Chapter 7

Overview of the inverse scattering method

7.1 Initial value problems

So far, we have seen methods to construct particular solutions.

Question: can we find a *general* solution to these p.d.e.s?

In more detail: given a wave equation and 'enough' initial data at $t = 0$, find $u(x, t)$ at all later times $t > 0$. For there to be a unique solution, sufficient initial data must be given.

• If p.d.e. is 1st order in time, (eg KdV) must specify $u(x, 0)$

• If 2nd order (eg sine-Gordon), need $u(x, 0)$ and $u_t(x, 0)$

• etc.

[why? because we can use the p.d.e. to solve for higher t derivatives. Eg for KdV, if I tell you $u(x, 0)$, you can use the p.d.e. to find out what $u_t(x, 0)$ must be – it's not independent data.]

But given that information, can we construct $u(x, t)$ for all $t > 0$? (analytically if possible). So far, the answer is no, unless the initial condition happens to be a snapshot of one of the special solutions seen before.

Eg in KdV, what if

(a) $u(x, 0) = 2\,\text{sech}^2(x)$
(b) $u(x, 0) = 2.1\,\text{sech}^2(x)$
(c) $u(x, 0) = 6\,\text{sech}^2(x)$?

Case (a) is a snapshot of a one-soliton solution at $t = 0$, so, assuming the uniqueness of solutions, the answer to (a) at all later times is

$$u(x, t > 0) = 2\text{sech}^2(x - 4t)\,.$$

But what about (b) and (c)?

It turns out that

(b) $\rightarrow$ { 2 solitons, 1 very small, both moving right, + some junk moving left }
(c) $\rightarrow$ { 2 solitons, both moving right, and that's all }

[so in fact, the initial condition for (c) is a snapshot of a "pure" 2-soliton solution]

Inverse scattering will allow us to understand situations like (b), and give a much more complete understanding of when things like (a) and (c) occur. In fact (as you might remember seeing "experimentally" at the start of last term) whenever the height is $N(N+1)$, $N = 1$, 2, 3…we are in a situation like (a) or (c)… but why?

Inverse scattering gives analytic insight into this question.

How might this go?

7.2 Linear initial value problems

For a linear wave equation, the general solution is a linear transformation of the initial data.

Examples

1. The heat equation

$$u_t + u_{xx} = 0\,, \quad -\infty < x < \infty, t > 0\,. \tag{7.1}$$

Given $u(x,0) \equiv u_0(x)$ (the initial data), $u(x,t)$ is

$$u(x,t) = \int_{-\infty}^{+\infty} \frac{1}{\sqrt{4\pi t}}\, e^{-(x-x')^2/(4t)} u_0(x')\, dx' \tag{7.2}$$

and this *is* a linear transform of $u_0(x)$ (it's actually a "Green's function" solution).

2. The Klein-Gordon equation

$$u_{tt} - u_{xx} + u = 0 \tag{7.3}$$

This is second-order in t, so we need to specify $u(x,0)$ ***and*** $u_t(x,0)$:

$$u(x,0) = \alpha(x)\,, \qquad u_t(x,0) = \beta(x)\,. \tag{7.4}$$

With luck, { (7.3) + (7.4) } is a "good" initial value problem.

It can be solved using a Fourier transform, which is like the Fourier series seen in AMV, but for functions on a infinite line.

Given $u(x,t)$, set

$$\begin{aligned} \widehat{u}(k,t) &= \int_{-\infty}^{+\infty} dx\, e^{-ikx} u(x,t) \\ u(x,t) &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} dk\, e^{+ikx} \widehat{u}(k,t) \end{aligned} \tag{7.5}$$

where the second equation shows how to get u back from $\widehat{u}$.

Working with $\widehat{u}(k,t)$ instead of $u(x,t)$ is a good move, because (7.3) for u implies

$$\widehat{u}_{tt} + (k^2+1)\widehat{u} = 0 \tag{7.6}$$

for $\widehat{u}$, and this equation is easier to solve – there are only t derivatives, so it can be treated as an *ordinary* differential equation rather than a partial one.

Solving (7.6),

$$\widehat{u}(k,t) = A(k)\, e^{i\omega t} + B(k)\, e^{-i\omega t} \tag{7.7}$$

where $\omega^2 = k^2 + 1$, and A and B can be fixed by matching with the initial condition at $t = 0$:

$$\begin{aligned} \widehat{u}(k,0) &= A(k) + B(k) = \widehat{\alpha}(k) \\ \widehat{u}_t(k,0) &= i\omega(A(k) - B(k)) = \widehat{\beta}(k)\,. \end{aligned} \tag{7.8}$$

Solving for A and B and simplifying the resulting expression for $\widehat{u}(k,t)$,

$$\widehat{u}(k,t) = \widehat{\alpha}(k)\cos\omega t + \frac{1}{\omega}\widehat{\beta}(k)\sin\omega t\,. \tag{7.9}$$

Finally, a reverse Fourier transform allows $u(x,t)$ to be found:

