45. **Note:** In this and subsequent exercises the Fourier transform will be denoted as $\mathbf{F}[f(x)] = \widehat{f}(k)$, where $\mathbf{F}[f(x)] = \widehat{f}(k) = \int_{-\infty}^{\infty} dx\, e^{-ikx} f(x)$ and $f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\, e^{ikx}\widehat{f}(k)$ . You can use results from the Fourier transform handout such as $\delta(y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dz\, e^{iyz}$ without proof.

Some properties of Fourier transforms:

(a) The *convolution* of $f$ and $g$ is defined as

$$(f * g)(x) = \int_{-\infty}^{\infty} dz\ f(z)\, g(x-z)\ .$$

Prove that $\mathbf{F}[fg] = \frac{1}{2\pi}\widehat{f}(k) * \widehat{g}(k)$ and $\mathbf{F}[f * g] = \widehat{f}(k)\widehat{g}(k)$.

(b) The *cross-correlation* of $f$ and $g$ is defined as

$$(f \otimes g)(x) = \int_{-\infty}^{\infty} dz\ f^*(z)\, g(x+z)\ .$$

Prove the *Weiner-Kinchin theorem,* that $\mathbf{F}[f \otimes g] = \widehat{f^*}(k)\widehat{g}(k)$.

(c) The *auto-correlation* of $f(x)$ is defined as

$$a(x) = (f \otimes f)(x).$$

Using the answer to part b, verify that $\mathbf{F}[a] = |\widehat{f}(k)|^2$. This is called the *energy spectrum* of $f$.

(d) Prove the FT version of *Parseval's theorem*, which you may have already seen for Fourier series:

$$\int_{-\infty}^{\infty} dx\, |f(x)|^2 = \int_{-\infty}^{\infty} \frac{dk}{2\pi}\, |\widehat{f}(k)|^2\ .$$

(Strictly speaking this is *Plancherel's theorem*; Parseval allows for two different functions $f$ and $g$ and turns into Plancherel when $f = g$.)

The locations of the factors of $2\pi$ in these formulae depend on the conventions used for the Fourier transform and its inverse, so they might look a little different in some textbooks.

**Solution** To save space all integrals will henceforth be assumed to run from $-\infty$ to $\infty$ unless otherwise stated.

(a) Convolution: $(f * g)(x) = \int dz\ f(z)\, g(x-z)$ .
We have

$$\begin{aligned}
\mathbf{F}[fg] &= \int dx\, e^{-ikx} f(x)g(x) \\
&= \int dx \iint \frac{dk_1 dk_2}{(2\pi)^2}\, e^{-i(k-k_1-k_2)x}\widehat{f}(k_1)\widehat{g}(k_2) \\
&= \iint \frac{dk_1 dk_2}{2\pi}\, \delta(k-k_1-k_2)\widehat{f}(k_1)\widehat{g}(k_2) \quad \text{(doing the } \textstyle\int dx) \\
&= \frac{1}{2\pi}\int dk_1\, \widehat{f}(k_1)\widehat{g}(k-k_1) = \frac{1}{2\pi}(\widehat{f} * \widehat{g})(k)\,.
\end{aligned}$$

and

$$\begin{aligned}
\mathbf{F}[f*g] &= \int dx\, e^{-ikx} \int dz\, f(z)g(x-z) \\
&= \int dx\, e^{-ikx} \int dz \iint \frac{dk_1 dk_2}{(2\pi)^2}\, e^{ik_1 z + ik_2(x-z)} \widehat{f}(k_1)\widehat{g}(k_2) \\
&= \int dz \iint \frac{dk_1 dk_2}{2\pi}\, e^{i(k_1-k_2)z} \delta(k_2-k)\widehat{f}(k_1)\widehat{g}(k_2) \quad \text{(doing the } \textstyle\int dx) \\
&= \iint dk_1 dk_2\, \delta(k_1-k_2)\delta(k_2-k)\widehat{f}(k_1)\widehat{g}(k_2) \quad \text{(doing the } \textstyle\int dz) \\
&= \widehat{f}(k)\widehat{g}(k)\,.
\end{aligned}$$

