Bounded solutions to T.I.S.E. $\left(-\frac{\partial^2}{\partial x^2} + V(x)\right)\psi(x) = k^2\psi(x)$ come in two flavours:

1. Positive energies $E = k^2 \in (0,\infty)$: "continuous spectrum" $\rightarrow$ "scattering solution"

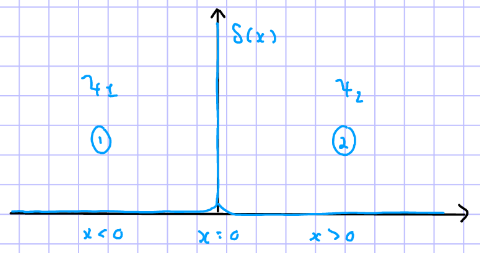
$$\psi(x) \approx \begin{cases} e^{ikx} + R(k)e^{-ikx}, & x \to -\infty \\ T(k)e^{ikx}. & x \to +\infty \end{cases} \qquad (11.13)$$

2. Negative energies $E = -\mu^2 \in \{-\mu_1^2, -\mu_2^2, \dots, -\mu_N^2\}$ (possibly empty set): "discrete spectrum" $\rightarrow$ "bound state solutions"

set $k = i\mu$ with $\mu > 0$

$$\psi(x) \approx \begin{cases} B\,e^{\mu x}, & x \to -\infty \\ C\,e^{-\mu x}, & x \to +\infty \end{cases}$$

Example 2: $V(x) = a\,\delta(x)$



$$\psi(x) = \begin{cases} \psi_1(x), & x<0 \\ \psi_2(x), & x>0 \end{cases}$$

$$\begin{aligned} \psi_1(x) &= Ae^{ikx} + Be^{-ikx}, & x<0 \\ \psi_2(x) &= Ce^{ikx} + De^{-ikx}, & x>0 \end{aligned} \qquad (11.23)$$

Matching condition # 1: $\psi_1(0) = \psi_2(0)$

Matching condition # 2: $\psi_2'(0) - \psi_1'(0) = a\,\psi(0)$

$$\Rightarrow \quad \begin{aligned} A &= \left(1 - \frac{a}{2ik}\right)C - \frac{a}{2ik}D \\ B &= \frac{a}{2ik}C + \left(1 + \frac{a}{2ik}\right)D \end{aligned} \qquad (11.28)$$

plug into (11.23) to give general solution for all x!

Two cases:

(a) Positive energies $E = k^2$ and $k > 0$. For scattering solution, set $D = 0$ and divide by $A = \left(1 - \frac{a}{2ik}\right)C$:

$$\psi(x) = \begin{cases} e^{ikx} + \dfrac{a}{2ik - a}e^{-ikx}, & x<0 \\ \dfrac{2ik}{2ik - a}e^{ikx}, & x>0 \end{cases} \qquad (11.29)$$

$$R(k) = \frac{a}{2ik - a}, \quad T(k) = \frac{2ik}{2ik - a}$$

(b) Bound states? $E = -\mu^2$ and WLOG $\mu > 0$. Set $k = i\mu$ in (11.23) and (11.28)

$$\psi(x) = \begin{cases} Ae^{-\mu x} + Be^{\mu x}, & x<0 \\ Ce^{-\mu x} + De^{\mu x}, & x>0 \end{cases}$$

ψ is bounded $\rightarrow$ $A = D = 0$. Then we have to check consistency with (11.28):

$$0 = \left(1 + \frac{a}{2\mu}\right)C$$

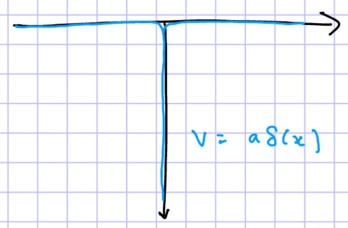
$$B = -\frac{a}{2\mu}C$$

two possibilities:

