* Last time *

11.3 Reflectionless potentials

$$V(x) = -a\,\mathrm{sech}^2 x$$

[For $a = n(n+1)$ with $n \in \mathbb{Z}_{>0}$ this is the initial data for pure n-soliton solutions to KdV]

The T.I.S.E. is:

$$-\psi''(x) - a\,\mathrm{sech}^2 x\, \psi(x) = E\,\psi(x) \qquad E = k^2$$

Make change of variables: $y = \tanh x$ and $k^2 = -m^2$

T.I.S.E. becomes

$$\frac{d}{dy}\left[(1-y^2)\frac{d\psi}{dy}\right] + \left(n(n+1) - \frac{m^2}{1-y^2}\right)\psi = 0 \qquad (11.48)$$

generalized (associated) Legendre equation

The solutions to this equation are known!

as $x \to \pm\infty$ we have $y \to \pm 1$ $\Rightarrow$ Extract solutions that are bounded as $y \to \pm 1$

If $n = 0, 1, 2, 3, \ldots$ bounded solutions exist for $m = 0, 1, 2, \ldots, n$:

"n^{th} Legendre polynomial"

$$\psi(y) = P^m{}_n(y) = (-1)^m (1-y^2)^{m/2} \frac{d^m}{dy^m} P_n(y)$$

associated Legendre "polynomial" of first kind

For $m > 0$, these go to zero as $y \to \pm 1$ and are bound state solutions (as we shall see!)

More generally: m and n non-integer (even complex!)

To write the solutions in this case we need to introduce certain special functions

$$\psi(y) = P^m{}_n(y) = \frac{1}{\Gamma(1-m)}\left(\frac{1+y}{1-y}\right)^{m/2} {}_2F_1\left(-n, n+1; 1-m; \frac{1-y}{2}\right)$$

Special Functions

1. Gamma function $\quad \Gamma(z) = \int_0^\infty dt\, t^{z-1} e^{-t} \qquad \mathrm{Re}(z) > 0$

Identities:

* $\Gamma(z+1) = z\,\Gamma(z) \Rightarrow \Gamma(z)$ has simple poles at $z = 0, -1, -2, -3, \ldots$

* $\Gamma(z)\Gamma(1-z) = \dfrac{\pi}{\sin(\pi z)}$ (11.56)

2. Gauss Hypergeometric function $\quad {}_2F_1(a, b; c; z)$

This is defined by the hypergeometric series for $|z| < 1$:

$${}_2F_1(a, b; c; z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \sum_{p=0}^{\infty} \frac{\Gamma(p+a)\Gamma(p+b)}{\Gamma(p+c)} \frac{z^p}{p!}$$

Useful identity to expand around $z \sim 1$:

$$\frac{\sin(\pi(c-a-b))}{\pi}\, {}_2F_1(a, b; c; z) = \frac{{}_2F_1(a, b; c; 1-z)}{\Gamma(c-a)\Gamma(c-b)\Gamma(a+b-c+1)} - (1-z)^{c-a-b} \frac{{}_2F_1(c-a, c-b; c-a-b+1; 1-z)}{\Gamma(a)\Gamma(b)\Gamma(c-a-b+1)}$$

Extracting the asymptotic behaviour of ψ as $x \to \pm\infty$

1. Positive $E = k^2 > 0$ and $k > 0$ (scattering solution), Recall: $m = ik$.

$$\psi \sim \begin{cases} \underbrace{\frac{\Gamma(-ik)}{\Gamma(1-ik+n)\Gamma(-ik-n)}}_{A} e^{ikx} + \underbrace{\frac{\Gamma(ik)}{\Gamma(-n)\Gamma(n+1)}}_{B} e^{-ikx}, & \text{as } x \to -\infty \\ \underbrace{\frac{1}{\Gamma(1-ik)}}_{C} e^{ikx} & \text{as } x \to +\infty \end{cases}$$

This is precisely the form of a scattering solution (up to normalization!)

