Chapter 12

The Marchenko equation

12.1 Introduction

To conclude the inverse scattering method, we need to reassemble the KdV field $u(x,t)$, or equivalently the Schrödinger potential $V(x;t) = -u(x,t)$, from the time-evolved scattering data. This is step **(c)**: "reassembly / inverse scattering".

This touches on a general question: if all you were allowed to do was sit at infinity and chuck particles at your potential, and measure how they come back, could you deduce the form of $V(x)$?

This question is of practical importance, for example when looking for oil using seismic reflection, or in medicine (one example there being deducing the shape of the inner ear from reflected sound waves). It belongs to the category of "inverse problems": deducing the form of an operator (here $D^2 + u$) from information about its spectrum (μ_i, c_n and so on): "can you hear the shape of a drum?"

For this one-dimensional (Schrödinger) case, the result was already known, found by Marchenko (following earlier work by Gelfand and Levitan), some years before GGKM.

In fact you don't need to know $T(k)$, just $R(k)$ for real k, together with the N discrete eigenvalues $-\mu_j^2$, $j = 1, \ldots N$, and the normalising coefficients c_j, $j = 1, \ldots N$. The full set $\left\{R(k), , \ \{\mu_n, c_n\}_{n=1}^N\right\}$ is precisely the scattering data we evolved forward in time in the last chapter.

There are two important special cases:

(1) $N = 0$: $V(x)$ has no bound states;

(2) $R(k) = 0 \;\; \forall k$: $V(x)$ is reflectionless, but there is still information about $V(x)$ hidden in the bound state eigenvalues and normalisation coefficients.

It turns out that

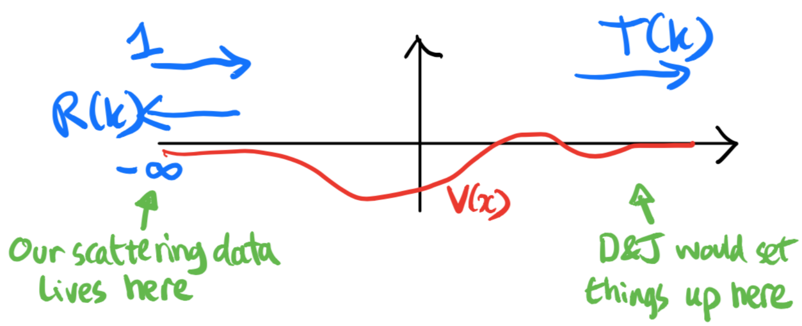
(1) $\Rightarrow$ initial data contains no solitons;

(2) $\Rightarrow$ initial data contains only solitons.

12.2 The recipe for inverse scattering: the Marchenko equation

We want to solve the inverse scattering problem for given scattering data at $x = -\infty$ to determine the potential $V(x)$, and hence the KdV field $u(x) = -V(x)$, at any fixed KdV time t.

The derivation is long and we'll skip it here – see for example section 3.3 of Drazin and Johnson. But a warning: everything in Drazin and Johnson is phrased in terms of scattering solutions with waves arriving from $+\infty$, and asymptotics also at $+\infty$, while we do the opposite:



Once the not inconsiderable quantity of dust has settled, the upshot is the following recipe:

(1) Construct the function

$$F(\xi) = \int_{-\infty}^{+\infty} \frac{dk}{2\pi} R(k)\, e^{-ik\xi} + \sum_{n=1}^{N} c_n^2\, e^{\mu_n \xi} \tag{12.1}$$

from the scattering data

$$\boxed{\Big\{ R(k)\,,\ \ \{\mu_n, c_n\}_{n=1}^{N} \Big\}} \tag{12.2}$$

(2) Solve the *Marchenko equation*

$$\boxed{K(x,z) + F(x{+}z) + \int_{-\infty}^{x} dy\, K(x,y)\, F(y{+}z) = 0} \tag{12.3}$$

to determine the unknown function $K(x,z)$ for all $z \leqslant x$ (and set $K(x,z) = 0$ for $x < z$).

(3) Finally determine the Schrödinger potential from

$$\boxed{V(x) = 2\frac{d}{dx} K(x,x)} \tag{12.4}$$

The KdV field is then given by $u = -V$.