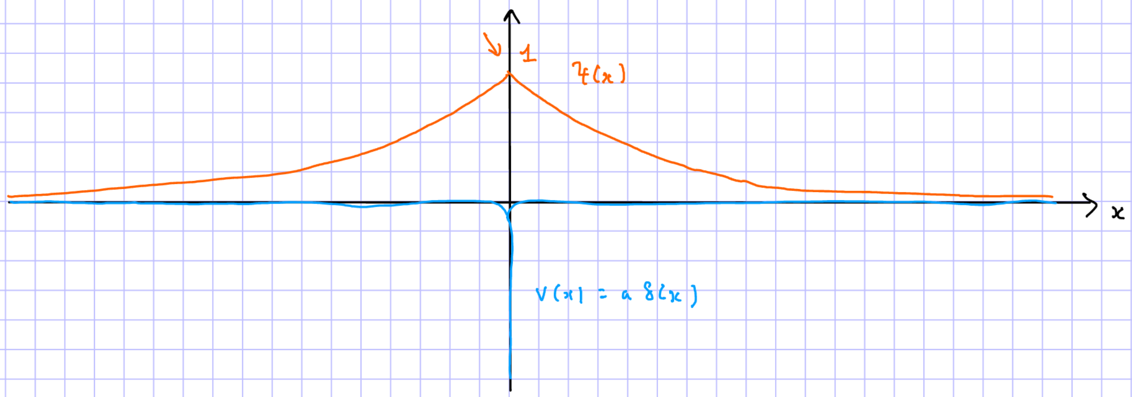
① $A = B = C = D = 0$ (trivial solution)

② $\mu = -a/2$ $\rightarrow$ $B = C$ and $A = D = 0$ (since $\mu > 0$ this only exists for $a < 0$)

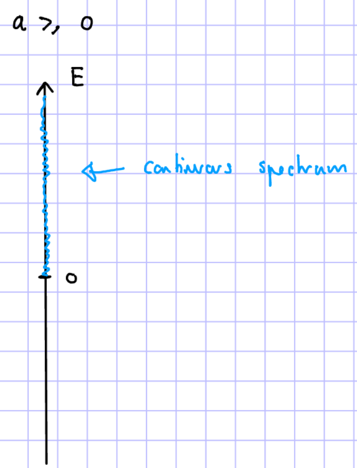
This is a single bound state solution for $a<0$ given by (normalize $C=1$):

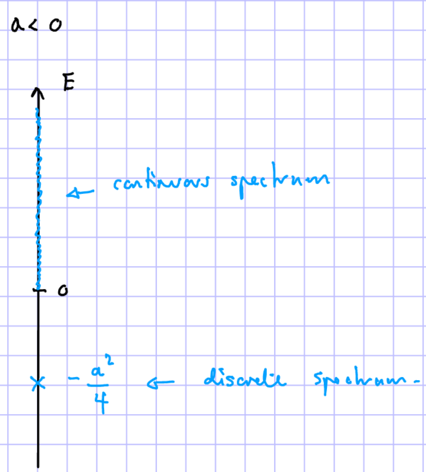


$$\psi(x) = \begin{cases} e^{-a/2\, x}, & x<0 \\ e^{a/2\, x}, & x>0 \end{cases} \qquad \text{with energy } E = -\frac{a^2}{4}$$



Summary of energy spectrum of bounded solutions:





Question: Can I obtain bound state solutions from the knowledge of the scattering solution?

Answer: Yes!

The general story

(a) For all $E = k^2 > 0$ and WLOG $k>0$ we find the scattering solution:

$$\psi(x) \approx \begin{cases} e^{ikx} + R(k)\, e^{-ikx}, & x \to -\infty \\ T(k)\, e^{ikx}, & x \to +\infty \end{cases} \qquad (11.37)$$

[N.B. R and T are meromorphic in the complex variable k and have simple poles on the imaginary axis].

(b) $E = -\mu^2$ and WLOG $\mu > 0$

Step #1 : Set $k = i\mu$ in (11.37)

$$\psi(x) \approx \begin{cases} e^{-\mu x} + R(i\mu)\, e^{\mu x}, & x \to -\infty \\ T(i\mu)\, e^{\mu x}, & x \to +\infty . \end{cases}$$

two problems :

1. This is not bounded ($e^{-\mu x} \to \infty$ as $x \to -\infty$)

2. R and T might contain poles at certain real values of μ

step #2 : Divide by $T(i\mu)$:

$$\psi(x) \approx \begin{cases} \dfrac{1}{T(i\mu)}\, e^{-\mu x} + \dfrac{R(i\mu)}{T(i\mu)}\, e^{\mu x}, & x \to -\infty \\ e^{-\mu x}, & x \to +\infty \end{cases}$$