(b) Cross-correlation: $(f \otimes g)(x) = \int dz\, f^*(z)\, g(x+z)$ .
Note that $(f \otimes g)(x) = \int dy\, f^*(-y)\, g(x-y)$, which is the convolution of $f^*(-x)$ and $g(x)$. So if we can show that $\mathbf{F}[f^*(-x)](k) = \widehat{f}^*(k)$ it will follow from part (a) that $\mathbf{F}[f \otimes g] = \widehat{f}^*(k)\widehat{g}(k)$. This holds since

$$\begin{aligned}
\mathbf{F}[f^*(-x)](k) &= \int dx\, f^*(-x)e^{-ikx} \\
&= \left(\int dx\, f(-x)e^{ikx}\right)^* \\
&= \left(\int dy\, f(y)e^{-iky}\right)^* \quad \text{(substituting } y=-x) \\
&= \widehat{f}^*(k)
\end{aligned}$$

(c) Auto-correlation: $a(x) = (f \otimes f)(x)$.
Using part (b), $\mathbf{F}[a] = \mathbf{F}[f \otimes f] = \widehat{f}^*(k)\widehat{f}(k) = |\widehat{f}(k)|^2$.

(d) Parseval's theorem: $\int dx\, |f(x)|^2 = \frac{1}{2\pi} \int dk\, |\widehat{f}(k)|^2$ .

$$\begin{aligned}
\int dx\, |f(x)|^2 &= \int dx \iint \frac{dk_1 dk_2}{(2\pi)^2}\, e^{-i(k_1-k_2)x} \widehat{f}^*(k_1)\widehat{f}(k_2) \\
&= \iint \frac{dk_1 dk_2}{2\pi}\, \delta(k_1-k_2)\widehat{f}^*(k_1)\widehat{f}(k_2) \quad \text{(doing the } \textstyle\int dz) \\
&= \int \frac{dk_1}{2\pi}\, \widehat{f}^*(k_1)\widehat{f}(k_1) = \int \frac{dk}{2\pi}\, |\widehat{f}(k)|^2\,.
\end{aligned}$$

46. Examples of Fourier transforms:

(a) Show that $e^{-x^2/2}$ is (up to a factor of $\sqrt{2\pi}$) its own FT.

(b) Find the FT of

$$f(x) = \begin{cases} 1/(2\varepsilon) & |x| \le \varepsilon \\ 0 & |x| > \varepsilon \end{cases}$$

and discuss the $\varepsilon \to 0$ limit.

(c) Find the FT of

$$f(x) = \begin{cases} 1 - x^2 & |x| < 1 \\ 0 & |x| > 1 \end{cases}.$$

**Solution**

(a)

$$\begin{aligned}
\mathbf{F}[e^{-x^2/2}](k) &= \int dx\, e^{-ikx} e^{-x^2/2} \\
&= \int dx\, e^{-(x^2+2ikx)/2} \\
&= \int dx\, e^{-((x+ik)^2+k^2)/2} \quad \text{(completing the square)} \\
&= e^{-k^2/2} \int dx\, e^{-(x+ik)^2/2} \\
&= e^{-k^2/2} \int dx\, e^{-x^2/2} \quad \text{(shifting } x \to x+ik \text{ as on the integrals sheet)} \\
&= \sqrt{2\pi}\, e^{-k^2/2} \quad \text{(using the definite integral from the integrals sheet).}
\end{aligned}$$

(b)

$$\begin{aligned}
\mathbf{F}[f(x)](k) &= \int_{-\epsilon}^{\epsilon} dx\, e^{-ikx} \frac{1}{2\epsilon} \\
&= \left[ e^{-ikx} \frac{-1}{2ik\epsilon} \right]_{-\epsilon}^{\epsilon} \\
&= \frac{1}{\epsilon k} \sin(k\epsilon)
\end{aligned}$$

As $\epsilon \to 0$ with $k$ fixed this tends to $1$, a constant. Now consider the Fourier transform of a Dirac delta function: $\mathbf{F}[\delta(x)](k) = \int dx\, e^{-ikx}\delta(x) = 1$ – it's the same! If you think about the shape of the original function $f(x)$ in the limit, this might seem reasonable.

