Chapter 10

Interlude: the KdV hierarchy and conservation laws

10.1 Deriving the KdV equation (and generalising it)

It's natural to ask whether there are any other evolution equations for $u(x,t)$ such that the eigenvalues of $L = D^2 + u(x,t)$ are constant. In more fancy language, we're looking for equations such that the $L(u)$'s at different times are *isospectral*; these are called an *isospectral flows.*

The answer is yes, there are more such equations, and the Lax pair idea allows us to find them.

Key point: the proof in section 9.1 only used the fact that, when u evolves by KdV, we have $L_t = [B, L]$ – no other details of B were needed, so some other $B(u)$ should work just as well. However, $B(u)$ is not completely arbitrary: since $L_t = u_t$, and is a multiplicative operator, $[B, L]$ must also be multiplicative. This means all the D's must cancel out when computing the commutator. If they *do* cancel, what's left in $[B, L]$ will be a polynomial in u, u_x, u_{xx} etc, and setting this equal to u_t will give us the desired evolution equation. We can see this in action via some examples.

Example (i)

Try $B(u) = \alpha(x)$ for some function $\alpha(x)$. Then, leaving it as an exercise to fill in the missing steps,

$$[L, B] = [D^2 + u, \alpha] = \cdots = \alpha_{xx} + 2\alpha_x D\,.$$

For this to be multiplicative, the "D" bit has to be zero, which requires $\alpha_x = 0$. Hence α is constant, $\alpha_{xx} = 0$, and $[L, B] = 0$. Hence the equation we get is

$$u_t = 0\,.$$

This clearly *is* an answer to the question of what evolution equation for u will leave the spectrum of $L(u)$ unchanged, but it's not a very interesting one!

Example (ii)

Next up, let's try $B(u) = \alpha(x)D + \beta(x)$. Then

$$\begin{aligned} &\text{(a)} \quad [D^2, \alpha D + \beta] = \cdots = 2\alpha_x D^2 + (\alpha_{xx} + 2\beta_x)D + \beta_{xx}\,; \\ &\text{(b)} \quad [u, \alpha D + \beta] = \cdots = -\alpha u_x\,, \end{aligned}$$

and (a) + (b) $\Rightarrow$

$$[L, B] = [D^2 + u, \alpha D + \beta] = 2\alpha_x D^2 + (\alpha_{xx} + 2\beta_x)D + \beta_{xx} - \alpha u_x\,.$$

For this to be multiplicative, the coefficients of D^2 and D must be zero.

$$\begin{aligned} (\;)D^2 &= 0 \quad \Rightarrow \quad 2\alpha_x = 0\,; \\ (\;)D\; &= 0 \quad \Rightarrow \quad 2\beta_x = 0\,. \end{aligned}$$

Hence α and β are both constants, and $[L, B] = -\alpha u_x$. The evolution equation is thus

$$0 = L_t + [L, B] = u_t - \alpha u_x\,.$$

Sadly this is also a bit trivial: it's the advection equation, and the solution for initial data $u(x, 0) = u_0(x)$ is

$$u(x, t) = u_0(x + \alpha t)\,.$$

This just translates the initial data sideways with velocity $-\alpha$, so the shape of the function is unchanged and it's easy to see that the same is true of the spectrum (exercise!).

Example (iii)

Finally, we try $B(u) = \alpha(x)D^2 + \beta(x)D + \gamma(x)$. Then

(a) $[D^2, \alpha(x)D^2 + \beta(x)D + \gamma(x)] = \cdots = 2\alpha_x D^4 + \alpha_{xx}D^3 + 2\beta_x D^2 + (\beta_{xx} + 2\gamma_x)D + \gamma_{xx}\,;$

(b) $[u, \alpha(x)D^2 + \beta(x)D + \gamma(x)] = \cdots = -\alpha(u_{xxx} + 3u_{xx}D + 3u_x D^2) + \beta u_x\,,$

and so

$$\begin{aligned} [L, B] &= \text{(a)} + \text{(b)} \\ &= 2\alpha_x D^4 \\ &\quad + \alpha_{xx} D^3 \\ &\quad + (2\beta_x - 3\alpha u_x)D^2 \\ &\quad + (\beta_{xx} + 2\gamma_x u - 3\alpha u_{xx})D \\ &\quad + \gamma_{xx} - \alpha u_{xxx} - \beta u_x\,. \end{aligned}$$