This all applies at one fixed KdV time t. But using the results of the last section of the last chapter, we know that

$$R(k;t) = R(k;0)\, e^{-8ik^3 t}$$
$$c_n(t) = c_n(0)\, e^{-4\mu_n^3 t}$$

while k^2 and μ_n^2 are independent of time.

So to find the field at time t, we just apply the above recipe starting from

$$\begin{aligned} F(\xi;t) &= \int_{-\infty}^{+\infty} \frac{dk}{2\pi}\, R(k;t)\, e^{-ik\xi} + \sum_{n=1}^{N} c_n(t)^2\, e^{\mu_n \xi} \\ &= \int_{-\infty}^{+\infty} \frac{dk}{2\pi}\, R(k;0)\, e^{-ik(\xi+8k^2 t)} + \sum_{n=1}^{N} c_n(0)^2\, e^{\mu_n(\xi - 8\mu_n^2 t)} \end{aligned} \tag{12.5}$$

At least in principle, this solves the problem! In practice the term involving R in the definition of F, with the integral over k, makes the calculation of F hard when $t > 0$. But for reflectionless potentials this term is absent, and $F(\xi, t)$ can be read off at any time t. This turns out to yield the 'pure' multisoliton solutins that can also be found via Bäcklund or Hirota. Even when R is nonzero, it can be shown that the term involving R goes to zero as $t \to \infty$. All of which leads to the following 'big picture':

(A) $\{\mu_n, c_n\}_{n=1}^{N} \leftrightarrow N$ right-moving solitons hidden inside the initial data:

(B) $R(k) \leftrightarrow$ a superposition of dispersive left-moving waves hidden inside the initial data:

The net result is a sort of "nonlinear Fourier analysis" (which reverts to the usual Fourier solution in the limit of small-amplitude waves).

12.3 Example 1: the single KdV soliton

Consider a reflectionless potential, so $R(k) = 0$, with just one bound state encoded in $\{\mu_1, c_1\} \equiv \{\mu, c\}$. Then (at fixed t)

$$F(\xi) = c^2 e^{\mu\xi} \tag{12.6}$$

and the Marchenko equation (12.3) reads

$$\boxed{K(x,z) + c^2 e^{\mu(x+z)} + \int_{-\infty}^{x} dy\, K(x,y)\, c^2 e^{\mu(y+z)} = 0} \tag{12.7}$$

This needs to be solved for $z \leqslant x$. As a first step, factorise $e^{\mu z}$ from the last two terms:

$$K(x,z) + e^{\mu z}\left(c^2 e^{\mu x} + \int_{-\infty}^{x} dy\, K(x,y)\, c^2 e^{\mu y}\right) = 0\,, \tag{12.8}$$

and note that the terms in brackets are independent of z, meaning that

$$\boxed{K(x,z) = h(x)\, e^{\mu z}} \tag{12.9}$$

for some $h(x)$. Substituting back into (12.8) and dividing through by $e^{\mu z}$, $h(x)$ must satisfy

$$0 = h(x) + c^2 e^{\mu x} + c^2 \int_{-\infty}^{x} dy\, h(x)\, e^{2\mu y} = h(x)\left(1 + c^2 \int_{-\infty}^{x} dy\, e^{2\mu y}\right) + c^2 e^{\mu x}$$

and hence

$$h(x) = -\frac{c^2 e^{\mu x}}{1 + \frac{c^2}{2\mu} e^{2\mu x}}\,. \tag{12.10}$$

If we set

$$c^2 = 2\mu\, e^{-2\mu x_0} \tag{12.11}$$

(thereby trading c for x_0) we obtain

$$h(x) = -2\mu \frac{e^{\mu(x-2x_0)}}{1+\frac{c^2}{2\mu}e^{2\mu(x-x_0)}} \tag{12.12}$$

and so

$$K(x,z) = -2\mu \frac{e^{\mu(x+z-2x_0)}}{1+\frac{c^2}{2\mu}e^{2\mu(x-x_0)}}\,. \tag{12.13}$$

Hence

$$\boxed{V(x) = 2\frac{d}{dx}K(x,x) = -2\mu^2 \mathrm{sech}^2(\mu(x-x_0))} \tag{12.14}$$

and $u = -V$ is indeed a snapshot of a single KdV soliton, at a time (say $t = 0$) when its centre is at $x = x_0$.

