Chapter 13

Integrable systems in classical mechanics

So far, we're (secretly) been looking at infinite-dimensional systems: classical field theories in one space and one time dimension, though these can often be thought of as the *continuum limits* (see last term) of systems with finitely-many degrees of freedom.

Many of the methods we've seen, in particular the idea of a Lax pair, can also apply to finite-dimensional systems, and more precisely to *finite-dimensional classical integrable Hamiltonian systems*. To understand what these words mean, some definitions are needed.

• A *finite-dimensional Hamiltonian system* is defined by:

- A set of (generalised) coordinates $q_{i=1\ldots n}$ and momenta $p_{i=1\ldots n}$, which completely specify the configuration of the system at time t (the space parametrised by these so-called *canonical coordinates* q, p is called the $2n$-dimensional *phase space* of the system);

- A function $H(q,p)$ defined on phase space called the *Hamiltonian*.

The time evolution equations are then, with the dots denoting time derivatives,

$$\boxed{\begin{aligned} \dot{q}_i &= \frac{\partial H}{\partial p_i} \\ \dot{p}_i &= -\frac{\partial H}{\partial q_i} \end{aligned}} \tag{13.1}$$

These are called *Hamilton's equations*.

Example: for n particles with masses m_i moving in one dimension under conservative forces associated with a potential energy $V(q_1, \ldots q_n)$, the Hamiltonian is

$$H(q,p) = \sum_{i=1}^{n} \frac{p_i^2}{2m_i} + V(q_1, \ldots q_n) \tag{13.2}$$

and Hamilton's equations are

$$\dot{q}_i = \frac{p_i}{m_i} \,, \quad \dot{p}_i = -\frac{\partial V(q_1, \ldots q_n)}{\partial q_i} \,. \tag{13.3}$$

These are the same as Newton's equations,

$$m_i \ddot{q}_i = -\frac{\partial V(q_1, \ldots q_n)}{\partial q_i} \,, \tag{13.4}$$

put into a first-order form.

• One can associate to a Hamiltonian system a *Poisson bracket* $\{,\}$, a bilinear antisymmetric form on the space of functions of q and p:

$$\boxed{\{f,g\} := \sum_{i=1}^{n} \left(\frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} - \frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} \right)} \tag{13.5}$$

Clearly $\{f,g\} = -\{g,f\}$ and $\{f,f\} = 0$.

• Hamilton's equations (13.1) imply that any $f(q,p)$ which does not depend explicitly on time, but only implicitly via $q(t)$ and $p(t)$, evolves as

$$\begin{aligned} \frac{d}{dt} f(q(t),p(t)) &= \sum_{i=1}^{n} \left(\dot{q}_i \frac{\partial f}{\partial q_i} + \dot{p}_i \frac{\partial f}{\partial p_i} \right) \\ &= \sum_{i=1}^{n} \left(\frac{\partial H}{\partial p_i} \frac{\partial f}{\partial q_i} - \frac{\partial H}{\partial q_i} \frac{\partial f}{\partial p_i} \right) \end{aligned}$$

That is,

$$\boxed{\frac{d}{dt} f(q,p) = \{H(q,p), f(q,p)\}} \tag{13.6}$$

(If f also depends explicitly on t, so $f = f(q(t),p(t),t)$, then $\frac{d}{dt} f(q,p) = \frac{\partial}{\partial t} f + \{H(q,p), f(q,p)\}$.)

• Functions $F(q,p)$ which don't depend explicitly on time and have vanishing Poisson bracket with the Hamilton $H(q,p)$ are conserved:

$$\boxed{\frac{d}{dt} F(q(t),p(t)) = \{H(q,p), F(q,p)\} = 0} \tag{13.7}$$

In particular, the antisymmetry of the Poisson bracket means that the Hamiltonian is always conserved:

$$\frac{d}{dt}H = \{H, H\} = 0\,. \tag{13.8}$$

Hence

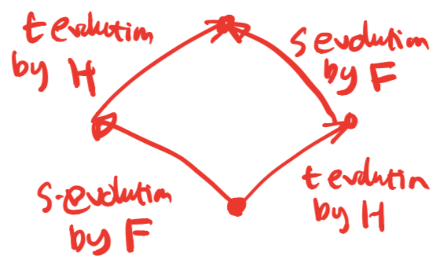
$$\boxed{H(q(t), p(t)) = E = \text{constant}} \tag{13.9}$$

which is nothing but the conservation of energy.