Equating the coefficients of the powers of D to zero:

$$
\begin{aligned}
&(\,)D^4 = 0 \;\;\Rightarrow\;\; 2\alpha_x = 0 \;\Rightarrow\; \alpha = \text{constant}\,;\\
&(\,)D^3 = 0 \quad \text{(now automatic)}\,;\\
&(\,)D^2 = 0 \;\;\Rightarrow\; 2\beta_x - 3\alpha u_x = 0 \;\Rightarrow\; (\beta - \tfrac{3}{2}\alpha u)_x = 0 \;\Rightarrow\; \beta = \tfrac{3}{2}\alpha u + k_1\,;\\
&(\,)D \;\,= 0 \;\;\Rightarrow\; \ldots \;\Rightarrow\; \gamma = \tfrac{3}{2}\alpha u_x + k_2
\end{aligned}
$$

where k_1 and k_2 are constants. The remaining (multiplicative) bit of $[L, B]$ is then $-\frac{1}{4}\alpha u_{xxx} - \frac{3}{2}\alpha u u_x - k_1 u_x$ and so in this case

$$L_t + [L, B] = 0 \quad \Leftrightarrow u_t - \tfrac{3}{2}\alpha u u_x - \tfrac{1}{4}\alpha u_{xxx} - k_1 u_x = 0\,.$$

For $\alpha = 0$ and $k_1 = 0$ this is the KdV equation.

10.2 Hints for the general case

To go further, introduce some new technology:

(i) (Hermitian) inner product

For two functions $\phi(x)$ and $\chi(x)$, define

$$(\phi, \chi) = \int_{-\infty}^{+\infty} \phi^*(x)\chi(x)\, dx\,. \tag{10.1}$$

(The complex conjugation on the first term ensures $(\phi, \phi) > 0$ for $\phi \neq 0$ even when ϕ is complex.)

In this notation, the key property of $L = D^2 + u$ used in the Lax proof was that

$$(\phi, L\chi) = (L\phi, \chi) \tag{10.2}$$

for all ϕ and χ.

(ii) The adjoint of an operator

If M is a differential operator, define $M^\dagger$ ("M dagger") to be the operator such that

$$(\phi, M\chi) = (M^\dagger\phi, \chi) \tag{10.3}$$

for all ϕ and χ. $M^\dagger$ is called the *adjoint* of M; it's a bit like a matrix transpose and, like the matrix transpose, satisfies

$$(M^\dagger)^\dagger = M\,, \quad (MN)^\dagger = N^\dagger M^\dagger$$

(exercise: check!). The key property of L was

$$L^\dagger = L \tag{10.4}$$

and such operators are called *self-adjoint* (or *symmetric*). Other important operators have

$$M^\dagger = -M \tag{10.5}$$

and are called *antisymmetric*, or *skew*.

Now if M is just multiplication by a (real) function, then $M^\dagger = M$. (Exercise: why?) This must be true of $[L, B]$ as it is supposed to be a (real) multiplicative operator, so B must be such that that $[L, B]^\dagger = [L, B]$.

What can we deduce about B from this?

We have $[L, B] = LB - BL$, so $[L, B] = [B, L]^\dagger$ implies

$$\begin{aligned} LB - BL &= (LB - BL)^\dagger \\ &= B^\dagger L^\dagger - L^\dagger B^\dagger \\ &= B^\dagger L - LB^\dagger \quad \text{(since } L \text{ is self-adjoint)} \end{aligned}$$

which implies

$$L(B + B^\dagger) - (B + B^\dagger)L$$

or

$$[L, (B + B^\dagger)] = 0\,.$$

Otherwise stated, the *symmetric part* of B must commute with L. (As with matrices, any B can be written as

$$B = \frac{1}{2}(B + B^\dagger) + \frac{1}{2}(B - B^\dagger)$$

where the first term is called the symmetric part of B, and the second the antisymmetric part.)

Since it's only the bit of B which doesn't commute with L that makes a difference to the equation $L_t + [L, B] = 0$, this means that <u>B can be assumed to be antisymmetric</u>.