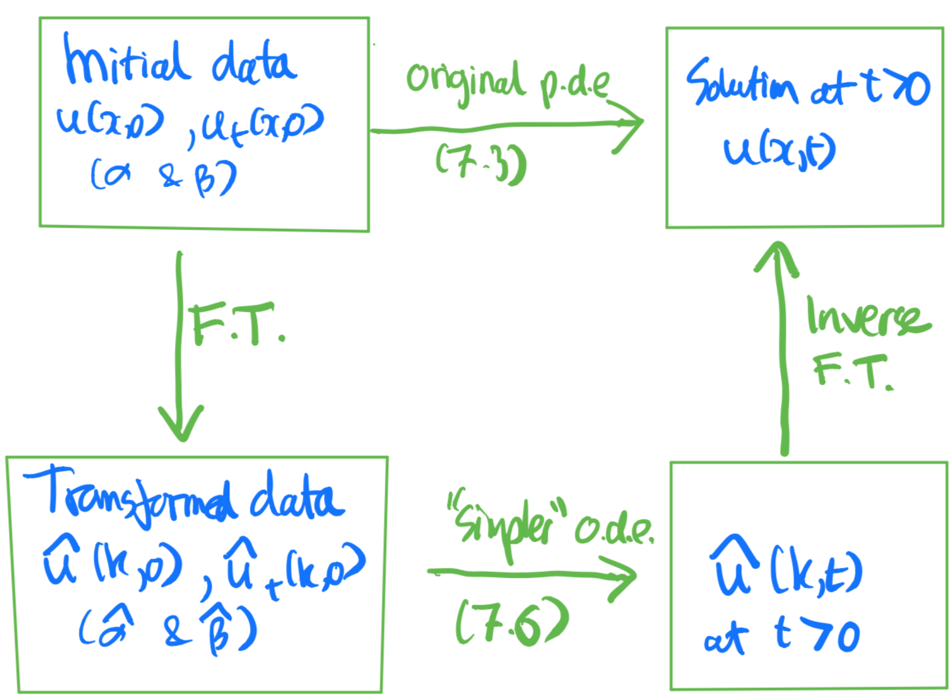
$$\begin{aligned} u(x,t) &= \frac{1}{2\pi}\int_{-\infty}^{+\infty}\widehat{u}(k,t)\,e^{ikx}\,dk \\ &= \ldots \\ &= \frac{1}{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{ik(x-x')}\left(u(x',0)\cos\omega t + \frac{1}{\omega}u_t(x',0)\sin\omega t\right)dx'dk \end{aligned} \tag{7.10}$$

with $\omega = \sqrt{k^2+1}$.

Again, this is a *linear* function of $u(x,0)$ and $u_t(x,0)$, the initial data [this won't be true for KdV].

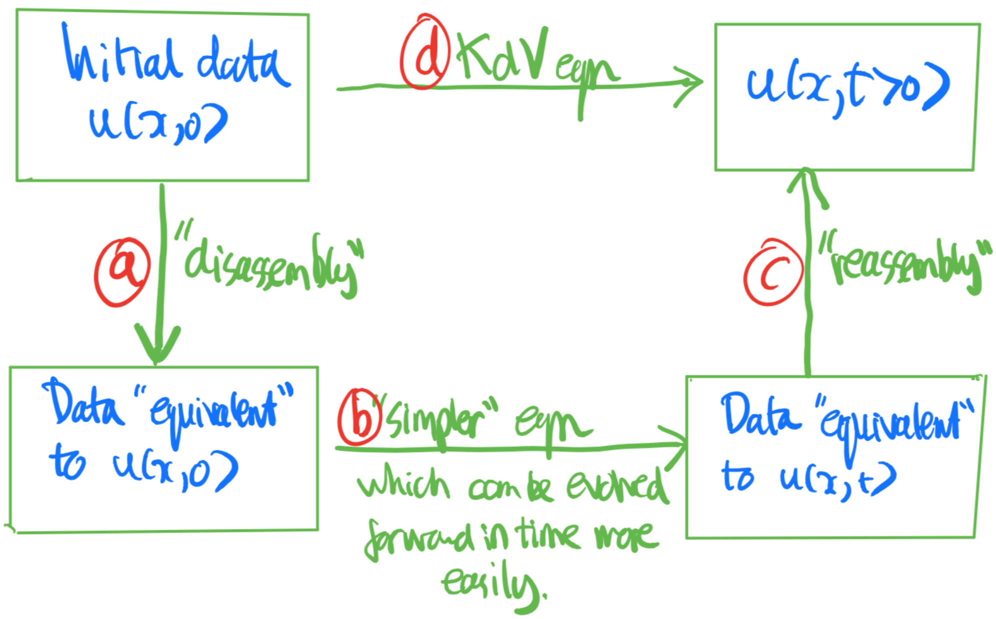
Key feature: the data for each value of k evolved separately, in a simple way, in the "transformed" equation (7.6) [something like this *will* be true for KdV].

Summarising, the general picture for Klein-Gordon is:



This will turn out to be the correct "big idea" for KdV also, but in a much more subtle way since KdV is nonlinear.

Map of the general strategy for KdV:



Instead of doing step (d) directly, we will go the roundabout route of (a) → (b) → (c).

This will be a long story, so it will be good to keep this "roadmap" in mind as we go, starting with step (a).