Time evolution is easily included using

$$c(t)^2 = c^(0)\, e^{-8\mu^3 t} = 2\mu\, e^{-2\mu(x_0-4\mu^2 t)} \tag{12.15}$$

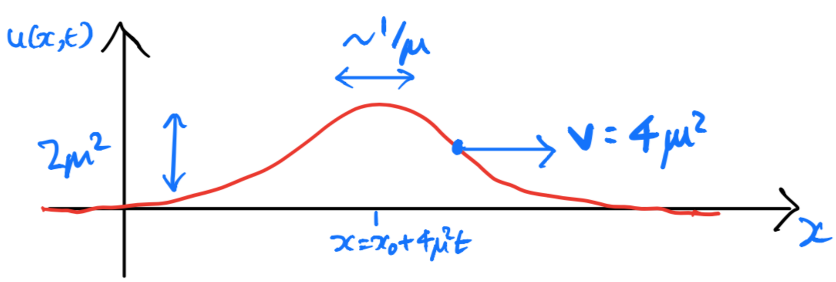
which has the effect of translating the centre of the soliton as

$$x_0 \to x_0 + 4\mu^2 t \tag{12.16}$$

and the KdV field at time t is

$$\boxed{u(x,t) = -V(x,t) = 2\mu^2 \mathrm{sech}^2(\mu(x-x_0-4\mu^2 t))} \tag{12.17}$$

which is a single moving soliton just as found earlier in the course:



12.4 Example 2: the N-soliton solution

Now let's consider a siutation with $R(k) = 0$ but with N bound states, encoded in $\{\mu_n, c_n\}_{n=1}^N$. Then

$$F(\xi) = \sum_{n=1}^{N} c_n^2\, e^{\mu_n \xi}\,. \tag{12.18}$$

Since

$$F(x+z) = \sum_{n=1}^{N} c_n^2 \, e^{\mu_n x} e^{\mu_n z}$$

is a sum of factorised terms, we will look for a solution where $K(x, z)$ is also a sum of factorised terms. This is best encoded using a vector and matrix notation, setting

$$E(x) = \begin{pmatrix} e^{\mu_1 x} \\ \vdots \\ e^{\mu_N x} \end{pmatrix}, \quad L(x) = \begin{pmatrix} c_1^2 e^{\mu_1 x} \\ \vdots \\ c_N^2 e^{\mu_N x} \end{pmatrix}, \quad H(x) = \begin{pmatrix} h_1(x) \\ \vdots \\ h_N(x) \end{pmatrix}, \tag{12.19}$$

where $H(x)$ is yet to be determined. With this notation set up, we have

$$F(x+z) = E^T(x) L(z) \tag{12.20}$$

(where the T superscript denotes a transpose) and we'll look for a $K(x, z)$ of the form

$$K(x, z) = H^T(x) L(z) \,. \tag{12.21}$$

Substituting into the Marchenko equation, we find

$$\begin{aligned} 0 &= K(x, z) + F(x+z) + \int_{-\infty}^{x} dy \, K(x, y) \, F(y+z) \\ &= H^T(x) L(z) + E^T(x) L(z) + H^T(x) \int_{-\infty}^{x} dy \, L(y) E^T(y) L(z) \\ &= \Big(H(x) + E(x) + \int_{-\infty}^{x} dy \, E(y) E^T(y) H(x) \Big)^T L(z) \,. \end{aligned} \tag{12.22}$$

If the term in brackets on the last line can be made to vanish, we'll have a solution. In turn this will be true if

$$\Gamma(x) \, H(x) = -E(x) \tag{12.23}$$

where $\Gamma(x)$ is not the gamma function seen earlier, but rather the $N \times N$ matrix

$$\Gamma(x) = \mathbb{1}_{N\times N} + \int_{-\infty}^{x} dy \, E(y) L^T(y) \tag{12.24}$$

with matrix elements

$$\begin{aligned} \Gamma(x)_{mn} &= \delta_{mn} + \int_{-\infty}^{x} dy \, e^{\mu_m y} c_n^2 e^{\mu_n y} \\ &= \delta_{mn} + c_n^2 \frac{e^{(\mu_m + \mu_n) y}}{\mu_m + \mu_n} \,. \end{aligned} \tag{12.25}$$

Note also we have

$$\frac{d}{dx} \Gamma(x) = E(x) L^T(x) \,, \tag{12.26}$$

a formula that will be useful shortly.