Normalizing the coefficient of e^{-ikx} as $x \to -\infty$ to 1, we can identify R and T:

(i.e. dividing by A)

$$\Rightarrow \quad R(k) = \frac{B}{A} = \frac{\Gamma(ik)\Gamma(1-ik+n)\Gamma(-ik-n)}{\Gamma(-ik)\Gamma(n+1)\Gamma(-n)} \underset{(11.56)}{=} -\frac{\sin(\pi n)}{\pi}\,\frac{\Gamma(ik)\Gamma(1-ik+n)\Gamma(-ik-n)}{\Gamma(-ik)}$$

$$T(k) = \frac{C}{A} = \frac{\Gamma(1-ik+n)\Gamma(-ik-n)}{\Gamma(1-ik)\Gamma(ik)}$$

Potentials of the form $V(x) = -n(n+1)\,\mathrm{sech}^2 x$ with $n \in \mathbb{Z}$ reflectionless potentials

- no particles are reflected for any k

2. $E = -\mu^2$ and WLOG $\mu > 0$ (bound state solutions)

Set $k = i\mu$ in the scattering solution and divide by $T(i\mu)$:

$$\psi(x) \approx \begin{cases} \frac{1}{T(i\mu)} e^{-\mu x} + \frac{R(i\mu)}{T(i\mu)} e^{\mu x}, & x \to -\infty \\ e^{-\mu x}, & x \to +\infty \end{cases}$$

This corresponds to a bound state solution for the values of $\mu > 0$ such that $1/T(i\mu) = 0$!

We have:

$$\frac{1}{T(i\mu)} = \frac{\Gamma(1+\mu)\Gamma(\mu)}{\Gamma(1+\mu+n)\Gamma(\mu-n)} \overset{!}{=} 0$$

This happens when:

(1) $1/\Gamma(1+\mu+n) = 0 \quad \Rightarrow \quad 1+\mu+n = -j, \quad j \in \mathbb{Z}_{\geq 0} \quad \to \quad \mu = -1-n-j \lesssim -1/2 - j$

(2) $1/\Gamma(\mu-n) = 0 \quad \Rightarrow \quad \mu - n = -\ell, \quad \ell \in \mathbb{Z}_{\geq 0} \quad \to \quad \mu = n - \ell$

Comments :

* There is no bound state solution if $n \notin \mathbb{R}$

* (1) and (2) are related by $n \to -1-n$ and are equal when $n = -1-n \to n = -1/2$

  $\Rightarrow$ can take $n \geq -1/2$

  In this way (1) has no solution for $\mu > 0$ and solutions with $\mu > 0$ exist for (2) if $n \geq 0$.

Bound state values of μ :

$$\mu = n,\ n-1,\ n-2,\ \ldots\ldots,\ n - \lfloor n \rfloor$$

where $\lfloor n \rfloor$ the largest integer smaller than or equal to n.

$\to$ total # bound state solutions is $\lceil n \rceil$ where $\lceil n \rceil$ is the smallest integer larger than or equal to n.

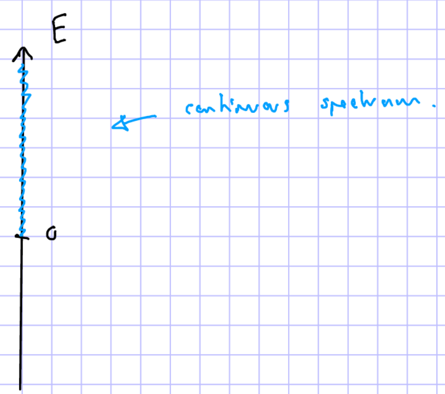
* if $n \in \mathbb{Z}$ then $\lceil n \rceil = \lfloor n \rfloor = n$ $\Rightarrow$ $\mu = 0$ is naively a bound state solution!

  But $\mu = 0$ is not a square integrable solution i.e. cannot be a bound state

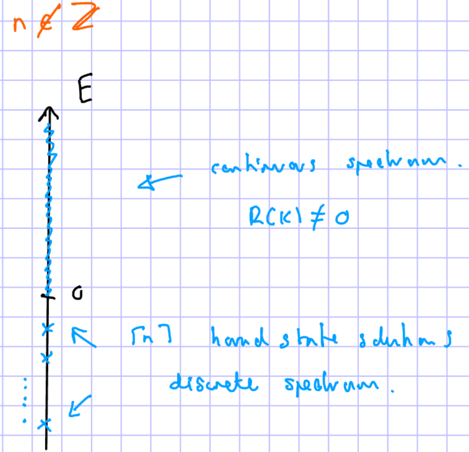
  We regard $\mu = 0$ as part of the continuous spectrum!