Note: If $\{F, H\} = 0$, then not only is $F(q, p)$ conserved under the time evolution (13.1), but also $H(q, p)$ is conserved under a different time evolution with a different time, s say, and Hamiltonian $F(q, p)$:

$$\left\{ \begin{aligned} \frac{d}{ds}q_i &= \frac{\partial F}{\partial p_i} \\ \frac{d}{ds}p_i &= -\frac{\partial F}{\partial q_i} \end{aligned} \right\} \quad \Rightarrow \quad \frac{d}{ds}H(q, p) = \{F(q, p), H(q, p)\} = 0\,. \tag{13.10}$$

It also means (via the Jacobi identity) that we can evolve along the two times, along t and then s, or vice versa, and we will end up at the same point in phase space:



In fancy language, F and H such that $\{F, H\} = 0$ are said to be *in involution* and they generate *commuting flows*, where one flow is t-evolution with Hamiltonian H, and the other flow is s-evolution with Hamiltonian F. We saw this idea earlier, in section 10.3, when discussing the KdV hierarchy.

Definition: A Hamiltonian system $\{\, q_{i=1\ldots n}, p_{i=1\ldots n}, H(q_i, p_i)\,\}$ is called *completely integrable* if it has n independent conserved quantities $Q_i(q, p)$ satisfying $\{Q_i, H\} = 0$, which are mutually *in involution*, that is

$$\boxed{\{Q_i, Q_j\} = 0 \quad \forall i, j = 1 \ldots n} \tag{13.11}$$

One of these conserved quantities is always the original Hamiltonian H.

For such systems it is possible to find a new set of coordinates φ_i and momenta Q_i on phase space such that the Hamiltonian only depends of the Q_i and not on the φ_i:

$$H = H(Q) \quad \Rightarrow \quad \begin{cases} \dot{\varphi} = \dfrac{\partial H}{\partial Q_i} \\ \dot{Q} = -\dfrac{\partial H}{\partial \varphi_i} = 0 \end{cases} \tag{13.12}$$

These are called *action-angle variables* (φ_i: angle variables; Q_i: action variables). The name is because if the surfaces of constant H are compact, then the φ_i parametrise periodic orbits and can therefore be thought of as angular variables.

• The n conserved quantities Q_i are the finite-dimensional analogues of the infinitely-many conserved charges of the KdV hierarchy discussed in section 10.3.

• What is interesting for us here is that the integrability of such classical systems can be established by constructing a Lax pair L, M, satisfying

$$\boxed{\dot{L} = [M, L]} \tag{13.13}$$

This is as we saw with L and B for KdV, but now L and M will be $n \times n$ matrices instead of differential operators. We'll see that the n conserved quantitites are the eigenvalues $\lambda_{i=1\ldots n}$ of the Lax matrix L (though as we'll also see, it may be more convenient sometimes to use some functions of those eigenvalues instead, such as the sums of their powers).

(To show that the conservation laws are in involution is a bit more tricky, and won't be discussed here.)

• In general, if there are n q's, $q_{i=1\ldots n}$, L and M will be $n \times n$ matrices and the n conserved quantities will be coded up in the n eigenvalues $\lambda_1 \ldots \lambda_n$ of the Lax matrix L.

• The Lax equation (13.13), with L and M functions of time, can be solved formally by

$$\boxed{L(t) = U(t)L(0)U(t)^{-1}} \tag{13.14}$$

where the time evolution operator $U(t)$ is the unique solution of the following (matrix) ordinary differential equation:

$$\boxed{\begin{aligned} \dot{U}(t) &= M(t)U(t) \\ U(0) &= \mathbb{1} \end{aligned}} \tag{13.15}$$

This can be proved as follows:

$$
\begin{aligned}
\dot{L} &= \frac{d}{dt}\left(UL(0)U^{-1}\right) \\
&= \dot{U}L(0)U^{-1} + UL(0)(\dot{U^{-1}}) \\
&= \dot{U}L(0)U^{-1} - UL(0)U^{-1}\dot{U}U^{-1} \\
&= \dot{U}U^{-1}UL(0)U^{-1} - UL(0)U^{-1}\dot{U}U^{-1} \\
&= ML - ML \\
&= [M, L]
\end{aligned}
\tag{13.16}
$$

(where the result $(\dot{U^{-1}}) = U^{-1}\dot{U}U^{-1}$ used in going from the second line to the third can be proved by differentiating $UU^{-1} = \mathbb{1}$).

