Chapter 11

The basics of scattering theory

Aim: to analyse the possible solutions to $L\psi = \lambda\psi$, that is

$$\left(\frac{d^2}{dx^2} + u(x)\right)\psi = \lambda\psi \tag{11.1}$$

with $\psi(x)$ bounded for all x (which restricts the possible λ's). Note this relaxes slightly the previous requirement that $\int_{-\infty}^{+\infty}|\psi|^2 dx < \infty$, ie $\psi \in L^2(\mathbb{R})$.

Note, in this chapter the KdV time t just appears as a parameter in $u(x,t)$ and stays fixed (and will be dropped from the notation).

11.1 Overview: the physical interpretation

FACT: the equation

$$\boxed{i\frac{\partial}{\partial\tau}\Psi(x,\tau) = \left(-\frac{\partial^2}{\partial x^2} + V(x)\right)\Psi(x,\tau)} \tag{11.2}$$

(the *time-dependent Schrödinger equation*) describes a particle (of mass $\frac{1}{2}$) moving on a line in a potential $V(x)$ in quantum mechanics. The *wavefunction* Ψ tells you where the particle is likely to be: $|\Psi(x,\tau)|^2$ is the probability to find it in the interval $[x, x+dx]$ at time τ. (Note, this time τ is not the same as the KdV time t.)

To solve (11.2), separate variables

$$\Psi(x,\tau) = \psi(x)\phi(\tau) \tag{11.3}$$

and substitute in and rearrange to find

$$i\frac{\dot{\phi}}{\phi} = \frac{-\psi'' + V\psi}{\psi} = \text{constant} \equiv k^2 \tag{11.4}$$

where the dot denotes $\frac{d}{d\tau}$, the dash $\frac{d}{dx}$, and the constant was called k^2 for later convenience. Solving first the equation for ϕ,

$$\dot{\phi} = -ik^2\phi \quad \Rightarrow \quad \boxed{\phi(\tau) = e^{-ik^2\tau}} \tag{11.5}$$

while $\psi(x)$ satisfies

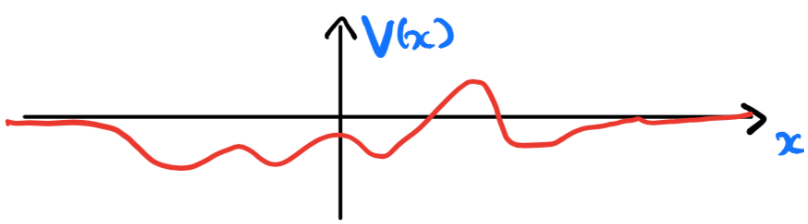
$$\boxed{\left(-\frac{d^2}{dx^2} + V(x)\right)\psi(x) = k^2\psi(x)} \tag{11.6}$$

(the *time independent Schrödinger equation*) which is the same as (9.1) with the identifications

$$u = -V \quad ; \qquad \lambda = -k^2\,. \tag{11.7}$$

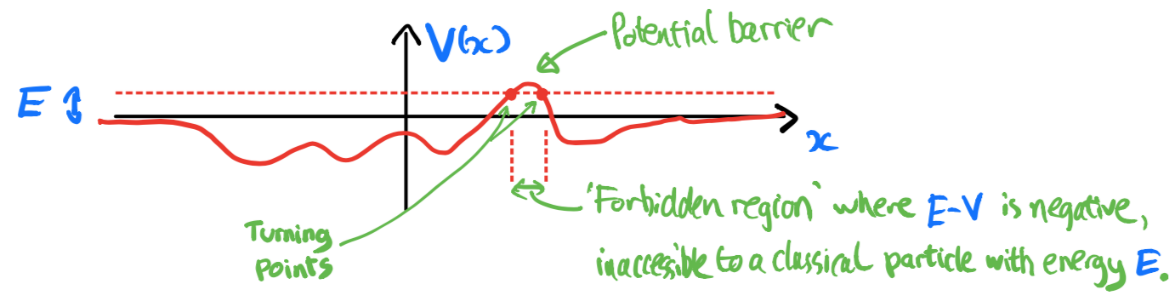
In quantum mechanics, (11.6) describes a particle with energy $E = k^2 = -\lambda$ moving in the potential $V(x) = -u(x)$.

With the link to KdV in mind, we'll consider potentials which tend to zero (sufficiently fast) as $x \to \pm\infty$:



In classical mechanics, a particle with total (kinetic plus potential) energy $E = T + V$ is localised, and bounces off the potential at the "turning points" x_* where $V(x_*) = E$.

By contrast, in quantum mechanics, there's a non-zero chance to find the particle anywhere (if V is finite), and it can 'tunnel' through potential barriers.



The *scattering data* will be encoded in the asymptotics (limiting behaviour) of $\psi(x)$ as $x \to \pm\infty$.

Since $V(x) \to 0$ as $x \to \pm\infty$, (11.6) in these regions reduces to

$$\boxed{-\frac{d^2}{dx^2}\psi = k^2\psi} \tag{11.8}$$

with two independent solutions $e^{\pm ikx}$.

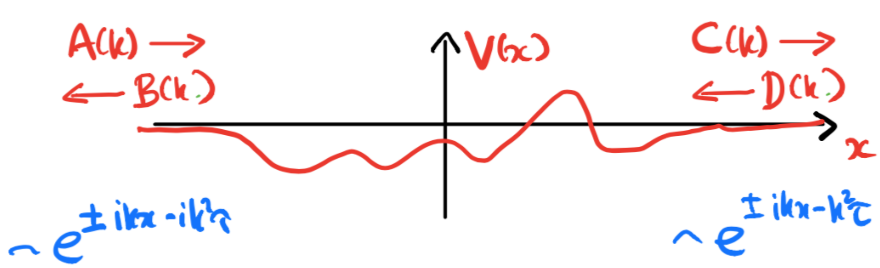
So the general solution with eigenvalue $E = k^2$ has the asymptotics

$$\begin{aligned} \psi(x) &\approx A(k)e^{ikx} + B(k)e^{-ikx} \qquad x \to -\infty \\ \psi(x) &\approx C(k)e^{ikx} + D(k)e^{-ikx} \qquad x \to +\infty \end{aligned} \tag{11.9}$$

and, restoring the τ-dependence,

$$\begin{aligned} \Psi(x,\tau) &\approx A(k)e^{ikx-ik^2\tau} + B(k)e^{-ikx-ik^2\tau} \qquad x \to -\infty \\ \Psi(x,\tau) &\approx C(k)e^{ikx-ik^2\tau} + D(k)e^{-ikx-ik^2\tau} \qquad x \to +\infty \end{aligned} \tag{11.10}$$

showing that for real $k > 0$ the 'A' and 'C' terms correspond to right-moving waves, while the 'B' and 'D' terms correspond to left-moving waves:



This solution will be bounded for any values for A, B, C and D if $E = k^2 > 0$.

