* Last time *

Lecture 34

Bound state solutions $\Leftrightarrow$ Poles in $T(i\mu)$ for $\mu > 0$

Algorithm to determine bound state solutions from scattering solution:

(1) Take the scattering solution for $E = k^2 > 0$ and $k > 0$:

$$\psi(x) \approx \begin{cases} e^{ikx} + R(k)e^{-ikx}, & x \to -\infty \\ T(k)e^{ikx}, & x \to +\infty \end{cases}$$

and set $k = i\mu$ with $\mu > 0$:

$$\psi(x) \approx \begin{cases} e^{-\mu x} + R(i\mu)e^{\mu x}, & x \to -\infty \\ T(i\mu)e^{-\mu x}, & x \to +\infty \end{cases}$$

solution with energy $E = -\mu^2$! (just it's not bounded)

(2) Divide by $T(i\mu)$:

$$\psi(x) \approx \begin{cases} \frac{1}{T(i\mu)} e^{-\mu x} + \frac{R(i\mu)}{T(i\mu)} e^{\mu x}, & x \to -\infty \\ e^{-\mu x}, & x \to +\infty \end{cases}$$

(3) Set μ equal to a pole of $T(i\mu)$ for $\mu > 0$ (for which $1/T(i\mu) = 0$):

$$\psi(x) \approx \begin{cases} \frac{R(i\mu)}{T(i\mu)} e^{\mu x}, & x \to -\infty \\ e^{-\mu x}, & x \to +\infty \end{cases}$$

Bound state solution!

(4) Repeat for all poles of $T(i\mu)$ for $\mu > 0$!

[exercise: check this reproduces the bound state solution for $V(x) = -a\,\delta(x)$ starting from the scattering solution with $R(k) = \frac{a}{2ik - a}$ and $T(k) = \frac{2ik}{2ik - a}$]

11.3 Reflectionless potentials

$$V(x) = -a \operatorname{sech}^2 x$$

[For $a = n(n+1)$ with $n \in \mathbb{Z}_{\geq 0}$ this is the initial data for pure n-soliton solutions to KdV]

Claim: For $n \in \mathbb{Z}$ this potential is reflectionless i.e. $R(k) = 0$.

The T.I.S.E. is:

$$-\psi''(x) - a \operatorname{sech}^2 x\, \psi(x) = E\,\psi(x) \qquad E = k^2$$

Make change of variables: $y = \tanh x$ and $k^2 = -m^2$

T.I.S.E. becomes

$$\frac{d}{dy}\left[(1-y^2)\frac{d\psi}{dy}\right] + \left(n(n+1) - \frac{m^2}{1-y^2}\right)\psi = 0 \qquad (11.48)$$

generalized (associated) Legendre equation

The solutions to this equation are known!

as $x \to \pm\infty$ we have $y \to \pm 1$ $\Rightarrow$ Extract solutions that are bounded as $y \to \pm 1$

Case 1 : $m = 0$ and $n = 0, 1, 2, 3, \dots$

$\hookrightarrow k = 0 \to E = 0$

general Legendre equation $\to$ Legendre equation

Bounded solutions

$$\psi(y) = P_n(y) = \frac{1}{n! 2^n} \frac{d^n}{dy^n} (y^2 - 1)^n$$

"n^{th} Legendre polynomial"

First few examples:

$$P_1(y) = y$$
$$P_2(y) = -\frac{1}{2} + \frac{3}{2} y^2$$
$$P_3(y) = -\frac{3}{2} y + \frac{5}{2} y^3$$
$$\vdots$$

these are normalized such that $P_n(1) = 1$. We have $P_n(-y) = (-1)^n P_n(y)$

$$\left.\begin{aligned} P_n(1) &= 1 \\ P_n(-1) &= (-1)^n \end{aligned}\right\} \Rightarrow$$ these are bounded but not bound state solutions, which would require $P_n(y) \to 0$ as $y \to \pm 1$.

Case 2 : If $n = 0, 1, 2, 3, \dots$ then we have bounded solutions for $m = 0, 1, 2, \dots, n$

$$\psi(y) = P^m{}_n(y) = (-1)^m (1 - y^2)^{m/2} \frac{d^m}{dy^m} P_n(y)$$

associated Legendre "polynomials" of the first kind.

These do go to zero as $y \to \pm 1$ (for $m > 0$) and correspond to bound state solutions.

General case : n and m are non-integer (even complex!)

In this case solutions can be written down explicitly by introducing certain special functions!

