_n}{\delta u} \propto K_n(u)$$

proportionality constant can be set to 1 by rescaling t:

$$u_t = \frac{\partial}{\partial x}\frac{\delta Q_n}{\delta u} \quad \longleftrightarrow \quad u_t = K_n(u)$$