As we'll see in examples, solving (11.6) in the middle region where $V(x) \neq 0$ interpolates between the two asymptotic regions and imposes two relations among A, B, C and D, leaving two undetermined coefficients, as expected for a 2^{nd} order ODE.

To fix these remaining coefficients, for $k^2 > 0$ we will impose

$$\boxed{A(k) = 1\,, \qquad D(k) = 0} \tag{11.11}$$

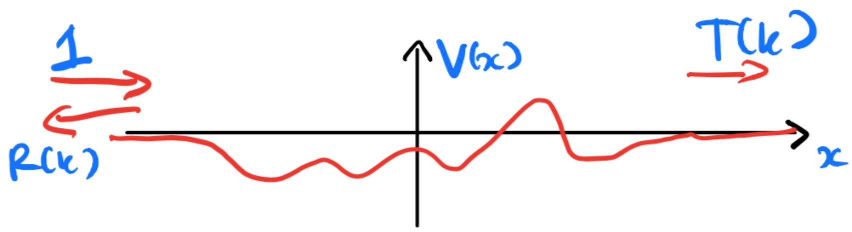
and write

$$\begin{aligned} B(k) &\equiv R(k) \qquad \text{(the } \textit{reflection coefficient}\text{)} \\ C(k) &\equiv T(k) \qquad \text{(the } \textit{transmission coefficient}\text{)} \end{aligned} \tag{11.12}$$

so that the resulting *scattering solution* has asymptotics

$$\begin{aligned} \psi(x) &\approx e^{ikx} + R(k)e^{-ikx} \qquad x \to -\infty \\ \psi(x) &\approx T(k)e^{ikx} \qquad x \to +\infty \end{aligned} \tag{11.13}$$

and represents a unit flux (since $A(k) = 1$) of incoming particles from the left, partially reflected from the potential and partially transmitted through it:



It can be shown (exercise) that

$$\boxed{|R(k)|^2 + |T(k)|^2 = 1} \tag{11.14}$$

meaning that with probability 1 the particle is either reflected or transmitted (conservation of probability).

Aside: the Wronskian

Results such as $|R(k)|^2 + |T(k)|^2 = 1$, proved in exercise 60, are proved using a gadget called the *Wronskian*. For two functions $f(x)$ and $g(x)$, their Wronskian is

$$\boxed{W[f,g](x) = f')x)g(x) - f(x)g'(x)\,.} \tag{11.15}$$

Two facts about W:

1) If f and g are linearly dependent, the $W[f,g]=0$ identically.

(It's easy to see that W is antisymmetric, and linear in each of its arguments. Then if, say, $f(x)=\alpha g(x)$ with α a constant, $W[f,g]=W[\alpha g,g]=\alpha W[g,g]=0$.)

2) The converse statement, that $W[f,g]=0$ implies that f and g are linearly dependent, is more tricky. The following is easily proved: if

a) $W[f,g](x)=0$ on some interval, and

b) one or other of f and g is nonzero on that interval,

then f and g are linearly dependent on that interval.

(Say it's g that is nonzero. Dividing $W[f,g](x)=0$ through by g^2 shows that $\frac{d}{dx}(f/g)=0$, so $f/g=$ constant, and f and g are linearly dependent.)

Notes:

• Some sort of extra condition such as b) *is* needed: consider, as suggested by Peano in 1889,

$$f(x)=x^2\,,\quad g(x)=x|x|=x^2\mathrm{sign}(x)\,.$$

Then f and g are not linearly dependent on $\mathbb{R}$, even though $W[f,g]=0$ everywhere. (Exercise: check this!)

• In fact, though it won't be proved here, the result that $W[f,g](x)=0$ everywhere implies f and g are linearly dependent *does* hold if both f and g are analytic. This is true of solutions to the ODEs we are dealing with here, so we will therefore assume that the converse statement to **1)** does hold in all cases we will need.

Now back to the time independent Schrödinger equation

$$\left(-\frac{d^2}{dx^2}+V(x)\right)\psi(x)=E\psi(x)=k^2\psi(x)\,.$$

So far we have looked at cases with $k^2=E>0$. For $k^2<0$, let $k=i\mu$ with $\mu>0$ real, so $E=-\mu^2$. Then the asymptotics of the general solution (11.9) become

$$\begin{aligned}\psi(x)&\approx a(\mu)e^{-\mu x}+b(\mu)e^{\mu x}\qquad x\to-\infty\\ \psi(x)&\approx c(\mu)e^{-\mu x}+d(\mu)e^{\mu x}\qquad x\to+\infty\end{aligned}\tag{11.16}$$

and it follows that

$$\boxed{\psi\text{ bounded}}\quad\Leftrightarrow\quad\boxed{a(\mu)=d(\mu)=0}\tag{11.17}$$

In such cases ψ is not only bounded, it also tends to zero at $\pm\infty$ and satisfies $\int|\psi|^2dx<0$.

Note that there might be no values for μ at which this happens. But if it does, the corresponding ψ is called a *bound state solution.*

Fact: Given a potential $V(x)$ tending to zero at $\pm\infty$, bound state solutions only exist for a finite (possibly empty) set of μ's:

$$\{\mu_k\}_{k=1}^{N} = \{\mu_1, \mu_2, \dots \mu_N\}, \qquad \mu_1 < \mu_2 < \dots \mu_N\,. \tag{11.18}$$

Summary

Bounded solutions to

$$\left(-\frac{d^2}{dx^2} + V(x)\right)\psi(x) = E\psi(x) = k^2\psi(x),$$

or equivalently $\left(\frac{d^2}{dx^2} + u(x)\right)\psi(x) = \lambda\psi(x)$ with $u(x) = -V(x)$ and $\lambda = -E$, come in two flavours when $V(x) \to 0$ as $x \to \pm\infty$:

1) $E = k^2 = -\lambda \in (0, +\infty)$: the "continuous spectrum", leading to *scattering solutions* which are bounded, and have oscillatory asymptotics;

2) $E = -\mu^2 = -\lambda \in \{-\mu_1^2, -\mu_2^2, \dots - \mu_N^2\}$: the "discrete spectrum", leading to *bound state solutions* which are square integrable (*i.e.* $\int_{-\infty}^{+\infty}|\psi(x)|^2 dx < \infty$), and have damped asymptotics.