How to write such a B?

Instead of writing a general B as

$$B = \sum_0^m \alpha_j(x) D^j\,,$$

we'll choose a different basis by writing

$$B = \sum_{0}^{m} (\beta_j(x) D^j + D^j \beta_j(x))\,.$$

(It can be checked that this is always possible.)

Now if $\beta(x)$ is real, $(\beta(x))^\dagger = \beta(x)$, and also $D^\dagger = -D$ (this is proved by integration by parts), which implies

$$\begin{aligned}(D^{2j})^\dagger &= D^{2j} \qquad \text{(self-adjoint)}\\ (D^{2j-1})^\dagger &= -D^{2j-1} \qquad \text{(skew)}\end{aligned}$$

and replacing B by its antisymmetric part, $\frac{1}{2}(B - B^\dagger)$, it becomes

$$B = \sum_{0<2j-1\leqslant m} (\beta_{2j-1}(x) D^{2j-1} + D^{2j-1}\beta_{2j-1}(x))\,. \tag{10.6}$$

It can also be checked that $[L, B]$ being multiplicative forces the coefficient of the leading term in D to be a constant, so the general guess is

$$B_n(u) = D^{2n-3} + \sum_{j=1}^{n-2} (\beta_j(x) D^{2j-1} + D^{2j-1}\beta_j(x))\,. \tag{10.7}$$

Notes:

• the degree $2n - 3$ of the leading term was picked for later convenience;

• the β_j's have been relabelled going from (10.6) and (10.7);

• setting the coefficient of the leading term to 1 in (10.7) does not lose any generality, since an overall rescaling of $B(u)$ can be "undone" in $L_t + [L, B] = 0$ by rescaling time.

There's now no alternative but to calculate. When the dust settles, $K_n(u) \equiv [B_n, L]$ will be a polynomial in u, u_x, u_{xx} etc, and setting $L_t + [L, B_n] = 0$, that is $u_t = K_n(u)$, will give a KdV-like equation with x derivatives up to order $2n - 3$.

The first few cases:

$$\begin{aligned}
n = 1 :&\quad u_t = 0\\
n = 2 :&\quad u_t + u_x = 0\\
n = 3 :&\quad u_t + 6uu_x + u_{xxx} = 0\\
n = 4 :&\quad u_t + 30u^2u_x + 20u_xu_{xx} + 10uu_{xxx} + u_{xxxxx} = 0
\end{aligned} \tag{10.8}$$

These are the first equations of the *KdV hierarchy*, and in each case, they evolve $u(x, t)$ forward in time in such a way as to leave the spectrum of $L(u) = D^2 + u$ unchanged.

10.3 Connection with conservation laws

Recall from last term that the KdV equation has an infinite sequence of conserved charges:

$$Q_n = \int_{-\infty}^{+\infty} T_n \, dx$$

where the conservation of Q_n, $\frac{dQ_n}{dt} = 0$, is proved by showing that $\frac{\partial T_n}{\partial t} + \frac{\partial X_n}{\partial x} = 0$ when the KdV equation holds, for some X_n with $[X_n]_{-\infty}^{\infty} = 0$. Normalising the T_n's as $T_n = u^n + \ldots$, the first few examples are

$$\begin{aligned} T_1 &= u \\ T_2 &= u^2 \\ T_3 &= u^3 - \tfrac{1}{2}u_x^2 \\ T_4 &= u^4 - 2uu_x^2 + \tfrac{1}{5}u_{xx}^2 \\ T_5 &= u^5 - \tfrac{105}{21}u^2u_x^2 + uu_{xx}^2 - \tfrac{1}{21}u_{xxx}^2 \end{aligned} \tag{10.9}$$

So we now have <u>two</u> infinite sequences:

- For the KdV equation itself, the sequence T_1, T_2, T_3,…
- Going beyond KdV, an infinite sequence K_1, K_2, K_3,… of polynomials in u and its x derivatives such that setting $u_t = K_n(u)$ leaves the eigenvalues of $D^2 + u(x,t)$ constant.

How do these two sequences tie together, if at all?