This is bounded at the values of μ where $T(i\mu)$ has a pole since $1/T(i\mu) = 0$

For such a value of μ :

$$\psi(x) \approx \begin{cases} \dfrac{R(i\mu)}{T(i\mu)}\, e^{\mu x}, & x \to -\infty \\ e^{-\mu x} & x \to \infty \end{cases}$$

This only exists for values of μ where $T(i\mu)$ has a pole for $\mu > 0$

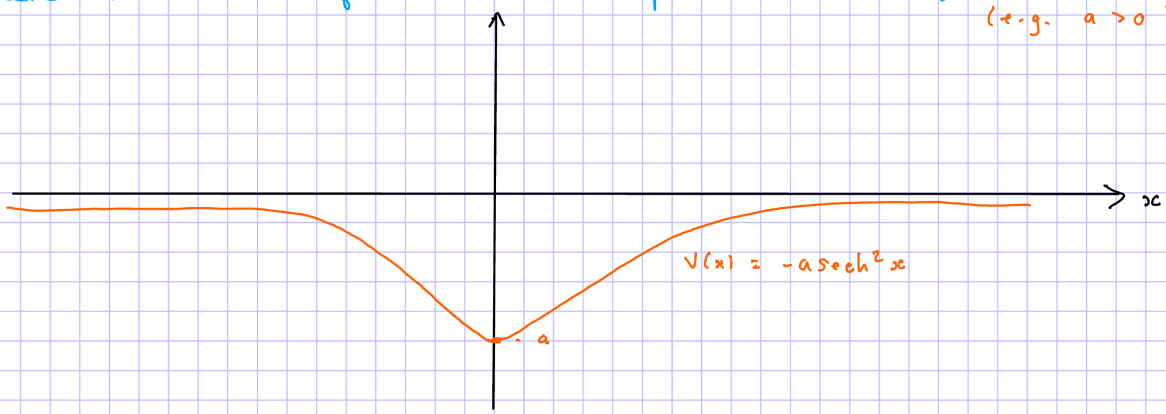
Poles in $T(i\mu)$ for $\mu > 0$ $\iff$ bound state solution.

11.3 Reflectionless potentials

Consider

$$V(x) = -a\,\mathrm{sech}^2 x \tag{11.42}$$

[recall : $u = -V = a\,\mathrm{sech}^2 x$ where for $a = n(n+1)$ and n is a positive integer this seemed to be the form of the initial data for pure n-soliton solutions to the KdV equation]
(e.g. $a > 0$)

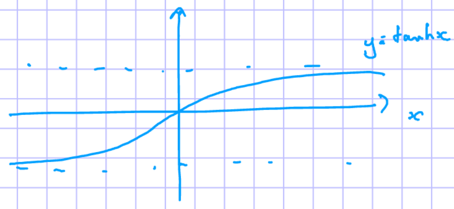


The T.I.S.E. takes the form :

$$-\psi''(x) - a\,\mathrm{sech}^2 x\, \psi = E\,\psi \qquad E = k^2$$

goal : look for bounded solutions

Key: make change of variables



$$y = \tanh x \in (-1, 1)$$

$$\frac{d}{dx} = \frac{dy}{dx}\frac{d}{dy} = \operatorname{sech}^2 x \frac{d}{dy}$$

$$= (1 - \tanh^2 x)\frac{d}{dy}$$

$$= (1 - y^2)\frac{d}{dy}$$

The T. I. S. E. becomes

$$\frac{d}{dy}\left[(1-y^2)\frac{d\psi}{dy}\right] + \left(\frac{k^2}{1-y^2} + a\right)\psi = 0$$

if we make the replacements:

$$k^2 = -m^2 \quad \text{and} \quad a = n(n+1)$$

This becomes

$$\frac{d}{dy}\left[(1-y^2)\frac{d\psi}{dy}\right] + \left(n(n+1) - \frac{m^2}{1-y^2}\right)\psi = 0$$

General (or associated) Legendre equation.