(Note: for some rather-special, slowly-decaying potentials, at least in higher dimensions, there may also be some square integrable solutions with $k^2 > 0$. These so-called 'bound states in the continuum' (BICs) crop up in a number of physical applications, but won't be relevant for the current discussion.)

11.2 Examples

Example 1

$$V(x) = 0\,.$$

This was already done, essentially, when looking at the asymptotics for general V. We must solve $-\frac{d^2}{dx^2}\psi = k^2\psi$ for all $x \in \mathbb{R}$. There are two cases to consider.

(a) $k^2 > 0$.

The general solution, valid for all x, not just asymptotically, is

$$\psi(x) = Ae^{ikx} + Be^{-ikx}\,. \tag{11.19}$$

Restoring the τ dependence, it's a left or right moving wave, bounded for all real values of k.

Comparing with (11.9) shows that in this case $C(k) = A(k)$ and $D(k) = B(k)$. Imposing $A(k) = 1$ and $D(k) = 0$ then gives us the scattering solution:

$$\psi(x) = e^{ikx}\,. \tag{11.20}$$

from which it follows that

$$\boxed{R(k) = 0\,, \quad T(k) = 1} \tag{11.21}$$

If you think about it this should seem reasonable – with no potential, a particle incident from the left is transmitted through to the right with probability 1.

(b) $k^2 = -\mu^2 < 0$.

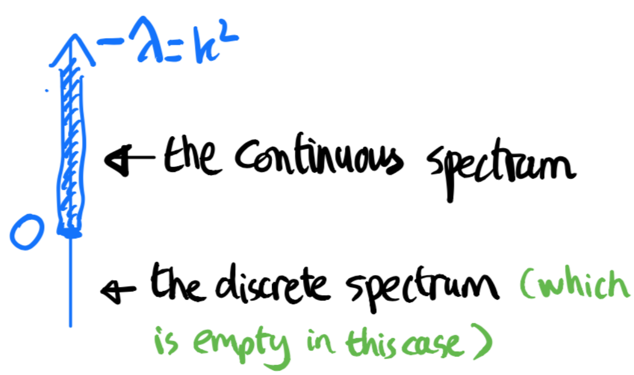
The general solution from part **(a)** turns into

$$\psi(x) = ae^{-\mu x} + be^{\mu x}\,. \tag{11.22}$$

and the only way to keep this bounded as $x \to \pm\infty$ is to set $a = b = 0$. Thus there are <u>no</u> bound state solutions for this problem.

Summary

For $u = 0$, the problem $L(u)\psi = \lambda\psi$, ψ bounded, has a 'scattering' solution for all real $\lambda < 0$, and no solutions for $\lambda > 0$:



Example 2

$$V(x) = a\delta(x)$$

where a is a real constant and $\delta(x)$ is the Dirac delta function. Recall that $\delta(x)$ can be viewed as the limit of a sequence (a 'delta sequence') of unit-area functions which are increasingly concentrated at the origin, so that for any (test) function $f(x)$,

$$\int_{-\infty}^{+\infty} \delta(x) f(x)\, dx = f(0)\,.$$

We seek a single solution $\psi(x)$, solving the equation in regions (1) and (2), and also consistent with the potential at $x = 0$.

(a) $k^2 > 0$.

In regions (1) and (2), $V(x) = 0$, so ψ satisfies $-\frac{d^2}{dx^2}\psi = k^2\psi$ and as in example 1, the solutions in the two regions are

$$\boxed{\begin{aligned} \psi_{(1)}(x) &= Ae^{ikx} + Be^{-ikx} \\ \psi_{(2)}(x) &= Ce^{ikx} + De^{-ikx} \end{aligned}} \tag{11.23}$$

To finish, we must match the two parts of the solution at $x = 0$, and this will determine the relation(s) between A, B, C and D.

• First, even for "funny" potentials like this one, $\psi(x)$ should be continuous at $x = 0$:

$$\boxed{[\,\psi(x)\,]_{0^-}^{0^+} = 0} \tag{11.24}$$

• But $\psi'(x)$ is forced by the equation to be discontinuous at $x = 0$. The equation is

$$-\psi''(x) + a\delta(x)\psi(x) = k^2\psi(x)\,. \tag{11.25}$$

Integrating from $x = -\epsilon$ to $x = +\epsilon$,

$$\int_{-\epsilon}^{+\epsilon} dx\,[-\psi''(x) + a\delta(x)\psi(x)] = k^2 \int_{-\epsilon}^{+\epsilon} dx\,\psi(x)$$

$$\Rightarrow \qquad -\,[\psi'(x)]_{-\epsilon}^{+\epsilon} + a\psi(0) = k^2 \int_{-\epsilon}^{+\epsilon} dx\,\psi(x)\,. \tag{11.26}$$

Provided that ψ is bounded (which it is), the RHS of this equation $\to 0$ as $\epsilon \to 0$, and taking this limit implies $-\,[\,\psi'(x)\,]_{0^-}^{0^+} + a\psi(0) = 0$, or

$$\boxed{[\,\psi'(x)\,]_{0^-}^{0^+} = a\psi(0)} \tag{11.27}$$

Applying the *matching conditions* (11.24) and (11.27) to (11.23) we have

$$A + B = C + D$$
$$ik(C - D) - ik(A - B) = a(A + B) = a(C + D)$$

which in turn implies

$$\begin{aligned} A + B &= C + D \\ A - B &= \big(1 - \frac{a}{ik}\big)C - \big(1 - \frac{a}{ik}\big)D \,. \end{aligned}$$

Adding and subtracting,

$$\begin{aligned} A &= \big(1 - \frac{a}{2ik}\big)C - \frac{a}{2ik}D \\ B &= \frac{a}{ik}C + \big(1 + \frac{a}{2ik}\big)D \,. \end{aligned} \tag{11.28}$$

Substituting into (11.23) gives the general solution, with, as expected, two undetermined constants.