The formal solution (13.14) can be used to prove that the eigenvalues of the Lax matrix L do not depend on time, just as was the case for KdV in infinitely-many dimensions. To see this, consider the *characteristic polynomial* of L:

$$
\boxed{P_L(\lambda) = \det(\lambda\mathbb{1} - L)}
\tag{13.17}
$$

This is a degree n monic polynomial ("monic": $\lambda^n + \dots$) whose roots are the n eigenvalues $\lambda_{i=1\dots n}$ of L. Now L is going to be a Hermitian – often real – matrix which can be diagonalised by conjugating it with some unitary matrix V:

$$
L = V\Lambda V^{-1}\,, \quad \Lambda = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix}
\tag{13.18}
$$

Thus (in a sequence of equalities that you might have seen before)

$$
\begin{aligned}
P_L(\lambda) &= \det(\lambda\mathbb{1} - L) \\
&= \det(\lambda\mathbb{1} - V\Lambda V^{-1}) \\
&= \det(\lambda V V^{-1} - V\Lambda V^{-1}) \\
&= \det(V(\lambda\mathbb{1} - \Lambda)V^{-1}) \\
&= \det(V)\det(\lambda\mathbb{1} - \Lambda)\det(V^{-1}) \\
&= \det(\lambda\mathbb{1} - \Lambda) \\
&= \prod_{i=1}^{n}(\lambda - \lambda_i) \\
&= \lambda^n - c_1\lambda^{n-1} + c_2\lambda^{n-2} - \dots + (-1)^n\prod_{i=1}^{n}\lambda_i\,.
\end{aligned}
\tag{13.19}
$$

(The signs of the coefficients on the last line are chosen for later convenience.)

Since time evolution is also given by conjugation (this time by $U(t)$ instead of V), the same argument shows that

$$\begin{aligned} P_{L(t)}(\lambda) &= \det(\lambda\mathbb{1} - U(t)L(0)U(t)^{-1}) \\ &= \det(\lambda\mathbb{1} - L(0)) \\ &= P_{L(0)}(\lambda) \end{aligned} \tag{13.20}$$

which implies that the eigenvalues λ_i of $L(t)$ are independent of time, as claimed.

Equivalently, we can take the n conserved quantities to be the coefficients c_k of the characteristic polynomial

$$c_k = \sum_{1\leqslant i_1 < i_2 \cdots < i_k \leqslant n} \lambda_{i_1}\lambda_{i_2}\ldots\lambda_{i_k}\,, \quad k = 1\ldots n\,, \tag{13.21}$$

or as

$$s_k = \sum_{i=1}^{n} \lambda_i^k = \mathrm{tr}(L^k)\,, \quad k = 1\ldots n\,. \tag{13.22}$$

Note that the conservation of s_k can be proved directly, taking d/dt of $\mathrm{tr}(L^k)$, expanding out, and using the Lax pair and then the cyclic property of the trace.

As a final remark about the general formalism, note that the eigenvalue equation for $L(t)$, namely

$$L(t)\psi(t) = \lambda\psi(t) \tag{13.23}$$

is solved formally by

$$\psi(t) = U(t)\psi(0) \tag{13.24}$$

where $\psi(0)$ is an eigenfunction at $t = 0$:

$$\begin{aligned} L(t)\psi(t) &= U(t)L(0)U(t)^{-1}\psi(t) \\ &= U(t)L(0)U(t)^{-1}U(t)\psi(0) \\ &= U(t)L(0)\psi(0) \\ &= U(t)\lambda\psi(0) \\ &= \lambda U(t)\psi(0) \\ &= \lambda\psi(t)\,. \end{aligned} \tag{13.25}$$

13.1 The Lax pair for the simple harmonic oscillator

The Hamiltonian for the S.H.O. (which has $n = 1$) is

$$H(q,p) = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 q^2 \,. \tag{13.26}$$

Hamilton's equations are then

$$\boxed{\dot{q} = \frac{p}{m}\,, \quad \dot{p} = -m\omega^2 q\,.} \tag{13.27}$$

These equations are equivalent to a Lax equation of the form (13.13) with

$$L = \begin{pmatrix} p & m\omega q \\ m\omega q & -p \end{pmatrix}\,, \quad M = \frac{\omega}{2}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\,. \tag{13.28}$$