The most boring possibility: each evolution equation $u_t = K_n(u)$ has its "own" set of T_n's, conserved densities for that equation alone. In fact the answer, found by Gardner, is more clever. To explain it, a new concept is needed…

The functional derivative

(Also known as the variational, or Fréchet, derivative.)

Suppose f is some function of u and its x derivatives. Then

$$\boxed{F[u] = \int_{-\infty}^{+\infty} dx f(u, u_x, u_{xx} \ldots)} \tag{10.10}$$

is an example of a *functional* of u: it takes a function $u(x)$ and yields a number $F[u]$. In practice u might also depend on the time t, in which case the formula should be taken at fixed t, which is <u>not</u> integrated over. Since t is a spectator for most of the following considerations, for now we won't write it explicitly in formulae.

Now consider a small variation $\delta u(x)$ of $u(x)$, so that $u(x) \to u(x) + \delta u(x)$, with $\delta u(x) \to 0$ as $x \to \pm\infty$.

This changes $F[u]$ to

$$\begin{aligned}
F[u+\delta u] &= \int_{-\infty}^{+\infty} dx\, f(u+\delta u, (u+\delta u)_x, (u+\delta u)_{xx}, \dots) \\
&= \int_{-\infty}^{+\infty} dx\, f(u+\delta u, u_x + \delta u_x, u_{xx} + \delta u_{xx}, \dots) \\
&= \int_{-\infty}^{+\infty} dx \left(f(u, u_x, u_{xx}, \dots) + \frac{\partial f}{\partial u}\,\delta u + \frac{\partial f}{\partial u_x}\,\delta u_x + \frac{\partial f}{\partial u_{xx}}\,\delta u_{xx} + \dots \right) \\
&\qquad \text{(Taylor expanding)} \\
&= F[u] + \int_{-\infty}^{+\infty} dx \left(\frac{\partial f}{\partial u}\,\delta u + \frac{\partial f}{\partial u_x}\,\delta u_x + \frac{\partial f}{\partial u_{xx}}\,\delta u_{xx} + \dots \right) + O(\delta u^2) \\
&= F[u] + \int_{-\infty}^{+\infty} dx \left(\frac{\partial f}{\partial u} - \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial u_x}\right) + \frac{\partial^2}{\partial x^2}\left(\frac{\partial f}{\partial u_{xx}}\right) + \dots \right) \delta u + O(\delta u^2) \\
&\qquad \text{(integrating by parts)}
\end{aligned}$$

and the term multiplying $\delta u(x)$ in the last line is called the *functional derivative* of $F[u]$, written as $\frac{\delta F[u]}{\delta u}$. More precisely, $\frac{\delta F[u]}{\delta u}$ is defined by

$$\boxed{F[u+\delta u] = F[u] + \int_{-\infty}^{+\infty} dx\, \frac{\delta F[u]}{\delta u}\,\delta u + O(\delta u^2)} \tag{10.11}$$

which is like $f(x+\delta x) = f(x) + \frac{df}{dx}\,\delta x + O(\delta x^2)$ for ordinary functions.

For functionals defined as in (10.10) the calculation just completed shows that

$$\boxed{\frac{\delta F[u]}{\delta u} = \frac{\partial f}{\partial u} - \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial u_x}\right) + \frac{\partial^2}{\partial x^2}\left(\frac{\partial f}{\partial u_{xx}}\right) + \dots} \tag{10.12}$$

Examples

(a) $f = u \quad \Rightarrow \quad \frac{\delta F[u]}{\delta u} = 1$

(b) $f = u^3 \quad \Rightarrow \quad \frac{\delta F[u]}{\delta u} = 3u^2$

(c) $f = u_x^2 \quad \Rightarrow \quad \frac{\delta F[u]}{\delta u} = -2u_{xx}$

(Exercise: check these results.)