To get to the *scattering solution*, set $D = 0$ and then divide through so that $A = 1$:

$$\psi(x) = \begin{cases} e^{ikx} + \frac{a}{2ik-a}\, e^{-ikx} & x < 0 \\ \frac{2ik}{2ik-a}\, e^{ikx} & x > 0 \end{cases} \tag{11.29}$$

and from this the reflection and transmission coefficients can be read off:

$$\begin{aligned} R(k) &= \frac{a}{2ik - a} \\ T(k) &= \frac{2ik}{2ik - a} \end{aligned} \tag{11.30}$$

and it's easy to see that

$$|R(k)|^2 + |T(k)|^2 = 1 \tag{11.31}$$

as expected.

(b) $k^2 = -\mu^2 < 0$.

Setting $k = i\mu$ in (11.23), (11.28) with $\mu > 0$ we obtain the general solution in this regime:

$$\psi(x) = \begin{cases} A(i\mu)e^{-\mu x} + B(i\mu)e^{\mu x} & x < 0 \\ C(i\mu)e^{-\mu x} + D(i\mu)e^{\mu x} & x > 0 \end{cases} \tag{11.32}$$

Given that we chose $\mu > 0$, this is <u>bounded</u> as $x \to \pm\infty$ iff

$$A(i\mu) = B(i\mu) = 0 \,. \tag{11.33}$$

Substituting into (11.28),

$$0 = \left(1 + \frac{a}{2\mu}\right)C$$
$$B = -\frac{a}{\mu}C$$

giving two options:

1) $C = B = A = D = 0$ (trivial)

2)

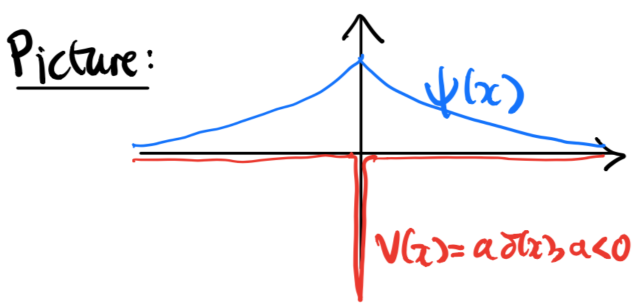
$$\boxed{\mu = -\frac{a}{2}, \quad B = C} \tag{11.34}$$

Given that we took $\mu > 0$, option **2** means that there is a bounded solution with $k^2 < 0$ only for $\underline{a < 0}$. The bound state solution is then

$$\boxed{\psi(x) = e^{\frac{a}{2}|x|} = \begin{cases} e^{-\frac{a}{2}x}, & x < 0 \\ e^{\frac{a}{2}x}, & x > 0 \end{cases}} \tag{11.35}$$

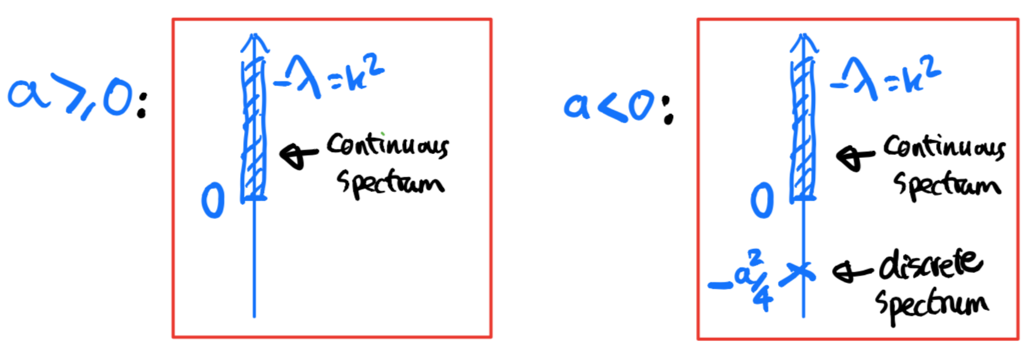
and for this case

$$\boxed{k^2 = -\frac{a^2}{4}} \tag{11.36}$$



Summary

For $V(x) = -u(x) = a\delta(x)$, the problem $L(u)\psi = \lambda\psi$, ψ bounded, has a scattering solution for all real $\lambda < 0$, and either no solutions for $\lambda > 0$ if $a \geqslant 0$, or one solution for $\lambda > 0$ if $a < 0$:



The general story

For potentials $V(x)$ which tend to zero as $x \to \pm\infty$, bounded solutions to $\left(-\frac{d^2}{dx^2} + V(x)\right)\psi = k^2\psi$ come in two flavours:

(a) For all $k^2 > 0$ (taking $k > 0$) we can find a bounded *scattering solution* with asymptotics

$$\boxed{\psi(x) \sim \begin{cases} e^{ikx} + R(k)\, e^{-ikx} & x \to -\infty \\ T(k)\, e^{ikx} & x \to +\infty \end{cases}} \tag{11.37}$$

(b) For $k^2 = -\mu^2 < 0$, set $k = i\mu$, $\mu > 0$, in the above scattering solution to find a solution to the ODE with asymptotics

$$\boxed{\psi(x) \sim \begin{cases} e^{-\mu x} + R(i\mu)\, e^{\mu x} & x \to -\infty \\ T(i\mu)\, e^{-\mu x} & x \to +\infty \end{cases}} \tag{11.38}$$

but since $e^{-\mu x}$ is unbounded as $x \to -\infty$ this looks to be unacceptable.

However, dividing through by $T(i\mu)$ gets to the following situation:

$$\boxed{\psi(x) \sim \begin{cases} \frac{1}{T(i\mu)} e^{-\mu x} + \frac{R(i\mu)}{T(i\mu)}\, e^{\mu x} & x \to -\infty \\ e^{-\mu x} & x \to +\infty \end{cases}} \tag{11.39}$$

and at a pole of $T(i\mu)$, $1/T(i\mu) = 0$ and (11.39) turns into a bounded (and in fact square integrable) solution:

$$\boxed{\psi(x) \sim \begin{cases} \frac{R(i\mu)}{T(i\mu)}\, e^{\mu x} & x \to -\infty \\ e^{-\mu x} & x \to +\infty \end{cases}} \tag{11.40}$$

(Exercise: check that this procedure recovers the bound state solution found above for the delta-function potential $V(x) = a\delta(x)$, $a < 0$.)

