Background notes on Fourier transforms

Please read the following notes - they will help with the exercises for section 7 on the problem sheet. We will begin with a quick and dirty derivation of Fourier transforms, starting with the Fourier series for a periodic function $f(x)$ defined on the interval $[-L/2,\, L/2]$, as seen in AMV:

$$\begin{aligned} f(x) &= \sum_{n=-\infty}^{+\infty} F_n \; e^{2\pi i n x/L} \\ F_n &= \frac{1}{L}\int_{-L/2}^{+L/2} dx \;\, f(x)\, e^{-2\pi i n x/L} \; . \end{aligned} \tag{1}$$

If we wish to consider a function that is not periodic we can take $L \to \infty$. In this limit the discrete variable n is replaced by a continuous variable k and the summation becomes an integral, with the correspondence

$$\begin{aligned} \frac{n}{L} &\;\to\; \frac{k}{2\pi} \\ L\, F_n &\;\to\; \widehat{f}(k) \\ \frac{1}{L}\sum_{n=-\infty}^{+\infty} &\;\to\; \int_{-\infty}^{+\infty} \frac{dk}{2\pi} \; . \end{aligned} \tag{2}$$

Substituting these definitions in (1) we get a pair of equations, giving the (inverse and direct) Fourier transforms

$$\begin{aligned} f(x) &= \int_{-\infty}^{+\infty} \frac{dk}{2\pi} \;\, \widehat{f}(k)\, e^{ikx} \\ \widehat{f}(k) &= \int_{-\infty}^{+\infty} dx \;\, f(x)\, e^{-ikx} \; . \end{aligned} \tag{3}$$

Note there's a factor of $1/(2\pi)$ in the first equation but not the second. Sometimes (including in last year's version of this course) an alternative more-symmetrical version is used, with a factor of $1/\sqrt{2\pi}$ in both equations instead of $1/(2\pi)$ in just one.

Connection with the Dirac delta function

Applying the previous formulae for the direct and inverse Fourier transform of a function we have

$$\begin{aligned} f(x) &= \int_{-\infty}^{+\infty} \frac{dk}{2\pi} \widehat{f}(k)\, e^{ikx} \\ &= \int_{-\infty}^{+\infty} \frac{dk}{2\pi}\, e^{ikx} \int_{-\infty}^{+\infty} dx'\; f(x')\, e^{-ikx'} \\ &= \iint_{-\infty}^{+\infty} \frac{dk\, dx'}{2\pi}\; f(x')\, e^{ik(x-x')} \\ &= \int_{-\infty}^{+\infty} dx'\; f(x')\, \delta(x-x')\;, \end{aligned} \tag{4}$$

where in the last step we bravely assumed it was OK to swap the k and x' integrals, and set

$$\delta(x-x') = \frac{1}{2\pi} \int_{-\infty}^{+\infty} dk\, e^{ik(x-x')}\;. \tag{5}$$

The 'function' $\delta(x)$ that we've just defined has the property that

$$\int_{-\infty}^{+\infty} dx'\; f(x')\, \delta(x-x') = f(x) \tag{6}$$

under integration – it's the Dirac delta function that was introduced in AMV last year. Loosely it can be thought of as a function which is everywhere 0 except where its argument vanishes, where it is infinite, and whose integral is 1. (Technically $\delta(x)$ is not a function, but rather a distribution.) Using the definition (5) and changing the sign of the integration variable, it is immediate to see that $\delta(x) = \delta(-x)$. Similar representations for the derivatives of the delta function can be found by differentiating (5), or using (6) and integrating by parts:

$$\begin{gathered} \frac{d^n}{dx^n}\delta(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} dk\, (ik)^n e^{ikx} \\ \int_{-\infty}^{+\infty} dx\; f(x)\, \frac{d^n}{dx^n}\delta(x-x_0) = (-1)^n \int_{-\infty}^{+\infty} dx\; \delta(x-x_0)\, \frac{d^n}{dx^n} f(x) = (-1)^n \frac{d^n f}{dx^n}(x_0)\;. \end{gathered} \tag{7}$$

Multidimensional version

The generalization to more dimensions is simple. Take a function $f(x_1, \ldots, x_n)$. For each dimension we can apply the FT separately defining a $k_{i=1\ldots n}$ for each transform. So we end up at

$$f(x_1, \ldots, x_n) = \int_{-\infty}^{+\infty} \frac{dk_1}{2\pi} \cdots \int_{-\infty}^{+\infty} \frac{dk_n}{2\pi}\; \widehat{f}(k_1, \ldots, k_n)\, e^{ik_1x_1 + \cdots + ik_nx_n}\;. \tag{8}$$