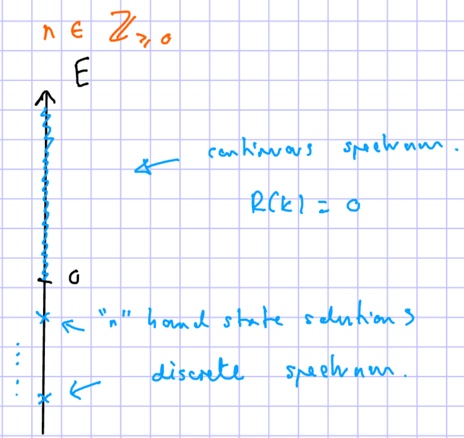
Summary

* $a = n(n+1) < 0$



* $a = n(n+1) > 0$ take $n \geq 0$





II.4 Scattering data for general potentials

Given localized initial data $u(x,0)$ for KdV, the T.I.S.E. with $V(x) = -u(x,0)$ has two types of bounded solution :

(a) Continuous spectrum ($E = k^2 > 0$ and $k > 0$)

$$\psi(x) \approx \begin{cases} e^{ikx} + R(k)\, e^{-ikx} & x \to -\infty \\ T(k)\, e^{ikx} & x \to \infty \end{cases}$$

(b) discrete spectrum ($E_n = -\mu_n^2$ and $\mu_n > 0$, $n = 1, \dots, N$)

$$\psi_n(x) \approx \begin{cases} c_n\, e^{\mu_n x} & x \to -\infty \\ d_n\, e^{-\mu_n x} & x \to +\infty \end{cases}$$

Normalization so far : $d_n = 1$

Normalization for the scattering data : choose d_n such that $(\psi_n, \psi_n) = 1$!

"scattering data" $\left\{ R(k),\ \{ \mu_n, c_n \}_{n=1}^{N} \right\}$

Simple example : $V(x) = a\,\delta(x)$

in this case $R(k) = \dfrac{a}{2ik - a}$

* For $a > 0$, thats all

* For $a < 0$ we have a bound state solution $\mu_1 = -a/2$:

$$\psi_1(x) = \begin{cases} c_1\, e^{\mu_1 x} & x < 0 \\ c_1\, e^{-\mu_1 x} & x > 0 \end{cases}$$

According to our definition of scattering data , c_1 is chosen so that $(\psi_1, \psi_1) = 1$.

$$(\psi_1, \psi_1) = \int_{-\infty}^{\infty} |\psi_1|^2\, dx$$

$$= \int_{-\infty}^{0} c_1^2\, e^{2\mu_1 x}\, dx + \int_{0}^{\infty} c_1^2\, e^{-2\mu_1 x}\, dx$$

$$= c_1^2 \underbrace{\left[\frac{1}{2\mu_1} e^{2\mu_1 x} \right]_{-\infty}^{0}}_{1/2\mu_1} + c_1^2 \underbrace{\left[-\frac{1}{2\mu_1} e^{-2\mu_1 x} \right]_{0}^{\infty}}_{1/2\mu_1}$$

$$= \frac{c_1^2}{\mu_1}$$

$$\overset{!}{=} 1$$

$$\Rightarrow \quad c_1 = \sqrt{\mu_1} = \sqrt{-a/2}$$

Summary of scattering data for $V(x) = a\,\delta(x)$

$$\left\{ R(k) = \frac{a}{2ik - a} \right\} \qquad \text{if } a > 0$$

$$\left\{ R(k) = \frac{a}{2ik - a} \,, \left\{ \mu_1 = -a/2 \,, \; c_1 = \sqrt{-a/2} \right\} \right\} \qquad \text{if } a < 0$$