Conclusion

Bound state solutions can be obtained from scattering solutions by

(1) dividing the scattering solution through by $T(k)$;

(2) setting

$$k = i\mu = \text{position of a pole of } T(k) \text{ on the positive imaginary axis}\,. \tag{11.41}$$

Depending on $T(k)$, there will be $0, 1, 2 \ldots$ such poles, and hence $0, 1, 2 \ldots$ bound states in the discrete spectrum.

More examples are on the problem sheet, and there's one more in the next section.

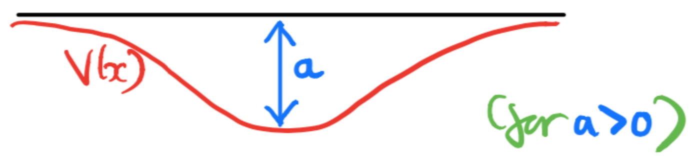
11.3 Reflectionless potentials

We return to the initial field configurations $u(x,0) = a\,\mathrm{sech}^2(x)$ that were tried for the KdV field earlier. These seemed to lead to interesting field evolutions whenever a was equal to $n(n+1)$ with n a positive integer, and it's natural to wonder whether interesting behaviour is also apparent in the correpsonding scattering problem.

The relevant potential is

$$\boxed{V(x) = -a\,\mathrm{sech}^2(x)} \tag{11.42}$$

as illustrated below:



The time independent Schödinger equation (T.I.S.E.) to be solved is

$$\boxed{-\psi''(x) - a\,\mathrm{sech}^2(x)\,\psi(x) = k^2\psi(x)} \tag{11.43}$$

and we're after bounded solutions to this problem.

Substituting

$$y = \tanh(x) \in (-1,1) \tag{11.44}$$

so that

$$\frac{d}{dx} = \mathrm{sech}^2(x)\frac{d}{dy} = (1-y^2)\frac{d}{dy} \tag{11.45}$$

the T.I.S.E. becomes

$$\boxed{\frac{d}{dy}\Big[(1-y^2)\frac{d\psi}{dy}\Big] + \Big(\frac{k^2}{1-y^2} + a\Big)\psi = 0} \tag{11.46}$$

and putting

$$\boxed{k^2 = -m^2\,, \quad a = n(n+1)} \tag{11.47}$$

this becomes

$$\boxed{\frac{d}{dy}\Big[(1-y^2)\frac{d\psi}{dy}\Big] + \Big(n(n+1) - \frac{m^2}{1-y^2}\Big)\psi = 0} \tag{11.48}$$

which is the standard form of the *general (or associated) Legendre equation.* This equation has been much studied, and in particular its solutions are known in general in terms of certain special functions.

Fact 1:

If $n = 1, 1, 2 \ldots$ (*i.e.* $n \in \mathbb{Z}_{\geqslant 0}$) and $m = 0$ (so $k = 0$), then (11.48) becomes the *Legendre equation* and its bounded solution for $y \in [-1, 1]$ is

$$\psi = P_n(y) = \frac{1}{n!\,2^n}\frac{d^n}{dy^n}(y^2 - 1)^n\,, \tag{11.49}$$

the n^{th} *Legendre polynomial of the first kind.* The first few examples are:

$$\begin{aligned} P_1(y) &= y \\ P_2(y) &= -\tfrac{1}{2} + \tfrac{3}{2}y^2 \\ P_3(y) &= -\tfrac{3}{2}y + \tfrac{5}{2}y^3 \\ P_4(y) &= \tfrac{3}{8} - \tfrac{15}{4}y^2 + \tfrac{35}{8}y^4 \end{aligned}$$

In general, $P_j(-x) = (-1)^j P_j(x)$ and $P_j(1) = 1$. Since $y = \pm 1$ corresponds to $x = \pm\infty$, this means that these are <u>bounded</u> solutions to the Schrödinger equation (tending to 1 or maybe -1 as $x \to \pm\infty$) but they are not <u>bound states</u> (for which ψ would have to tend to zero as $x \to \pm\infty$).

(The second solutions, the *Legendre functions of the second kind*, $Q_n(y)$, blow up at $y = \pm 1$.)

Fact 2:

If $n \in \mathbb{Z}_{\geqslant 0}$, bounded solutions to (11.48) only exist for

$$\boxed{m = 0, 1, 2 \ldots n} \tag{11.50}$$

and are

$$P_n^m(y) = (-1)^m (1 - y^2)^{m/2} \frac{d^m}{dy^m} P_n(y)\,. \tag{11.51}$$

These are the *associated Legendre 'polynomials' of the first kind* (the word polynomials is in quotes since for m odd, $m/2$ is not an integer so they aren't strictly speaking polynomials).

Fact 3:

Even when m and n are not integers (and in fact even when they are complex), solutions to (11.48) can be written explicitly using certain special functions. We have that

$$\boxed{P_n^m(y) = \frac{1}{\Gamma(1-m)}\left(\frac{1+y}{1-y}\right)^{m/2} {}_2F_1\big(-m, n{+}1; 1{-}m; \tfrac{1-y}{2}\big)} \tag{11.52}$$

solves (11.48), and reduces to (11.51) if $n \in \mathbb{Z}_{\geqslant 0}$ and $m = 0, 1, \ldots n$.