The conserved quantities $Q_n = \int T_n \, dx$ are examples of functionals of u, and so we can also calculate their functional derivatives:

(a) $\frac{\delta Q_1}{\delta u} = \frac{\delta u}{\delta u} = 1$

(b) $\frac{\delta Q_2}{\delta u} = \frac{\delta u^2}{\delta u} = 2u$

(c) $\frac{\delta Q_3}{\delta u} = \frac{\delta}{\delta u}\left(u^3 - \frac{1}{2}u_x^2\right) = 3u^2 + u_{xx}$

Taking $\frac{\partial}{\partial x}$ of each of these,

$$\frac{\partial \textbf{(a)}}{\partial x} = 0 \,, \qquad \frac{\partial \textbf{(b)}}{\partial x} = 2u_x \,, \qquad \frac{\partial \textbf{(c)}}{\partial x} = 6uu_x + u_{xxx}$$

and these match, up to an overall scale, the first three equations of the KdV hierarchy:

$$u_t = 0 \,, \qquad u_t = -u_x \,, \qquad u_t = -6uu_x - u_{xxx} \,.$$

The normalisations of the charges, or else the scale of t, can be adjusted to make these matches precise. They are the first three examples of Gardner's general result:

$$\boxed{u_t = \frac{\partial}{\partial x}\left(\frac{\delta Q_n}{\delta u}\right)} \quad \longleftrightarrow \quad \boxed{u_t = K_n(u)}$$

connecting the n^{th} KdV conservation law with the n^{th} equation of the KdV hierarchy. Thus the two sequences are the same!

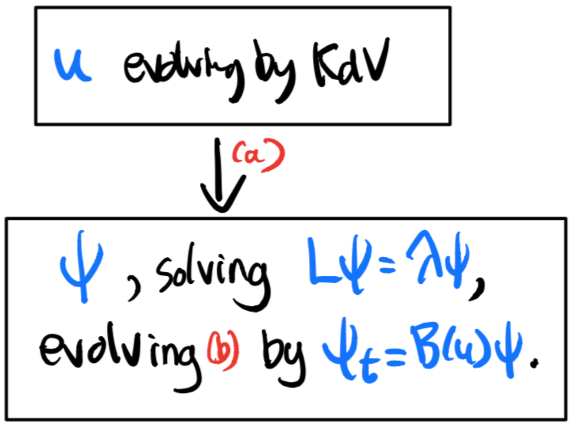
Furthermore:

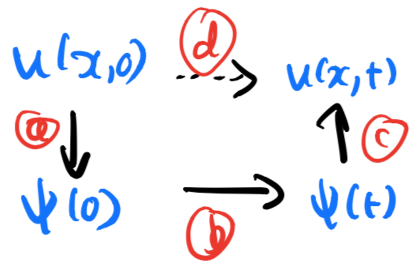
• If $u_m(x,t)$ evolves by the m^{th} KdV equation, all the T_n's are conserved densities for it.

• Imagine we have one "time" for each equation in the hierarchy, so that instead of $u_m(x,t)$ with $\frac{\partial}{\partial t}u_m = K_m(u)$ we have $u(x, t_t, t_2, t_3 \dots)$ with $\frac{\partial}{\partial t_m}u_m = K_m(u)$. Then if we evolve (or 'flow') $u(x, t_t, t_2, t_3 \dots)$ for a while in t_i, then for a while in t_j, we end up with the <u>same</u> function of x as if we'd evolved in t_j first followed by t_i. This is the idea of *commuting flows*: it's very important in "modern" soliton theory.

Now back to KdV…

The story so far





The 'big idea' was to encode $u(x,0)$ as ψ, [step (a)], then evolve it forward in time by $\psi_t = B(u)\psi$, [step (b)], then decode $u(x,t)$ [step (c)] a time t later.

One problem: to evolve ψ, we seem to need to know how $u(x,t)$ evolves, since B depends on $u(x,t)$ (recall, for KdV, $B(u) = -4D^3 - 6uD - 3u_x$).

This looks to be fatal…

But, just how much of $\psi(x,t)$ do we really need to reconstruct $u(x,t)$? The method might be saved if we only needed to know $\psi(x,t)$ at $|x| \to \infty$, since in this limit $u \to 0$ and $B(u) \to -4D^3$, which *is* independent of u.

If we could get away with only this, the idea would be saved. In fact GGKM already knew this to be true, which is perhaps why they persisted. To understand how it goes, some more information on the solutions to problems like $(\frac{d^2}{dx^2} + u(x))\psi = \lambda\psi$ is required, and this is the subject of the next chapter.