Indeed

$$\dot{L} = \begin{pmatrix} \dot{p} & m\omega\dot{q} \\ m\omega\dot{q} & -\dot{p} \end{pmatrix}\,, \quad [M,L] = \begin{pmatrix} -m\omega^2 q & \omega p \\ \omega p & m\omega^2 q \end{pmatrix} \tag{13.29}$$

and so $\dot{L} = [M,L] \quad \leftrightarrow \quad$ (13.27).

• Since in this case M is independent of t, the time evolution operator defined by (13.15) is simply

$$\boxed{U(t) = e^{Mt}} \tag{13.30}$$

where the exponential of the matrix Mt is defined by its Taylor expansion:

$$e^{Mt} = \sum_{n=0}^{\infty} \frac{t^n}{n!}\, M^n \,. \tag{13.31}$$

This can be calculated explicitly, noting that

$$M^2 = -(\frac{\omega}{2})^2\, \mathbb{1}$$

and so

$$M^{2k} = (-1)^k (\frac{\omega}{2})^{2k}\, \mathbb{1}\,, \quad M^{2k+1} = (-1)^k (\frac{\omega}{2})^{2k}\, M \tag{13.32}$$

and so (splitting (13.31) into sums over even and odd terms and then spotting the Taylor series for cosine and sine)

$$U(t) = \begin{pmatrix} \cos(\omega t/2) & -\sin(\omega t/2) \\ \sin(\omega t/2) & \cos(\omega t/2) \end{pmatrix}\,. \tag{13.33}$$

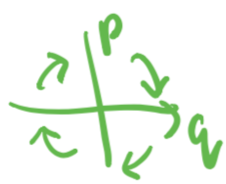
Hence

$$
\begin{aligned}
L(t) &= \begin{pmatrix} p(t) & m\omega q(t) \\ m\omega q(t) & -p(t) \end{pmatrix} = U(t)L(0)U(t)^{-1} \\
&= \begin{pmatrix} \cos(\omega t/2) & -\sin(\omega t/2) \\ \sin(\omega t/2) & \cos(\omega t/2) \end{pmatrix} \begin{pmatrix} p(0) & m\omega q(0) \\ m\omega q(0) & -p(0) \end{pmatrix} \begin{pmatrix} \cos(\omega t/2) & \sin(\omega t/2) \\ -\sin(\omega t/2) & \cos(\omega t/2) \end{pmatrix} \\
&= \dots \\
&= \begin{pmatrix} p(0)\cos(\omega t) - m\omega q(0)\sin(\omega t) & p(0)\sin(\omega t) + m\omega q(0)\cos(\omega t) \\ p(0)\sin(\omega t) + m\omega q(0)\cos(\omega t) & -p(0)\cos(\omega t) + m\omega q(0)\sin(\omega t) \end{pmatrix}
\end{aligned}
\tag{13.34}
$$

and hence

$$
\begin{aligned}
q(t) &= q(0)\cos(\omega t) + \frac{p(0)}{m\omega}\sin(\omega t) \\
p(t) &= p(0)\cos(\omega t) - m\omega\, q(0)\, \sin(\omega t)
\end{aligned}
\tag{13.35}
$$

This shows that, up to a scaling of the axes, the time evolution is uniform rotation in the S.H.O. phase space:



In this case $n = 1$, and there is just one nontrivial conserved quantity, which should be the Hamiltonian. Indeed $\mathrm{tr}(L) = 0$ (so this is trivially conserved) while

$$
\mathrm{tr}(L^2) = \mathrm{tr}\begin{pmatrix} p^2 + m^2\omega^2 q^2 & 0 \\ 0 & p^2 + m^2\omega^2 q^2 \end{pmatrix} = 2(p^2 + m^2\omega^2 q^2 = 4m\, H(q,p) \tag{13.36}
$$

is the only independent conserved quantity. While this case is a bit easy, it does illustrate the general point that it's simpler to work with traces of powers of the Lax matrix, rather than with the individual eigenvalues themselves.