From (12.23) we have

$$H(x) = -\Gamma(x)^{-1}E(x) \tag{12.27}$$

and so

$$\begin{aligned} K(x,z) &= L^T(z)H(x) = -L^T(z)\Gamma(x)^{-1}E(x) \\ &= -\operatorname{tr}\big(\Gamma(x)^{-1}E(x)L^T(z)\big)\,. \end{aligned} \tag{12.28}$$

Therefore

$$\begin{aligned} K(x,x) &= -\operatorname{tr}\big(\Gamma(x)^{-1}E(x)L^T(x)\big) \\ &= -\operatorname{tr}\big(\Gamma(x)^{-1}\frac{d}{dx}\Gamma(x)\big) \\ &= -\operatorname{tr}\big(\frac{d}{dx}\log\Gamma(x)\big) \\ &= -\frac{d}{dx}\operatorname{tr}\big(\log\Gamma(x)\big) \\ &= -\frac{d}{dx}\log\big(\det\,\Gamma(x)\big) \end{aligned} \tag{12.29}$$

using the matrix identities

$$\frac{d}{dx}\log\Gamma = \Gamma^{-1}\frac{d}{dx}\Gamma\,, \quad \operatorname{tr}(\log\Gamma) = \log(\det\,\Gamma)\,. \tag{12.30}$$

This implies that the KdV field is

$$u = -2\frac{d}{dx}K(x,x) = 2\frac{d^2}{dx^2}\log(\det\,\Gamma(x)) \tag{12.31}$$

or, putting back the t-dependence hidden in Γ (through the c_n),

$$u(x,t) = 2\frac{\partial^2}{\partial x^2}\log(\det\,\Gamma(x;t)) \tag{12.32}$$

with

$$\Gamma(x;t)_{mn} = \delta_{mn} + c_n^2(t)\frac{e^{(\mu_m+\mu_n)x}}{\mu_m+\mu_n}\,. \tag{12.33}$$

These formulae are very similar to the N-soliton KdV solutions found by Hirota. To see that they are in fact exactly the same, we can use *Sylvester's determinant theorem*, which states that

$$\det(\mathbb{1}_{N\times N} + AB) = \det(\mathbb{1}_{N\times N} + BA) \tag{12.34}$$

for any pair of $N\times N$ matrices A, B.

Taking

$$A_{mn} = e^{\mu_m x}\delta_{mn}\ , \quad B_{mn} = \frac{c_n^2 e^{\mu_n x}}{\mu_m + \mu_n}$$

we have

$$(AB)_{mn} = \frac{c_n^2 e^{(\mu_m+\mu_n)x}}{\mu_m + \mu_n}\ , \quad (BA)_{mn} = \frac{c_n^2 e^{2\mu_n x}}{\mu_m + \mu_n}\ ,$$

and so we can equivalently write

$$\boxed{u(x,t) = 2\frac{\partial^2}{\partial x^2}\log(\det S(x;t))} \tag{12.35}$$

with

$$\begin{aligned} S(x;t)_{mn} &= \delta_{mn} + \frac{1}{\mu_m + \mu_n}\, c_n^2(t)\, e^{2\mu_n x} \\ &= \delta_{mn} + \frac{2\mu_n}{\mu_m + \mu_n}\, e^{2\mu_n(x - x_{0,n} - 4\mu_n^2 t)} \end{aligned} \tag{12.36}$$

where, just as done above for the one-soliton solution, we traded $c_n(0)$ for $x_{0,n}$ by setting

$$c_n(0)^2 = 2\mu_n e^{-2\mu_n x_{0,n}}\ . \tag{12.37}$$

These equations give the general form of the N-soliton solution of the KdV equation.