(c) In this case $\mathbf{F}[f(x)](k) = \int_{-1}^{1} dx\, e^{-ikx}(1 - x^2)$. As a shortcut which avoids integrating by parts, define

$$I(k) = \int_{-1}^{1} dx\, e^{-ikx} = \frac{1}{-ik} \left[ e^{-ikx} \right]_{-1}^{1} = \frac{2}{k} \sin(k)$$

and notice, differentiating inside the integral for the first equality, that

$$\frac{d^2}{dk^2} I(k) = -\int_{-1}^{1} dx\, x^2 e^{-ikx} = \frac{d^2}{dk^2} \left( \frac{2}{k} \sin(k) \right) = \frac{4}{k^3} \sin(k) - \frac{4}{k^2} \cos(k) - \frac{2}{k} \sin(k)\,.$$

Thus

$$\mathbf{F}[f(x)](k) = \left( I(k) + \frac{d^2}{dk^2} I(k) \right) = \frac{4}{k^3} \Big( \sin(k) - k \cos(k) \Big)\,.$$

47. Solving the heat equation using Fourier transforms:

(a) Find the general solution of the heat equation $u_t = u_{xx}$ in the form

$$u(x,t) = \int_{-\infty}^{+\infty} dk\ \widehat{u}(k,0) f(k,x,t)\ ,$$

where $\widehat{u}(k,0)$ is the Fourier transform of the initial condition $u(x,0)$ and $f(k,x,t)$ is a function of $k$, $x$ and $t$ that you should determine.

(b) Evaluate the previous integral over $k$ in the case where the initial condition is $u(x,0) = \delta(x)$, to obtain the corresponding solution $u(x,t)$ for $t > 0$ explicitly. [**Hint**: look at the definite integrals on the useful integrals sheet and read the note below.]

(c) Finally, derive the general solution as in equation (7.2) in the lecture notes.

**Solution**

(a) Taking the Fourier transform of the heat equation and integrating by parts twice on the $u_{xx}$ term, $\widehat{u}(k,t)$ must solve

$$\widehat{u}_t + k^2 \widehat{u} = 0$$

which is a first-order ODE, easily solved for any value of $k$:

$$\widehat{u}(k,t) = \widehat{u}(k,0)\, e^{-k^2 t}\, .$$

Transforming back,

$$u(x,t) = \frac{1}{2\pi} \int dk\, \widehat{u}(k,t)\, e^{ikx} = \frac{1}{2\pi} \int dk\, \widehat{u}(k,0)\, e^{-k^2 t + ikx}\, .$$

(b) For $u(x,0) = \delta(x)$ it's easy to compute that $\widehat{u}(k,0) = 1$ . Substituting this into the result from part (a),

$$u(x,t) = \frac{1}{2\pi} \int dk\, e^{-k^2 t + ikx} = \frac{1}{2\pi} \int dk\, e^{-t(k - i\frac{x}{2t})^2} e^{-\frac{x^2}{4t}} = \frac{1}{2\sqrt{\pi t}}\, e^{-\frac{x^2}{4t}}$$

using the Gaussian integral from the integrals sheet for the definite integral, after shifting the integration variable by a finite imaginary amount as in the note below the integral.

(c) In the general case we want $u(x,0) = u_0(x)$, so (using $x'$ instead of $x$ for the integration variable in the FT) $\widehat{u}(k,0) = \int dx' e^{-ikx'} u_0(x')$ . Inserting this into the formula found in part (a) and then doing the $k$ integral just as in part (b), though with $x$ replaced by $x - x'$,

$$\begin{aligned} u(x,t) &= \frac{1}{2\pi} \iint dk\, dx'\, e^{-ikx'} u_0(x')\, e^{-k^2 t + ikx} \\ &= \frac{1}{2\pi} \iint dk\, dx'\, u_0(x')\, e^{-k^2 t + ik(x - x')} \\ &= \frac{1}{2\sqrt{\pi t}} \int dx'\, u_0(x')\, e^{-\frac{(x-x')^2}{4t}} \end{aligned}$$

which is indeed formula (7.2) from lectures. Note, you can think of this as "adding up" (using an integral) lots of solutions to the problem from part (b) – this is the motivation for the idea of a *Green's function*.

48. Find the general solution of the linearised KdV equation $u_t + u_{xxx} = 0$. Your answer should be in the form of an integral involving $\widehat{u}(k, 0)$, the Fourier transform of the initial condition $u(x, 0)$.

**Solution** Taking the Fourier transform of the linearised KdV equation $u_t + u_{xxx} = 0$,

$$\widehat{u}_t = ik^3\widehat{u}$$

which has the solution $\widehat{u}(k,t) = e^{ik^3t}\widehat{u}(k,0)$ . Transforming back,

$$\begin{aligned} u(x,t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \widehat{u}(k,t)\, e^{ikx}\, dk \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ik^3t}\widehat{u}(k,0)\, e^{ikx}\, dk \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ik(k^2t+x)}\widehat{u}(k,0)\, dk \end{aligned}$$

where $\widehat{u}(k,0) = \int_{-\infty}^{\infty} u(x,0)\, e^{-ikx}\, dx$ . Note that this is a superposition of waves travelling leftwards, in line with numerical simulations of small-amplitude waves in the full KdV equation.