Gathering everything into a vector notation $\mathbf{x} = (x_1, \dots x_n)$ and $\mathbf{k} = (k_1, \dots k_n)$ we have

$$
\begin{aligned}
f(\mathbf{x}) &= \int \frac{d^n\mathbf{k}}{(2\pi)^n}\ \widehat{f}(\mathbf{k})\, e^{i\mathbf{k}\cdot\mathbf{x}} \\
\widehat{f}(\mathbf{k}) &= \int d^n\mathbf{x}\ f(\mathbf{x})\, e^{-i\mathbf{k}\cdot\mathbf{x}}
\end{aligned} \tag{9}
$$

where all the integrals are understood to be over $\mathbb{R}^n$.

Fourier transform of derivatives

Consider

$$
f(x) = \int_{-\infty}^{+\infty} \frac{dk}{2\pi}\ \widehat{f}(k)\ e^{ikx}\ . \tag{10}
$$

Taking the derivative of this equation with respect to x we have

$$
\begin{aligned}
f'(x) &= \int_{-\infty}^{+\infty} \frac{dk}{2\pi}\ \widehat{f}(k)\ \frac{d}{dx} e^{ikx} \\
&= \int_{-\infty}^{+\infty} \frac{dk}{2\pi}\ ik\widehat{f}(k)\ e^{ikx}\ .
\end{aligned} \tag{11}
$$

So $f'(x)$ has Fourier transform $ik\widehat{f}(k)$. Continuing in the same way,

$$
\frac{d^n}{dx^n} f(x) = \int_{-\infty}^{+\infty} \frac{dk}{2\pi}\ (ik)^n \widehat{f}(k)\ e^{ikx}\ . \tag{12}
$$

so $f^{(n)}(x)$ has Fourier transform $(ik)^n\widehat{f}(k)$. This makes for a very useful tool in solving differential equations. The example in lectures was the Klein-Gordon wave equation

$$
\frac{\partial^2}{\partial x^2} u - u = \frac{\partial^2}{\partial t^2} u\ . \tag{13}
$$

Assume the initial condition $u(x, 0) = \alpha(x)$ and $\dot{u}(x, 0) = \beta(x)$. We can always write this equation in terms of the Fourier transforms at some time t and equate those instead. We find

$$
-(k^2 + 1)\, \widehat{u}(k, t) = \frac{\partial^2}{\partial t^2} \widehat{u}(k, t)\ , \tag{14}
$$

This equation is easily solved for $\widehat{u}(k, t)$, because it is an ODE for any fixed k. The general solution is

$$
\widehat{u}(k, t) = A(k) e^{i\omega(k)t} + B(k) e^{-i\omega(k)t}\ , \tag{15}
$$

where

$$
\omega(k) = \sqrt{k^2 + 1} \tag{16}
$$

and where $A(k)$ and $B(k)$ are constants of integration. They can be found from the initial values of u and u_t at $t = 0$:

$$\begin{aligned}
\alpha(x) &= \int_{-\infty}^{+\infty} \frac{dk}{2\pi} \ (A(k) + B(k)) \ e^{ikx} \\
\beta(x) &= \int_{-\infty}^{+\infty} \frac{dk}{2\pi} \ i\omega(k)(A(k) - B(k)) \ e^{ikx} \ .
\end{aligned} \tag{17}$$

So we need to take the inverse FT to get them

$$\begin{aligned}
A(k) + B(k) &= \int_{-\infty}^{+\infty} dx \ \alpha(x) \ e^{-ikx} \\
A(k) - B(k) &= \frac{1}{i\omega(k)} \int_{-\infty}^{+\infty} dx \ \beta(x) \ e^{-ikx} \ ,
\end{aligned} \tag{18}$$

which can be solved for $A(k)$ and $B(k)$. Our final expression for the solution is then

$$u(x,t) = \int_{-\infty}^{+\infty} \frac{dk}{2\pi} \ \left[A(k)e^{i(kx+\omega(k)t)} + B(k)e^{i(kx-\omega(k)t)}\right] \tag{19}$$

with $A(k)$ and $B(k)$ given by solving the previous pair of equations.