Chapter 9

Time evolution of the scattering data

9.1 The idea of a Lax pair

We'll think of $\psi_{xx} + u\psi = \lambda\psi$ at some fixed time t as an *eigenvalue problem*:

$$L(u)\psi = \lambda\psi \tag{9.1}$$

where $L(u)$ is the following differential operator:

$$L(u) = \frac{d^2}{dx^2} + u(x,t)\,. \tag{9.2}$$

Notes:

1. You should think of differential operators such as L, or d/dx, or whatever, as acting on all functions sitting to their right.

2. (9.1) *does* pick out "special" values of λ (the *eigenvalues*) since we require that $\psi(x)$ is square integrable (ie $\int_{-\infty}^{+\infty} |\psi(x)|^2 dx < \infty$) which in particular means $\psi(x) \to 0$ both as $x \to -\infty$ and as $x \to +\infty$. [Later, we will relax this a little to allow solutions ψ that are merely *bounded*, but for now we will require that the stronger condition holds.]

3. The "t" in (9.2) has nothing to do with the time in the time-dependent Schrödinger equation you might see in quantum mechanics; rather, it's the KdV time.

4. Since L depends on u, and u depends on t, the *eigenfunctions* ψ and (in principle) the *eigenvalues* λ might be different at different times.

But, we have two remarkable facts:

THEOREM:
(i) If $u = u(x,t)$ evolves by the KdV equation, then the set of eigenvalues $\{\lambda\}$ of $L(u)$ (the *spectrum* of $L(u)$) is independent of t;
(ii) There is a set of eigenfunctions ψ of $L(u)$ which evolves in t simply, as $\psi_t = B(u)\psi$, where $B(u)$ is another differential operator.

The result (i) is particularly striking – it says that the spectra of $d^2/dx^2 + u(x,0)$ and $d^2/dx^2 + u(x,t)$ are the same, which is very unexpected since $u(x,0)$ and $u(x,t)$ might look very different.

PROOF:

First, we'll assume that a $B(u)$ can be found such that the time evolution of $L(u(x,t))$ is given by

$$\begin{aligned} L(u)_t &= B(u)L(u) - L(u)B(u) \\ &= [B(u), L(u)] \end{aligned} \tag{9.3}$$

when u evolves by KdV (we'll find B later).

Here, $[B,L] := BL - BL$ is called the *commutator* of the two operators B and L. Since B and L can both involve x derivatives, $BL \neq LB$ is possible – see later for examples.

Now let λ and ψ be an eigenvalue and eigenfunction of L, so that $L\psi = \lambda\psi$. Taking $\partial/\partial t$ of this equation,

$$L_t\psi + L\psi_t = \lambda_t\psi + \lambda\psi_t\,.$$

Rearranging,

$$\begin{aligned} \lambda_t\psi &= \lambda_t\psi + L\psi_t - \lambda\psi_t \\ &= (BL - LB)\psi + (L-\lambda)\psi_t \qquad \text{(using (9.3))} \\ &= (B\lambda - LB)\psi + (L-\lambda)\psi_t \qquad \text{(using } L\psi = \lambda\psi\text{)} \\ &= (L-\lambda)\big(\psi_t - B\psi\big) \end{aligned}$$

Now multiply both sides by $\psi(x)^*$ and integrate $\int_{-\infty}^{+\infty} dx$ to find

$$\lambda_t \int_{-\infty}^{+\infty} |\psi|^2 dx = \int_{-\infty}^{+\infty} \psi(x)^*(L-\lambda)\big(\psi_t - B\psi\big)\psi(x)\,dx\,. \tag{*}$$

(Note, the integral on the LHS of this equation is finite since that was part of the specification of the eigenvalue problem.)

Now we'll use a key property of L: for *any* pair of functions ϕ and χ, both tending to zero at $x = \pm\infty$,

$$\int_{-\infty}^{+\infty} \phi(x)^* \, L\, \chi(x)\, dx = \int_{-\infty}^{+\infty} (L\phi(x))^* \, \chi(x)\, dx\,.$$

Such an L is called self-adjoint; more on this in the next section.

Proof of the key property: Recall $L = d^2/dx^2 + u(x)$ and $u(x)$ is real. Then compute

$$\int \phi^* L\chi \, dx = \int \phi^*(x) \frac{d^2}{dx^2}\chi(x) + \phi^*(x)u(x)\chi(x)\, dx$$

$$= \int \Big(\frac{d^2}{dx^2}\phi^*(x)\Big)\chi(x) + \big(u(x)\phi(x)\big)^*\chi(x)\, dx$$

(integrating by parts twice for the first term, and using reality of u for the second)

$$= \int \big(L\phi(x)\big)^*\chi(x)\, dx$$

as required. □

Since λ is real (the proof of this fact is left as an exercise!) the key property also holds for $L - \lambda$. Thus the earlier result (*) can be rewritten as

$$\lambda_t \int_{-\infty}^{+\infty} |\psi|^2 dx = \int_{-\infty}^{+\infty} \big((L-\lambda)\psi\big)^* \big(\psi_t - B\psi\big)\psi(x)\, dx\,.$$