49. Try to solve the full (non-linear) KdV equation using the same method, Fourier transform. [Do not try too hard as it is impossible! Just convince yourself that it is impossible and understand what goes wrong/why the Fourier transform doesn't work in the nonlinear case.]

50. Show that if $u(x, t)$ satisfies the KdV equation $u_t + 6uu_x + u_{xxx} = 0$, and $u = \lambda - v^2 - v_x$ where $\lambda$ is a constant and $v(x, t)$ some other function, then $v$ satisfies

$$\big(2v + \frac{\partial}{\partial x}\big)\big(v_t + 6(\lambda - v^2)v_x + v_{xxx}\big) = 0\,.$$

(You might recognise this problem from last term!)

**Solution** Differentiating $u = \lambda - v^2 - v_x$ yields:

$$\begin{aligned} u_t &= -2vv_t - v_{tx} \\ u_x &= -2vv_x - v_{xx} \\ u_{xx} &= -2v_x^2 - 2vv_{xx} - v_{xxx} \\ u_{xxx} &= -6v_xv_{xx} - 2vv_{xxx} - v_{xxxx}\,. \end{aligned}$$

Substituting into the KdV equation, and noting that $(v^2v_x)_x = v^2v_{xx} + 2vv_x^2$ , we find

$$-2v\left[v_t + 6\lambda v_x - 6v^2v_x + v_{xxx}\right] - \frac{\partial}{\partial x}\left[v_t + 6\lambda v_x - 6v^2v_x + v_{xxx}\right] = 0,$$

and thus

$$\left(2v + \frac{\partial}{\partial x}\right)\left(v_t + 6\lambda v_x - 6v^2 v_x + v_{xxx}\right) = 0\,.$$

51. If $\lambda$ is an eigenvalue of $\frac{d^2}{dx^2}\psi(x)+u(x)\psi(x) = \lambda\psi(x)$, where we require $\int_{-\infty}^{\infty}|\psi(x)|^2dx < \infty$, and $u(x)$ is real, prove that $\lambda$ must also be real. (Hint: start by multiplying by $\psi(x)^*$ and integrating.)

**Solution** Following the hint, we find

$$\int_{-\infty}^{\infty}\psi(x)^*\frac{d^2}{dx^2}\psi(x) + \psi(x)^*u(x)\psi(x)\,dx = \lambda\int_{-\infty}^{\infty}|\psi(x)|^2dx$$

and integrating the first term on the LHS by parts once,

$$\int_{-\infty}^{\infty}-|d\psi(x)/dx|^2 + u(x)|\psi(x)|^2\,dx = \lambda\int_{-\infty}^{\infty}|\psi(x)|^2dx\,.$$

Since $u(x)$ is real, the LHS is real; and dividing through by $\int_{-\infty}^{\infty}|\psi(x)|^2dx$ (which is finite, real and nonzero) shows that $\lambda$ is real.

52. Let $D = d/dx$ and let $g(x)$ be a general function of $x$.

(a) Show that, as differential operators,

$$Dg = g_x + gD\,, \qquad D^2g = g_{xx} + 2g_xD + gD^2\,.$$

(b) Show more generally that

$$D^n g = \sum_{m=0}^{n}\binom{n}{m}\frac{d^m g}{dx^m}D^{n-m}\,.$$

[**Hint**: to show that two differential operators are equal, you just have to show that they have the same effect on any function $f(x)$. For part (b), either try induction or think about the formula for the differentiation of a product.]

**Solution**

(a) $g$ as an operator sends $f(x)$ to $g(x)f(x)$; $D$ sends $f(x)$ to $\frac{d}{dx}f(x)$. $Dg$ means 'do $g$ then do $D$ on the result', so $Dg\,f = \frac{d}{dx}(gf) = g_xf + gf_x = (g_x + gD)f$. Hence on any function $f$, the action of $Dg$ is the same as that of $g_x + gD$, which implies

$$Dg = g_x + gD\,.$$

Likewise

$$D^2g\,f = \frac{d^2}{dx^2}(gf) = \frac{d}{dx}(g_xf + gf_x) = g_{xx}f + 2g_xf_x + gf_{xx}$$

which is the same as $(g_{xx} + 2g_x D + gD^2)f$, from which the desired identity follows.