13.2 The Lax pair for the Toda lattice

The last example was a bit trivial. Much less trivial, and still the subject of research, is the *finite Toda lattice* which describes in particles on a line, each one interacting with its nearest neighbours. Let's take the particles to have equal masses, $m_i = 1$. Toda's Hamiltonian is

$$
H(q,p) = \sum_{i=1}^{n} \left(\frac{p_i^2}{2} + e^{-(q_i - q_{i-1})} \right) \tag{13.37}
$$

where, at least at $t = 0$,

$$q_0 \equiv -\infty < q_1 < q_2 \cdots < q_n < q_{n+1} \equiv +\infty\,.$$

Hamilton's equations for this system are:

$$\boxed{\begin{aligned} \dot{q}_i &= p_i \\ \dot{p}_i &= e^{-(q_i - q_{i-1})} - e^{-(q_{i+1} - q_i)} \end{aligned}} \tag{13.38}$$

<u>Note</u> that it follows from these equations that $\frac{d}{dt}\sum_{i=1}^n p_i = 0$, so $\sum_{i=1}^n p_i = \text{constant} = P$, say, and $\frac{d}{dt}\sum_{i=1}^n q_i = P$. This in turn implies that $\sum_{i=1}^n q_i = Pt + \text{const}$, thus solving a part of the equations of motion.

The Lax pair is most simply formulated in terms of *Flaschka's variables*:

$$\boxed{a_i = \tfrac{1}{2}\, e^{-(q_{i+1} - q_i)/2}\,, \qquad b_i = -\tfrac{1}{2}\, p_i} \tag{13.39}$$

which satisfy

$$\begin{aligned} \dot{a}_i &= \tfrac{1}{4}\, e^{-(q_{i+1} - q_i)/2}(p_{i+1} - p_i) = a_i(b_{i+1} - b_i) \\ \dot{b}_i &= -\tfrac{1}{2}\left(e^{-(q_i - q_{i-1})} - e^{-(q_{i+1} - q_i)}\right) = 2(a_i^2 - a_{i-1}^2) \end{aligned} \tag{13.40}$$

(It might be objected that Flaschka's variables only encode the differences of the q_is, but given the <u>note</u> above, we already know their overall sum, so the differences are all that we need.)

Then the Lax pair is

$$\begin{aligned} L &= \begin{pmatrix} b_1 & a_1 & & & & \\ a_1 & b_2 & a_2 & & & \\ & a_2 & b_3 & a_3 & & \\ & & \ddots & \ddots & \ddots & \\ & & & a_{n-2} & b_{n-1} & a_{n-1} \\ & & & & a_{n-1} & b_n \end{pmatrix} \\ M &= \begin{pmatrix} 0 & a_1 & & & & \\ -a_1 & 0 & a_2 & & & \\ & -a_2 & 0 & a_3 & & \\ & & \ddots & \ddots & \ddots & \\ & & & -a_{n-2} & 0 & a_{n-1} \\ & & & & -a_{n-1} & 0 \end{pmatrix} \end{aligned} \tag{13.41}$$

(Exercise: check for yourself that $\dot{L} = [M, L] \Rightarrow$ (13.40).)

This implies that the eigenvalues of L, or equivalently the traces of the powers of L, are all conserved! This gives us n conserved quantities,

$$\boxed{Q_k = \mathrm{tr}(L^k)\,, \quad k = 1 \ldots n\,.} \tag{13.42}$$

The first few are

$$\begin{aligned}
Q_1 &= \mathrm{tr}(L) \\
&= \sum_{i=1}^{n} b_i = -\tfrac{1}{2}\sum_{i=1}^{n} p_i \qquad \text{(total momentum)} \\
Q_2 &= \mathrm{tr}(L^2) \\
&= \sum_{i=1}^{n} b_i^2 + 2\sum_{i=1}^{n-1} a_i^2 \\
&= \tfrac{1}{2}\left(\tfrac{1}{2}\sum_{i=1}^{n} p_i^2 + \sum_{i=1}^{n-1} e^{-(q_{i+1}-q_i)}\right) \qquad \text{(the Hamiltonian, or total energy)} \\
Q_3 &= \mathrm{tr}(L^3) \\
&= \sum_{i=1}^{n} b_i^3 + 3\sum_{i=1}^{n-1} a_i^2(b_i + b_{i+1}) \\
&= \tfrac{1}{8}\left(\sum_{i=1}^{n} p_i^3 - 3\sum_{i=1}^{n-1} e^{-(q_{i+1}-q_i)}(p_i + p_{i+1})\right)
\end{aligned} \tag{13.43}$$

Interestingly, the limit $n \to \infty$ yields the *infinite Toda lattice*, which describes an infinite number of particles on a line, and this system has solitons.