But $L\psi = \lambda\psi$, so $(L-\lambda)\psi = 0$, and the RHS of this equation is zero. Since $\int |\psi|^2 dx$ is finite and nonzero, we deduce $\lambda_t = 0$, which is result (i). □

For result (ii), we need to show that $(L-\lambda)\psi = 0$ continues to be true if ψ changes according to $\psi_t = B\psi$. Calculating,

$$\begin{aligned}
\frac{\partial}{\partial t}\big((L-\lambda)\psi\big) &= L_t\psi + L\psi_t - \lambda_t\psi - \lambda\psi_t \\
&= L_t\psi + L\psi_t - \lambda\psi_t \qquad \text{(since we already know } \lambda_t = 0\text{)} \\
&= L_t\psi + LB\psi - \lambda B\psi \qquad \text{(using } \psi_t = B\psi\text{)} \\
&= L_t\psi + LB\psi - B\lambda\psi \qquad \text{(since } \lambda \text{ is a number)} \\
&= L_t\psi + LB\psi - BL\psi \qquad \text{(using } L\psi = \lambda\psi\text{)} \\
&= \big(L_t - [B,L]\big)\psi \\
&= 0 \qquad \text{(using (9.3))}
\end{aligned}$$

This shows that if $\psi_t = B\psi$ and ψ starts of at $t = 0$ as an eigenfunction, then it stays that way, which is result (ii). □

L and B are called a *Lax pair*. All that remains now is to find a suitable $B(u)$.

9.2 The Lax pair for KdV

We've already decided that $L(u) = \frac{d^2}{dx^2} + u(x,t)$. For now we'll just make an inspired guess for $B(u)$, and check that it works; in the next chapter a more systematic approach will be explained. The guess is to set

$$B(u) = -\left(4D^3 + 6uD + 3u_x\right) \tag{9.4}$$

where to save ink the notation $D \equiv \frac{d}{dx}$, $D^2 \equiv \frac{d^2}{dx^2}$, etc has been adopted.

Notice that operators like D act on *everything* to their right, and that differential operators are defined by their actions on functions. So for example $[u, D]$ is defined by how it would act on any function $f(x)$. Calculating,

$$\begin{aligned}
[u, D]f &= \left(u\frac{d}{dx} - \frac{d}{dx}u\right) f \\
&= u\frac{df}{dx} - \frac{d}{dx}(uf) \\
&= u\frac{df}{dx} - \left(\frac{du}{dx}\right)f - u\frac{df}{dx} \qquad \text{(using the product rule)} \\
&= -\left(\frac{du}{dx}\right)f
\end{aligned}$$

Thus the effect of $[u, D]$ on $f(x)$ is to multiply it by $-u_x(x)$. Since this is true for all functions $f(x)$ we have that $[u, D] = -u_x$ as an identity between differential operators. Perhaps more usefully, this can be rephrased as

$$Du = uD + u_x$$

which shows how to "shuffle" Ds past other functions. This can be used to rewrite expressions in a form where all Ds are on the far right in all terms, making cancellations easier to spot.

Now just calculate! We have

$$L = D^2 + u\,, \quad B = -(4D^3 + 6uD + 3u_x)$$

so

$$\begin{aligned}
LB &= -D^2(4D^3 + 6uD + 3u_x) - u(4D^3 + 6uD + 3u_x) \\
&= \ldots \\
&= -\big(4D^5 + 6uD^3 + 12u_xD^2 + 6u_{xx}D + 3u_xD^2 \\
&\qquad + 6u_{xx}D + 3u_{xxx} + 4uD^3 + 6u^2D + 3uu_x\big)
\end{aligned}$$

while

$$\begin{aligned} BL &= -4D^3(D^2+u) - 6uD(D^2+u) - 3u_x(D^2+u) \\ &= \dots \\ &= -(4D^5 + 4uD^3 + 12u_xD^2 + 12u_{xx}D + 4u_{xxx} \\ &\qquad + 6uD^3 + 6u^2D + 6uu_x \\ &\qquad + 3u_xD^2 + 3uu_x)\,. \end{aligned}$$

Hence

$$LB - BL = u_{xxx} + 6uu_x$$

and somewhat surprisingly all of the Ds have gone.

Also, $L_t = u_t$ and so

$$L_t + [L, B] = u_t + 6uu_x + u_{xxx}$$

which is zero, as required, if $u(x, t)$ satisfies the KdV equation.

This completes the proof of properties (i) and (ii) of the associated linear problem for solutions of the KdV equation. $\square$

Notes

1. L and B were both *differential* operators, since they involved $D = \frac{d}{dx}$, but in some senses $[L, B]$ wasn't: $[L, B]$ acting on some function $f(x)$ doesn't do any differentiating, but just multiplies f pointwise by $(u_{xxx} + 6uu_x)$. For this reason $[L, B]$ is called *multiplicative*.

2. The equation for the time evolution of ψ, $\psi_t = B(u)\psi$, is linear (good news!), but since B depends on $u(x, t)$, the thing we're trying to find, it's not yet clear we have made too much progress on step (b) (bad news). We will fix this later, once we have developed a better understanding of the scattering data. But first, a diversion to explore other options for $B(u)$…