(b) The relevant formula for differentiating a product is

$$\frac{d^n}{dx^n}(gf) = \sum_{m=0}^{n} \binom{n}{m} g^{(m)} f^{(n-m)} .$$

53. Let $D = \partial/\partial x$, and

$$L(u) = D^2 + u(x,t) , \qquad B(u) = -(4D^3 + 6uD + 3u_x) .$$

Check that

$$L(u)_t + [L(u), B(u)] = u_t + 6uu_x + u_{xxx} .$$

**Solution** This goes as in the lecture notes.

54. Let $L(u) = D^2 + u(x,t)$ and $B(u) = \alpha D$ for some constant $\alpha$.

(a) Check that

$$L(u)_t = [B(u), L(u)] \quad \Longleftrightarrow \quad u_t = \alpha u_x .$$

(b) Let $\psi(x,0)$ be an eigenfunction of $L(u)$ at $t = 0$ with eigenvalue $\lambda$, so that

$$(D^2 + u(x,0))\psi(x,0) = \lambda\psi(x,0) .$$

If $u(x,t)$ evolves according to the equation of part 1, find an eigenfunction $\psi(x,t)$ for each later time $t$, with the same eigenvalue $\lambda$, so that

$$(D^2 + u(x,t))\psi(x,t) = \lambda\psi(x,t) .$$

Verify that $\psi(x,t)$ can be arranged to satisfy $\psi_t = B(u)\psi$. (You can assume that the eigenfunction is non-degenerate, namely that there is a single eigenfunction with that eigenvalue. This is the case both for bound state solutions and for scattering solutions.)

**Solution**

(a) We have $L(u)_t = u_t$, and

$$[B(u), L(u)] = \alpha[D, D^2 + u] = \alpha[D, u] = \alpha u_x .$$

Hence $L(u)_t = [B(u), L(u)] \Leftrightarrow u_t = \alpha u_x$ as required.

(b) If $u_t = \alpha u_x$ then $u(x,t) = f(x + \alpha t)$; matching to the initial condition at $t = 0$, $u(x,t) = u(x + \alpha t, 0)$. Now suppose that

$$(D^2 + u(x,0))\psi(x,0) = \lambda\psi(x,0) .$$

Replacing $x$ by $x + \alpha t$ throughout,

$$(D^2 + u(x + \alpha t, 0))\psi(x + \alpha t, 0) = \lambda\psi(x + \alpha t, 0)$$

but since $u(x,t) = u(x+\alpha t, 0)$ this is the same as

$$(D^2 + u(x,t))\psi(x+\alpha t, 0) = \lambda\psi(x+\alpha t, 0)$$

and hence $(D^2 + u(x,t))\psi(x,t) = \lambda\psi(x,t)$ is solved by setting $\psi(x,t) = \psi(x+\alpha t, 0)$. For this solution we have

$$\psi(x,t)_t = \frac{\partial}{\partial t}\psi(x+\alpha t, 0) = \alpha\frac{\partial}{\partial x}\psi(x+\alpha t, 0) = \alpha\frac{\partial}{\partial x}\psi(x,t) = \alpha D\psi(x,t) = B(u)\psi(x,t)$$

as required.

55. (a) Show that the differential operator $D = \partial/\partial x$ is anti-symmetric with respect to the inner product

$$(\psi_1, \psi_2) := \int_{-\infty}^{+\infty} dx\ \psi_1(x)^*\psi_2(x)$$

on the space $L^2(\mathbb{R})$ of square integrable functions, that is $(\psi_1, D\psi_2) = -(D\psi_1, \psi_2)$ for all $\psi_1,\ \psi_2 \in L^2(\mathbb{R})$.

(b) Show that $L(u) = D^2 + u(x,t)$ is self-adjoint, given that $u$ is real.

(c) Given a Lax pair $L(u)$, $B(u)$, show that the symmetric part of $B(u)$ commutes with $L(u)$ and therefore drops out of the Lax equation $L(u)_t = [B(u), L(u)]$.

(d) Now assume that $B(u)$ is anti-symmetric. Show that $(\psi_1, \psi_2)$ is independent of time $t$ if $\psi_i(x;t)$ evolves according to the equation $(\psi_i)_t = B(u)\psi_i$.

