The story so far:

main goal: $u(x,0) \xrightarrow{\text{KdV equation}} u(x,t>0)$

so far: $u(x,t)$ evolves by KdV

$$\downarrow$$

$$\left(\frac{\partial^2}{\partial x^2} + u\right)\psi = \lambda\psi \quad \text{with simpler time evolution} \quad \psi_t = B(u)\psi$$

This is great, but there are some hurdles to overcome:

1. Since $\psi_t = B(u)\psi$ depends on u, to know how ψ evolves means we need to know how u evolves!
2. How do we extract u from ψ? (this is answered in chapter 12)

It turns out that understanding how to overcome 2 solves problem 1! In particular:

Q: How much of $\psi(x,t)$ do we need to know to reconstruct $u(x,t)$?

A: We only need to know ψ at $x \to \pm\infty$ i.e. the scattering data

This solves 1 because $u \to 0$ as $x \to \pm\infty$ $\Rightarrow$ $B \to -4D^3$ i.e. independent of u $\Rightarrow$ ψ_t as $x \to \pm\infty$ is independent of u!

Chapter 11: The basics of scattering theory

11.1 Overview: The physical interpretation

In quantum mechanics (QM) the fundamental object is the wavefunction $\Psi(x,\tau)$

τ is time w.r.t. which QM particle evolves and is not KdV time t.

Probability density: $\rho = |\Psi(x,\tau)|^2 = \Psi^*(x,\tau)\Psi(x,\tau)$

The probability to find particle in the interval $[x, x+dx]$ at time τ:

$$\rho\, dx$$

The time evolution in τ is governed by the (time-dependent) Schrödinger equation:

$$i\frac{\partial}{\partial\tau}\Psi(x,\tau) = \Big(-\frac{\partial^2}{\partial x^2} + \underbrace{V(x)}_{\text{potential}}\Big)\Psi(x,\tau) \qquad (11.2)$$

Gives QM description of a particle (mass 1/2) moving on a line (x) in a potential $V(x)$.

How to make contact with associated linear problem $L\psi = \lambda\psi$?

The key is to separate variables: $\Psi(x,\tau) = \phi(\tau)\psi(x)$

Substitute into (11.2): $\left[\ \dot{} = \frac{\partial}{\partial\tau}, \quad ' = \frac{\partial}{\partial x}\right]$

$$i\dot\phi\psi = (-\psi'' + V\psi)\phi$$

$$i\frac{\dot\phi}{\phi} = \frac{(-\psi'' + V\psi)}{\psi} = \text{constant} = k^2$$

only depends on τ only depends on x

Equation for ϕ:

$$\dot{\phi} = -ik^2 \phi \quad \to \quad \phi(\tau) = e^{-ik^2\tau}$$

Equation for ψ:

$$\left(-\frac{d^2}{dx^2} + V(x) \right) \psi(x) = k^2 \psi(x) \qquad (11.6)$$

" time independent Schrödinger equation for a particle of energy $E = k^2$ in potential V "

This is equation (9.1) for associated linear problem $L\psi = \lambda\psi$ upon identifying:

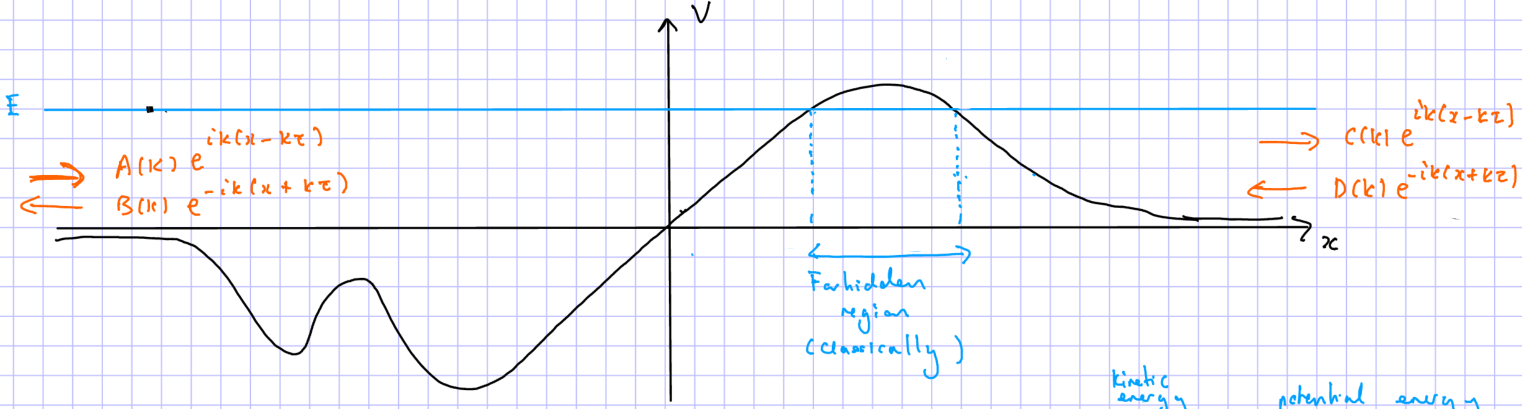
KdV field $\to$ $u = -V$ $\leftarrow$ minus the potential

Eigenvalue of L $\to$ $\lambda = -k^2$ $\leftarrow$ minus the total energy of QM particle.

" scattering solution " $\Leftrightarrow$ Relax requirement $\int_{-\infty}^{\infty} |\psi|^2 \, dx < \infty$ to just require ψ is bounded

still restricts possible values of λ

Potential $V \to 0$ as $x \to \pm\infty$ $\Leftrightarrow$ KdV field $u \to 0$ as $x \to \pm\infty$



kinetic energy potential energy

In classical mechanics, a particle of total energy $E = T + V$.

If $E < \max\{V(x)\}$ the particle bounces from potential at "turning points" x^* where $V(x^*) = E$.

In quantum mechanics (if V is finite) there is a non-zero chance to find the particle anywhere and can "tunnel" through the forbidden region.

The scattering data is encoded in the asymptotics (limiting value) of ψ as $x \to \pm\infty$.

Since $V \to 0$ as $x \to \pm\infty$ the time independent Schrödinger equation reduces to:

$$-\frac{d^2}{dx^2}\psi \approx k^2 \psi$$

two independent solutions : $e^{\pm ikx}$

as $x \to -\infty$: $\psi(x) \approx A(k)\, e^{ikx} + B(k)\, e^{-ikx}$

as $x \to +\infty$: $\psi(x) \approx C(k)\, e^{ikx} + D(k)\, e^{-ikx}$

Restoring τ (multiplying by $\phi(\tau)$) the asymptotic form of $\Psi(x,\tau)$ (WLOG $k>0$) :

as $x \to -\infty$: $\Psi(x,\tau) \approx A(k)\, e^{ik(x-k\tau)} + B(k)\, e^{-ik(x+k\tau)}$

as $x \to +\infty$ $\Psi(x,\tau) \approx C(k)\, e^{ik(x-k\tau)} + D(k)\, e^{-ik(x+k\tau)}$

Consider first the case that $E = k^2 > 0$ and $k > 0$ (always bounded)

To study scattering, we impose (by choice of convention)

$D(k) = 0$ (incoming particles from left)

$A(k) = 1$ (unit flux of particles from left).

With this choice, we can the re-define:

$B(k) = R(k)$ " reflection coefficient "

$C(k) = T(k)$ " transmission coefficient "