Here

$$\boxed{\Gamma(z) = \int_0^\infty dt\, t^{z-1} e^{-t}} \tag{11.53}$$

is *Euler's Gamma function* and satisfies

$$\Gamma(N+1) = N! \text{ if } N \in \mathbb{Z}_{\geqslant 0} \tag{11.54}$$

(and, for general N, $\Gamma(N+1) = N\Gamma(N)$)

$$\Gamma(z) \neq 0 \;\; \forall z\,, \quad \frac{1}{\Gamma(z)} = 0 \text{ iff } z \in \{0, -1, -2, \dots\} \tag{11.55}$$

$$\Gamma(z)\Gamma(1-z)\frac{\pi}{\sin(\pi z)} \tag{11.56}$$

while ${}_2F_1$ is the *hypergeometric function* and has the Taylor expansion

$$\boxed{{}_2F_1(a,b;c;z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)}\sum_{k=0}^{\infty}\frac{\Gamma(k+a)\Gamma(k+b)}{\Gamma(k+c)}\frac{z^k}{k!}} \tag{11.57}$$

for $|z| < 1$. The first few terms are

$${}_2F_1(a,b;c;z) = 1 + \frac{ab}{c}\frac{z}{1!} + \frac{a(a{+}1)b(b{+}1)}{c(c{+}1)}\frac{z^2}{2!} + \dots\,.$$

So, up to normalisation, a potentially bounded solution to (11.46) is

$$\psi = P_n^m(y) \tag{11.58}$$

with

$$m = ik\,, \quad n = \frac{\sqrt{1+4a}}{2} - \frac{1}{2}\,. \tag{11.59}$$

(a) $k^2 > 0$ – the continuous spectrum.

• $\underline{x \to +\infty:}$ In this limit $y = \tanh(x) \sim 1 - 2e^{-2x} \to 1^-$ and so

$${}_2F_1(\dots;\tfrac{1-y}{2}) \to {}_2F_1(\dots;0) = 1\,; \quad \frac{1+y}{1-y} \sim e^{2x}\,.$$

Putting these bits together,

$$\boxed{\psi \sim \frac{1}{\Gamma(1-ik)}\,e^{ikx}} \tag{11.60}$$

as $x \to +\infty$.

• $\underline{x \to -\infty:}$ In this limit $y = \tanh(x) \sim -1 + 2e^{2x} \to -1^+$ and $\frac{1+y}{1-y} \sim e^{2x}$, and it turns out that

$$\frac{1}{\Gamma(1{-}m)}\,{}_2F_1(-n, n{+}1; 1{-}m; \tfrac{1-y}{2}) \sim \frac{\Gamma(-m)}{\Gamma(1{-}m{+}n)\Gamma(-m{-}n)} + \frac{\Gamma(m)}{\Gamma(-n)\Gamma(n{+}1)}\,e^{-2mx}\,.$$

This asymptotic can be proved using the already-mentioned properties of the hypergeometric function together with the identity

$$\frac{\sin(\pi(c-a-b))}{\pi}\,{}_2F_1(a,b;c;z) = \frac{{}_2F_1(a,b;c;1-z)}{\Gamma(c-a)\Gamma(c-b)\Gamma(a+b-c+1)} - (1-z)^{c-a-b}\,\frac{{}_2F_1(c-a,c-b;c-a-b+1;1-z)}{\Gamma(a)\Gamma(b)\Gamma(c-a-b+1)}\,.$$

Hence

$$\boxed{\psi \sim \frac{\Gamma(-ik)}{\Gamma(1-ik+n)\Gamma(-ik-n)}\,e^{ikx} + \frac{\Gamma(ik)}{\Gamma(-n)\Gamma(n+1)}\,e^{-ikx}} \tag{11.61}$$

as $x \to -\infty$.

Normalising this solution so that the coefficient of e^{ikx} at $-\infty$ is 1, we can read off the values of $R(k)$ and $T(k)$:

$$\begin{aligned} R(k) &= \frac{\Gamma(ik)\Gamma(1-ik+n)\Gamma(-ik-n)}{\Gamma(-ik)\Gamma(1+n)\Gamma(-n)} = -\frac{\sin(\pi n)}{\pi}\,\frac{\Gamma(ik)\Gamma(1-ik+n)\Gamma(-ik-n)}{\Gamma(-ik)} \\ T(k) &= \frac{\Gamma(1-ik+n)\Gamma(-ik-n)}{\Gamma(1-ik)\Gamma(-ik)}\,. \end{aligned} \tag{11.62}$$

Note: The $\sin(\pi n)$ factor in $R(k)$ means that it vanishes for all k if n is an integer. The corresponding potentials

$$\boxed{V(x) = -n(n+1)\,\mathrm{sech}^2(x)} \tag{11.63}$$

with $n \in \mathbb{Z}_{\geqslant 0}$ are called *reflectionless*: no particles are reflected for any value of k.

(b) $k^2 < 0$ – the discrete spectrum.

To find the discrete spectrum, set $k = i\mu$, $\mu > 0$ and divide the scattering solution through by $T(i\mu)$ to find a possible eigenfunction

$$\boxed{\psi(x) \sim \begin{cases} \frac{1}{T(i\mu)}e^{-\mu x} + \frac{R(i\mu)}{T(i\mu)}\,e^{\mu x} & x \to -\infty \\ e^{-\mu x} & x \to +\infty \end{cases}} \tag{11.64}$$

This is automatically bounded as $x \to +\infty$; it will be bounded as $x \to -\infty$ if (and only if) $\mu \geqslant 0$ is such that $1/T(i\mu) = 0$. (In fact we'll require $\mu > 0$, since $\int_{-\infty}^{+\infty} |\psi|^2 dx$ should be finite for the discrete spectrum.) This in turn requires

$$\frac{1}{T(i\mu)} = \frac{\Gamma(1+\mu)\Gamma(mu)}{\Gamma(1+\mu+n)\Gamma(\mu-n)} = 0\,.$$

Given that μ must be a posiitve real number, there are two options:

(1) $1 + \mu + n = -j,\ j \in \mathbb{Z}_{\geqslant 0}$
(2) $\mu - n = -h,\ h \in \mathbb{Z}_{\geqslant 0}$

• If $n \notin \mathbb{R}$ then there are no real solutions for μ.