Potential bounded solution :

$$\psi(y) = P^m{}_n(y) = \frac{1}{\Gamma(1-m)} \left(\frac{1+y}{1-y}\right)^{m/2} {}_2F_1\left(-n, n+1; 1-m; \frac{1-y}{2}\right)$$

Special functions

1. Gamma function : $\Gamma(z) = \int_0^\infty dt\, t^{z-1} e^{-t}$, $\quad Re(z) > 0$

when $z = N + 1$, $N \in \mathbb{Z}_{\geq 0}$ then $\Gamma(N+1) = N! = N \times (N-1) \times \dots \times 1$

identities : $\quad \Gamma(1) = 0! = 1$

 $\Gamma(z+1) = z\,\Gamma(z)$

From this identity, can show that $\Gamma(z)$ has simple poles for $z = 0, -1, -2, \ldots$

$$\Gamma(z) = \frac{\Gamma(z+1)}{z} \quad \Rightarrow \quad \Gamma(0) = \frac{\Gamma(1)}{0} = \infty$$

$$\Gamma(-1) = \frac{\Gamma(0)}{-1} = -\infty$$

$$\Gamma(-2) = \frac{\Gamma(-1)}{-2} = \infty$$

$$\vdots \qquad \vdots$$

More generally: $\quad 1/\Gamma(z) = 0 \quad \forall \quad z = 0, -1, -2, -3, \ldots$

$$* \quad \Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)} \qquad (11.56)$$

2. Gauss hypergeometric function: ${}_2F_1(a, b; c; z)$

If $|z| < 1$ this is defined by the hypergeometric series:

$${}_2F_1(a, b; c; z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \sum_{p=0}^{\infty} \frac{\Gamma(p+a)\Gamma(p+b)}{\Gamma(p+c)} \frac{z^p}{p!}$$

$$= 1 + \frac{ab}{c} z + \frac{1}{2} \frac{a(a+1)b(b+1)}{c(c+1)} z^2 + \ldots$$

Extracting the asymptotic behaviour of ψ as $x \to \pm\infty$

1. Positive $E = k^2 > 0$ and $k > 0$ (scattering solution). Recall: $m = ik$.

$*$ limit $x \to +\infty$

$$y = \tanh x = \frac{1 - e^{-2x}}{1 + e^{-2x}} \sim 1 - 2e^{-2x} \to 1^- \quad \text{as} \quad x \to +\infty$$

$${}_2F_1\left(a, b; c; \frac{1-y}{2}\right) \to {}_2F_1(a, b; c; 0) = 1$$

$$\left(\frac{1+y}{1-y}\right) \to \frac{2}{2e^{-2x}} = e^{2x}$$

$$\left(\frac{1+y}{1-y}\right)^{m/2} \to e^{mx}$$

All together: $$\psi \sim \underbrace{\frac{1}{\Gamma(1-ik)}}_{C} e^{ikx} \qquad x \to +\infty$$

$*$ limit $x \to -\infty$:

$$y = \tanh x = \frac{e^{2x} - 1}{e^{2x} + 1} \sim -1 + 2e^{2x} \to -1^+ \quad \text{as} \quad x \to -\infty$$

$$\left(\frac{1+y}{1-y}\right) \to \frac{2e^{2x}}{2} = e^{2x}$$

$$\left(\frac{1+y}{1-y}\right)^{m/2} \to e^{mx}$$

$$\frac{1}{\Gamma(1-m)}\,{}_2F_1\left(-n, n+1; 1-m; \frac{1-y}{2}\right) \sim \frac{\Gamma(-m)}{\Gamma(1-m+n)\Gamma(-m-n)} + \frac{\Gamma(m)}{\Gamma(-n)\Gamma(n+1)}e^{-2mx}$$

All together : $\psi \sim \underbrace{\frac{\Gamma(-ik)}{\Gamma(1-ik+n)\Gamma(-ik-n)}}_{A} e^{ikx} + \underbrace{\frac{\Gamma(ik)}{\Gamma(-n)\Gamma(n+1)}}_{B} e^{-ikx}$, as $x \to -\infty$

This is precisely the form of a scattering solution (up to normalization!)
Normalizing the coefficient of e^{-ikx} as $x \to -\infty$ to 1, we can identify R and T :
(i.e. dividing by A)

$$\Rightarrow R(k) = \frac{B}{A} = \frac{\Gamma(ik)\Gamma(1-ik+n)\Gamma(-ik-n)}{\Gamma(-ik)\Gamma(n+1)\Gamma(-n)}$$

$$T(k) = \frac{C}{A} = \frac{\Gamma(1-ik+n)\Gamma(-ik-n)}{\Gamma(1-ik)\Gamma(ik)}$$