Chapter 8

The KdV-Schrödinger connection

We'll follow the route of the original discoverers of the method, Gardner, Greene, Kruskal and Miura (GGKM), in the late 1960s. Their aim was to solve the KdV equation

$$u_t + 6uu_x + u_{xxx} = 0 \tag{8.1}$$

for $t > 0$ on $-\infty < x < \infty$, with the initial condition

$$u(x, 0) = f(x)\,, \qquad -\infty < x < \infty\,.$$

Recall first the (generalised) Miura transformation: if $v(x, t)$ solves

$$v_t + 6(\lambda - v^2)v_x + v_{xxx} = 0 \tag{8.2a}$$

then

$$u = \lambda - v^2 - v_x \tag{8.2b}$$

solves the KdV equation (8.1). Now think about this backwards: take u to be known, and try to solve (8.2b) for v. There's a standard trick for this: write

$$v = \psi_x/\psi$$

for some other function ψ, and try to find ψ first. With a small amount of rearrangement (8.2b) becomes

$$\psi_{xx} + u\psi = \lambda\psi\,. \tag{8.3}$$

Now (8.3) is interesting (and this is what attracted GGKM's attention) because it's a well-known equation: the time-independent Schrödinger equation, the quantum-mechanical equation for a particle moving in a potential $V(x) = -u(x)$.

The QM interpretation of the equation won't be too important here, apart from the fact that a great deal was known about its solutions, and GGKM were able to exploit this.

The important thing is that any field profile u can be associated with another function ψ by solving (8.3), which is sometimes called the *associated linear problem.*

Some key facts:

• t appears in (8.3) only as a parameter, in $u(x,t)$;

• for any t, it turns out to be possible to reconstruct u from a limited amount of information about ψ at the different values of λ. This information is called the *scattering data* ;

• it further turns out that, if $u(x,t)$ evolves by the KdV equation, then the scattering data evolves in a particularly simple way.

The scattering data is "like" $\widehat{\alpha}$ and $\widehat{\beta}$ in the linear (Klein-Gordon) case, while λ is like k.

Finding the scattering data for the initial configuration $u(x,0)$ constitutes the 'disassembly' step (a); we'll come back to it later. But first we turn to step (b), the time dependence, using an idea due to Peter Lax.