• If $n \in \mathbb{R}$ we can take $n \geqslant -1/2$ without losing generality, since **(1)** $\leftrightarrow$ **(2)** when $n \to -1-n$. Then **(1)** never holds, while solutions for poisitive μ do exist for option **(2)** provided $n \geqslant 0$:

$$\boxed{\mu = n, n-1, n-2 \ldots n - \lfloor n \rfloor} \tag{11.65}$$

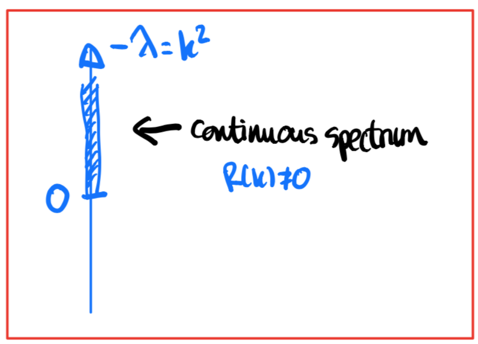
where $\lfloor n \rfloor$ = 'floor' of n = {largest integer $\leqslant n$}. So

$$\boxed{\text{Total number of bound states} = \lceil n \rceil} \tag{11.66}$$

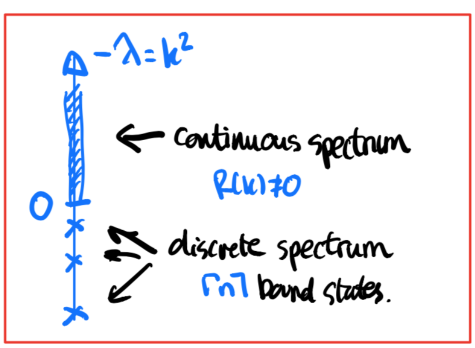
where $\lceil n \rceil$ = {smallest integer $\geqslant n$}. (If n is an integer, then the last eigenvalue, for $\mu = 0$, should be discarded as the corresponding ψ is not squuare integrable and so is not a bound state – it's in the continuous spectrum instead.)

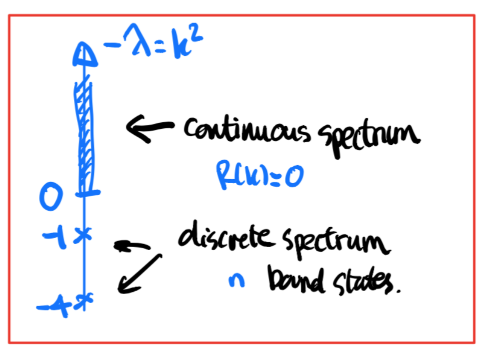
Summary for $V(x) = -a\,\mathrm{sech}^2(x) = -n(n{+}1)\,\mathrm{sech}^2(x)$:

• $a < 0$:



• $a = n(n+1) > 0$:
(n not an integer (say $n = 2.5$) on the left, $n \in \mathbb{Z}_{>0}$ (say $n = 2$) on the right)





11.4 Scattering data for general potentials

So far we've seen that for any localised initial data $u(x,0)$ for KdV, the auxiliary time-independent Schrödinger equation

$$-\psi''(x) + V(x)\psi(x) = k^2\psi(x) \tag{11.67}$$

with potential $V(x) = -u(x,0)$ has

(a) A *continuous spectrum* of non-negative eigenvalues E with $E = k^2 \geqslant 0$ and eigenfunctions

$$\boxed{\psi(x) \sim \begin{cases} e^{ikx} + R(k)\, e^{-ikx} & x \to -\infty \\ T(k)\, e^{ikx} & x \to +\infty \end{cases}} \tag{11.68}$$

normalised so that the incoming flux is one;

(b) A (maybe empty) *discrete spectrum* of negative eigenvalues $E = k^2 = -\mu_n^2 < 0$, indexed by $n = 1, 2 \ldots N$. These look like

$$\boxed{\psi_n(x) \sim \begin{cases} c_n\, e^{\mu_n x} & x \to -\infty \\ d_n\, e^{-\mu_n x} & x \to +\infty \end{cases}} \tag{11.69}$$

So far the ψ_n's we've found have been normalised so that $d_n = 1$, but now we will instead normalise them so that

$$(\psi_n, \psi_n) = \int_{-\infty}^{+\infty} |\psi_n(x)|^2\, dx = 1\,. \tag{11.70}$$

Once ψ_n has been normalised in this way, the number c_n is called the *normalising coefficient* and it will be needed later, to reconstruct $V(x) = -u(x)$. More precisely, to reconstruct $V(x)$ we will need to know

$$\boxed{\Big\{R(k)\,,\ \ \{\mu_n, c_n\}_{n=1}^{N}\Big\}} \tag{11.71}$$

This is called the *scattering data*, refining the notion of scattering data given earlier.

• Clearly, u (or $V = -u$) determines the scattering data completely (this was step **(a)**, disassembly, of the roadmap).

• Amazingly, the converse also holds: u (or $V = -u$) can be reconstructed uniquely from the scattering data (step **(c)**, reassembly).

• The next major task is to return to step **(b)**, time evolution, to see precisely how the scattering data evolves.

Before going there, let's make precise the scattering data for two sets of potentials studied earlier.

Examples of scattering data

1) $V(x) = a\,\delta(x)$:

• For all values of a we have

$$\boxed{R(k) = \frac{a}{2ik - a}}$$

• For $a \geqslant 0$ that's all.

• For $a < 0$ there is also a single bound state $\psi(x) = Ae^{-\mu|x|}$ with $\mu = -a/2 > 0$. Normalising determines $A^2/\mu = 1$ so $A = \sqrt{\mu} = \sqrt{-a/2}$.

Thus the general scattering data for $u(x,0) = -a\delta(x)$, $V(x) = a\delta(x)$, is

$$\boxed{\begin{cases} \left\{ R(k) = \frac{a}{2ik-a} \right\} & \text{if } a \geqslant 0 \\ \left\{ R(k) = \frac{a}{2ik-a} \left\{\mu_1 = -a/2, c_1 = \sqrt{-a/2}\right\} \right\} & \text{if } a < 0 \end{cases}} \tag{11.72}$$

2) $V(x) = -n(n{+}1)\,\mathrm{sech}^2(x)$, $n \in \mathbb{Z}_{\geqslant 0}$:

(a) Scattering states: $R(k) = 0$ (since the potential is reflectionless).

(b) Bound states: we have $\psi_m(x) = A\,P_n^m(\tanh(x))$, $m = 1, 2 \ldots n$, where A is a normalisation constant that can be fixed by imposing

$$1 = \int_{-\infty}^{+\infty} |\psi_m(x)|^2\,dx = A^2 \int_{-1}^{1} P_n^m(y)^2\,\frac{dy}{1-y^2} = A^2\,\frac{(n{+}m)!}{m(n{-}m)!}$$

where the last equality makes use of one of the standard properties of P_n^m.