The index $i \in \mathbb{Z}$ for the infinite Toda lattice is analogous to $x \in \mathbb{R}$ for KdV, while $q_i(t) \in \mathbb{R}$ corresponds to $u(x, t) \in \mathbb{R}$. Thus space has been discretised, while time remains continuous, as does the field value. (In the ball and box model the process of discretisation goes two steps further, with both time and the field values also becoming discrete.)

The solitons of the infinite Toda lattice can be derived in a number of ways, including inverse scattering. The following turns out to be a solution, for any γ, $k > 0$:

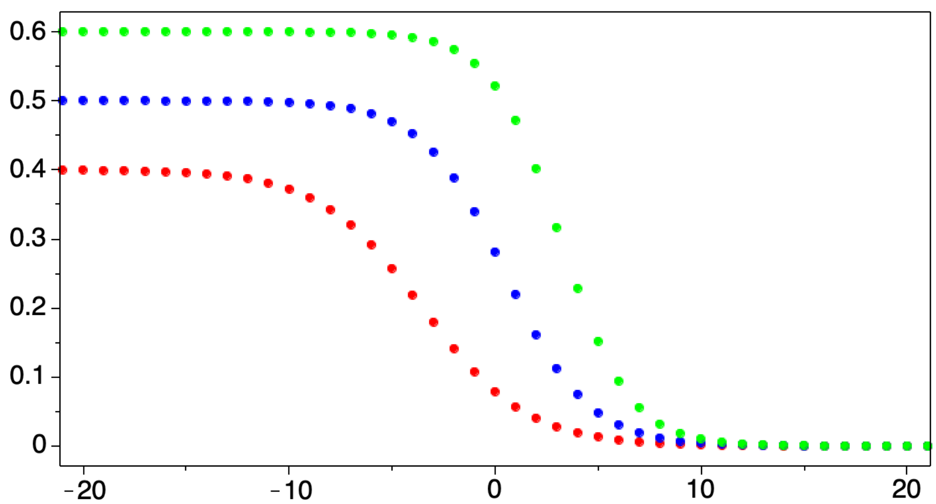
$$\boxed{q_l(t) = q_0 - \log \frac{1 + \gamma\, e^{-2kl \pm 2\sinh(k)t}}{1 + \gamma\, e^{-2k(l-1) \pm 2\sinh(k)t}}} \tag{13.44}$$

This is a single soliton moving through $\mathbb{Z}$ with

$$\begin{aligned} \text{velocity} &= \pm \sinh(k)/k\,, \\ \text{width} &\sim 1/k\,. \end{aligned} \tag{13.45}$$

As for KdV, the faster a soliton is moving, the narrower it becomes.

Here's a plot comparing three of these solitons at $t = 0$, taking the '+' option with $q_0 = 0$ in (13.44), with $(k, \gamma) = (0.2, 0.2)$ (red), $(k, \gamma) = (0.25, 1)$ (blue) and $(k, \gamma) = (0.3, 5)$ (green):



Note that the horizontal axis here is the index l, while in the sketch between equations (13.37) and (13.38) it was the 'field value' q_l.

It is also possible to find N-soliton solutions, which turn out to have a form similar to those we found earlier for the KdV equation:

$$\boxed{q_l(t) = q_0 - \log \frac{\det(\mathbb{1}_{N\times N} + C_l(t))}{\det(\mathbb{1}_{N\times N} + C_{l-1}(t))}} \tag{13.46}$$

where $\mathbb{1}_{N\times N}$ is the $N \times N$ identity matrix, and $\{C_l(t)\}$ is a family of $N \times N$ matrices depending on the space coordinate l and the time coordinate t as follows:

$$\boxed{(C_l(t))_{ij} = \frac{\sqrt{\gamma_i \gamma_j}}{1 - e^{-(k_i+k_j)}}\, e^{-(k_i+k_j)l-(\sigma_i \sinh(k_i)+\sigma_j \sinh(k_j))t}} \tag{13.47}$$

with $k_i, \gamma_i > 0$ and $\sigma_i = \pm 1$.

The end