associated linear problem — time independent Schrödinger equation (T.I.S.E.)

$$\left(\frac{\partial^2}{\partial x^2} + u\right)\psi = \lambda \psi \quad \Longleftrightarrow \quad \left(-\frac{\partial^2}{\partial x^2} + V\right)\psi = E\psi$$

KdV field $u = -V$ (QM potential)

Eigenvalue $\lambda = -E$ (total energy of QM particle)

If $u$ evolves by KdV, then $\psi$ has a simple time evolution w.r.t. the KdV time $t$. $V$ (and hence $u$) is then reconstructed from the scattering data which is the asymptotic form of bounded solutions to the T.I.S.E.

$$\psi(x) \approx \begin{cases} A(k)e^{ikx} + B(k)e^{-ikx}, & x \to -\infty \\ C(k)e^{ikx} + D(k)e^{-ikx}, & x \to +\infty \end{cases}$$

(setting $E = k^2$) (11.9)

Bounded solutions come in two flavours:

1. Positive energies $E = k^2 \in (0,\infty)$: "continuous spectrum" → "scattering solution"

$$\psi(x) \approx \begin{cases} e^{ikx} + R(k)e^{-ikx}, & x \to -\infty \\ T(k)e^{ikx}, & x \to +\infty \end{cases}$$

(11.13)

(by convention: particles only incoming from the left ($D=0$) and with unit flux)

$R(k)$ : reflection coefficient
$T(k)$ : transmission coefficient

assignment 7 : $|R(k)|^2 + |T(k)|^2 = 1$

2. Negative energies $E = -\mu^2 \in \{-\mu_1^2, -\mu_2^2, \dots, -\mu_N^2\}$ (possibly empty set): "discrete spectrum" → "bound state solutions"

set $k = i\mu$ with $\mu > 0$

$$\psi(x) \approx \begin{cases} B\, e^{\mu x}, & x \to -\infty \\ C\, e^{-\mu x}, & x \to +\infty \end{cases}$$

(boundedness ⇒ $A = D = 0$)

this might not always be possible (as we shall see!)

## 11.2 Examples

Example 1 : $V(x) = 0$ (everywhere)

T.I.S.E. is : $-\dfrac{d^2}{dx^2}\psi = E\psi$

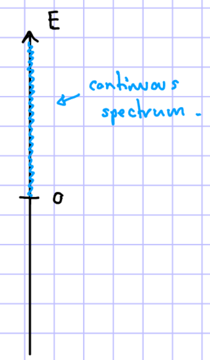
General solution: (setting $E = k^2$) $\psi(x) = Ae^{ikx} + Be^{-ikx}$ (for all $x$)

1. Scattering solution : $\psi(x) = e^{ikx}$ $\forall$ $E = k^2 \in (0,\infty)$

(particles moving to right, transmitted w. probability 1 ($T(k) = 1$))

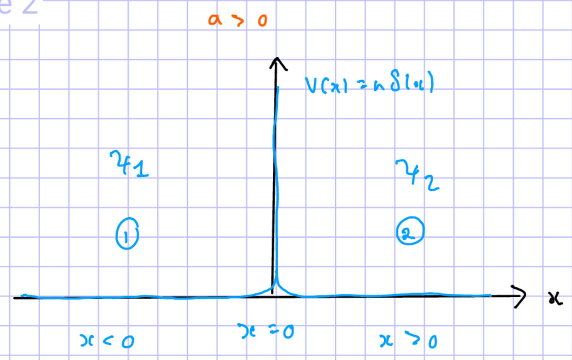
2. Bound state solution: Boundedness requires $A = B = 0$ → No solution.

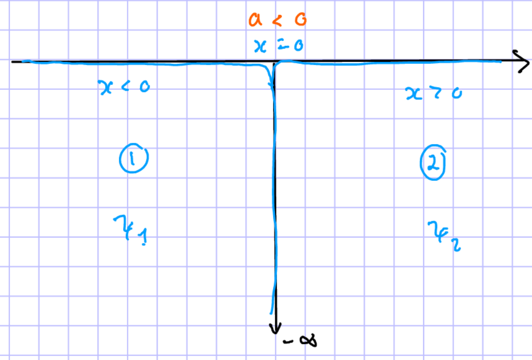
Energy spectrum of bounded solutions:




Example 2 : $V(x) = a\,\delta(x)$ where $\delta(x) = \begin{cases} 0, & x \neq 0 \\ \infty, & x = 0 \end{cases}$ such that $\int_{-\infty}^{\infty} \delta(x)\, dx = 1$

$$\int_{-\infty}^{\infty} \delta(x) f(x)\, dx = f(0)$$




In regions ① and ② we have $V(x) = 0$ and T.I.S.E. is:

$$-\frac{d^2}{dx^2}\psi = E\psi \qquad \text{for} \quad x \neq 0$$

which we know how to solve (see example 1). Denote solutions in regions ① ② by $\psi_1$ and $\psi_2$ respectively

$$\psi(x) = \begin{cases} \psi_1(x) & x < 0 \\ \psi_2(x) & x > 0. \end{cases}$$

How to obtain general solution for <u>all</u> $x$ (including $x = 0$)? To do that we impose matching conditions:

\# 1: The wavefunction $\psi$ is continuous:

Continuity is ensured within regions ① and ②, but across regions ① and ② we have to put it in by hand:

The discontinuity of $\psi$ across $x = 0$: $\quad \left[\psi(x)\right]_{0^-}^{0^+} = \psi(x=0^+) - \psi(x=0^-)$

$$= \psi_2(0) - \psi_1(0)$$

Continuity of $\psi$ across $x = 0 \Rightarrow \quad \psi_2(0) = \psi_1(0) \qquad (11.24)$

\# 2 The derivative $\psi'$ of the wave function $\psi$ is a certain discontinuity across regions ① and ②

$$-\frac{d^2}{dx^2}\psi + a\,\delta(x)\,\psi(x) = E\,\psi(x)$$

integrate in $x$

$$\underbrace{\int_{0^-}^{0^+} -\frac{d^2\psi}{dx^2}\,dx}_{\left[-\frac{d\psi}{dx}\right]_{0^-}^{0^+}} + \underbrace{\int_{0^-}^{0^+} dx\; a\,\delta(x)\,\psi(x)}_{a\,\psi(0)} = \underbrace{E\int_{0^-}^{0^+} \psi(x)\,dx}_{=0 \text{ b.c. } \psi \text{ is continuous.}}$$

$$\rightarrow \quad \left[ \frac{d\psi}{dx} \right]_{0^-}^{0^+} = \psi_2'(0) - \psi_1'(0) = a\,\psi(0) \qquad (11.27)$$

Matching conditions #1 and #2 allow to relate the general solutions in regions ① and ② to give the general solution for <u>all</u> $x$!

Before imposing the matching conditions we have:

$$\text{region ①}: \quad \psi_1(x) = A e^{ikx} + B e^{-ikx} \qquad x < 0 \qquad (11.23)$$

$$\text{region ②}: \quad \psi_2(x) = C e^{ikx} + D e^{-ikx} \qquad x > 0$$

This <u>and</u> the matching conditions hold for both $k$ real and $k$ imaginary. Applying the matching conditions #1 and #2 gives relations between $A$, $B$, $C$ and $D$ reducing the number of independent arbitrary constants to the required <u>two</u>.
(for 2nd order ode)

Matching #1: $\psi_1(0) = \psi_2(0) \quad \Rightarrow \quad A + B = C + D$

Matching #2: $\psi_2'(0) - \psi_1'(0) = a\psi(0) \quad \Rightarrow \quad ik(C - D) - ik(A - B) = a(A+B) = a(C+D)$

Rearranging:

$$A + B = C + D$$

$$A - B = \left(1 - \frac{a}{ik}\right) C - \left(1 + \frac{a}{ik}\right) D$$

$$\Rightarrow \quad A = \left(1 - \frac{a}{2ik}\right) C - \frac{a}{2ik} D$$

$$B = \frac{a}{2ik} C + \left(1 + \frac{a}{2ik}\right) D$$

Plugging into (11.23) gives the general solution for <u>all</u> $x$!

two cases:

(a) Positive energies $E = k^2$ and $k > 0$. For the scattering solution, set $D = 0$ and divide by $A = \left(1 - \frac{a}{2ik}\right) C$ to get unit flux from the left:

$$\psi(x) = \begin{cases} e^{ikx} + \dfrac{a}{2ik - a} e^{-ikx} & x < 0 \\[2ex] \dfrac{2ik}{2ik - a} e^{ikx} & x > 0 \end{cases} \qquad (11.29)$$

$$\rightarrow \quad \left. \begin{aligned} R(k) &= \frac{a}{2ik - a} \\ T(k) &= \frac{2ik}{2ik - a} \end{aligned} \right\} \quad |R(k)|^2 + |T(k)|^2 = \frac{a^2}{(2k)^2 + a^2} + \frac{(2k)^2}{(2k)^2 + a^2} = 1$$