In addition P has the asymptotic

$$P_n^m(\tanh(x)) \sim (-1)^n\,\frac{(n{+}m)!}{m!(n{-}m)!}\,e^{mx}\,, \quad x \to -\infty\,.$$

Hence the asymptotic of the normalised bound state is

$$\psi_m(x) \sim (-1)^n\,\frac{1}{m!}\sqrt{\frac{m(n{+}m)!}{(n{-}m)!}}\,e^{mx}\,, \quad x \to -\infty$$

and the full scattering data is

$$\boxed{\left\{ R(k) = 0\,,\ \left\{\mu_m = m\,,\ c_m = (-1)^n\,\frac{1}{m!}\sqrt{\frac{m(n{+}m)!}{(n{-}m)!}}\right\}_{m=1}^{n} \right\}} \tag{11.73}$$

11.5 Time evolution of the scattering data – concluded

We have seen that if u evolves by the KdV equation, then

1) the eigenvalues λ of $L(u) = D^2 + u$ remain constant in t;

2) the eigenfunctions ψ evolve by $\psi_t = B(u)\psi$.

Question: how does the scattering data associated to $V = -u$ evolve in time?

Answer: We need to look at the asymptotics of the time-evolution equation $\psi_t = B(u)\psi$ as $x \to \pm\infty$. Recall that for KdV

$$B(u) = -(4D^3 + 6uD + 3u_x)$$

and so, since u, $u_x \to 0$ as $x \to \pm\infty$ for all t, as follows from the boundary conditions on u,

$$\boxed{B(u) \sim -4D^3 \quad \text{as } x \to \pm\infty} \tag{11.74}$$

and is independent of $u(x,t)$. This is the key point: we can evolve the scattering data forward in t without knowing in advance what u evolves to!

[You might worry about the bound state normalisation condition $(\psi_m, \psi_m) = 1$. Is this preserved under time evolution? It turns out that the answer is yes: this follows, with a little work, from the antisymmetry of B, that is $B(u)^\dagger = -B(u)$.]

Next, we need to work out explicitly the t evolution of the asymptotics of the scattering and bound state solutions.

(a) The continuous spectrum ($-\lambda = k^2 > 0$)

Start with an un-normalised scattering solution:

$$\boxed{\psi_k(x;t) \sim \begin{cases} A(k;t)\,e^{ikx} + B(k;t)\,e^{-ikx} & x \to -\infty \\ C(k;t)\,e^{ikx} & x \to +\infty \end{cases}} \tag{11.75}$$

Imposing $\frac{\partial}{\partial t}\psi_k(x;t) = B(u)\psi_k(x;t) \sim -4D^3\psi_k(x;t)$ as $x \to \pm\infty$, we have

$$\begin{aligned} A_t(k;t)\,e^{ikx} + B_t(k;t)\,e^{-ikx} &= 4ik^3\left[A(k;t)\,e^{ikx} - B(k;t)\,e^{-ikx}\right] \\ C_t(k;t)\,e^{ikx} &= 4ik^3 C(k;t)\,e^{ikx} \end{aligned}$$

and, hence, equating coefficients of $e^{\pm ikx}$,

$$\begin{aligned} A_t(k;t) &= 4ik^3 A(k;t) \\ B_t(k;t) &= -4ik^3 B(k;t) \\ C_t(k;t) &= 4ik^3 C(k;t) \end{aligned} \tag{11.76}$$

Solving,

$$\begin{aligned} A(k;t) &= A(k;0)\,e^{4ik^3t} \\ B(k;t) &= B(k;0)\,e^{-4ik^3t} \\ C(k;t) &= C(k;0)\,e^{4ik^3t} \end{aligned} \tag{11.77}$$

Dividing the un-normalised solution at time t through by $A(k;t)$ so that it continues to be correctly normalised with unit incoming flux, $R(k;t)$ and $T(k;t)$ can be read off as follows:

$$\begin{aligned} R(k;t) &= R(k;0)\,e^{-8ik^3t} \\ T(k;t) &= T(k;0)\,. \end{aligned} \tag{11.78}$$

This can be summed up in the asymptotics of the normalised scattering solution:

$$\boxed{\psi_k(x;t) \sim \begin{cases} e^{ikx} + R(k;0)\,e^{-ik(x+8k^2t)} & x \to -\infty \\ T(k;0)\,e^{ikx} & x \to +\infty \end{cases}} \tag{11.79}$$

The reflected waves for ψ_k, encoded in $R(k;t)$, translate into a dispersive component of $u(x,t)$, moving to the left as t increases.

(b) The discrete spectrum $(-\lambda = -\mu_n^2 < 0)$

The n^{th} bound state wave function has asymptotics

$$\psi_n(x;t) \sim \begin{cases} c_n(t)\,e^{\mu_n x} & x \to -\infty \\ d_n(t)\,e^{-\mu_n x} & x \to +\infty \end{cases} \tag{11.80}$$

Imposing $\frac{\partial}{\partial t}\psi_n(x;t) = B(u)\psi_k(x;t) \sim -4D^3\psi_n(x;t)$ as $x \to \pm\infty$, we have

$$\begin{cases} \frac{\partial}{\partial t}\,c_n(t) &= -4\mu_n^3\,c_n(t) \\ \frac{\partial}{\partial t}\,d_n(t) &= +4\mu_n^3\,d_n(t) \end{cases}$$

and, solving,

$$\begin{aligned} c_n(t) &= c_n(0)\,e^{-4\mu_n^3t} \\ d_n(t) &= d_n(0)\,e^{+4\mu_n^3t} \end{aligned} \tag{11.81}$$

Again, this can be summarised as

$$\psi_n(x;t) \sim \begin{cases} c_n(0)\,e^{\mu_n(x-4\mu_n^2t)} & x \to -\infty \\ d_n(0)\,e^{-\mu_n(x-4\mu_n^2t)} & x \to +\infty \end{cases} \tag{11.82}$$

This will translate into a soliton for $u(x,t)$, moving to the right with velocity $4\mu_n^2$.

These results describe the time evolution of the scattering data, completing step **(b)** of the inverse scattering method.