**Solution**

(a) We have

$$(\psi_1, D\psi_2) = \int_{-\infty}^{+\infty} dx\ \psi_1(x)^*\tfrac{\partial}{\partial x}\psi_2(x) = -\int_{-\infty}^{+\infty} dx\ (\tfrac{\partial}{\partial x}\psi_1(x)^*)\psi_2(x) = -(D\psi_1, \psi_2)$$

integrating by parts for the middle equality and using the fact that $\psi_1$ and $\psi_2$ tend to zero at $\pm\infty$, so there's no boundary term.

(b) This follows in much the same way as part (a), integrating by parts twice.

(c) Given that $L$ and $B$ form a Lax pair, we know that $[L, B]$ is multiplicative (and real) and so $[L, B] = [L, B]^\dagger$, and also $L = L^\dagger$. Hence

$$\begin{aligned}
0 &= [L, B] - [L, B]^\dagger \\
&= LB - BL - (LB - BL)^\dagger \\
&= LB - BL - (B^\dagger L^\dagger - L^\dagger B^\dagger) \\
&= LB - BL - (B^\dagger L - LB^\dagger) \\
&= L(B + B^\dagger) - (B + B^\dagger)L = [L, (B + B^\dagger)]
\end{aligned}$$

Since the symmetric part of $B$ is $\frac{1}{2}(B + B^\dagger)$ (and the commutator is linear) the result follows.

(d) We have

$$\begin{aligned}\frac{\partial}{\partial t}(\psi_1,\psi_2) &= (\frac{\partial}{\partial t}\psi_1,\psi_2)+(\psi_1,\frac{\partial}{\partial t}\psi_2)\\ &= (B\psi_1,\psi_2)+(\psi_1,B\psi_2)\\ &= (B\psi_1,\psi_2)-(B\psi_1,\psi_2) \qquad \text{(using antisymmetry of } B)\\ &= 0\end{aligned}$$

as required.

56. (a) Show that the differential operator of order $2m-1$

$$B(u)=\sum_{j=1}^{m}\left(\beta_j(x)D^{2j-1}+D^{2j-1}\beta_j(x)\right)$$

is anti-symmetric if the functions $\beta_j(x)$ are real.

(b) If $L(u) = D^2 + u(x,t)$, compute the leading term of $[L(u),B(u)]$ in the form $\gamma(x)D^{2m}$. If $[L(u),B(u)]$ is to be purely multiplicative (forcing $\gamma(x)$ to be zero), deduce that $\beta_m(x)$ must be a constant.

**Solution**

(a) As in lectures, integration by parts shows that $D^\dagger = -D$, and hence $(D^{2j-1})^\dagger = (-1)^{2j-1}D^{2j-1} = -D^{2j-1}$ if $j\in\mathbb{N}$. Thus

$$\begin{aligned}B(u)^\dagger &= \sum_{j=1}^{m}\left(\beta_j(x)D^{2j-1}+D^{2j-1}\beta_j(x)\right)^\dagger\\ &= \sum_{j=1}^{m}\left((D^{2j-1})^\dagger\beta_j(x)^*+\beta_j(x)^*(D^{2j-1})^\dagger\right)\\ &= -\sum_{j=1}^{m}\left((D^{2j-1})^\dagger\beta_j(x)+\beta_j(x)(D^{2j-1})^\dagger\right) \qquad \text{(since } \beta_j\in\mathbb{R})\\ &= -B(u)\,.\end{aligned}$$

(b) For $L(u) = D^2+u$, $B(u)$ as above, we have

$$\begin{aligned}[L(u),B(u)] &= [D^2+u,\sum_{j=1}^{m}\left(\beta_jD^{2j-1}+D^{2j-1}\beta_j\right)]\\ &= [D^2,\beta_mD^{2m-1}+D^{2m-1}\beta_m]+\text{(terms involving } D^n \text{ with } n<2m)\\ &= [D^2,\beta_m]D^{2m-1}+D^{2m-1}[D^2,\beta_m]+\text{(terms involving } D^n \text{ with } n<2m)\end{aligned}$$

where the last step can be checked by writing out the terms. Since $[D^2,\beta_m]=\beta_{m,xx}+2\beta_{m,x}D$ we deduce

$$\begin{aligned}[L(u),B(u)] &= 2\beta_{m,x}\,D^{2m}+2D^{2m-1}\beta_{m,x}\,D+\text{(terms involving } D^n \text{ with } n<2m)\\ &= 2\beta_{m,x}\,D^{2m}+2\beta_{m,x}\,D^{2m}+\text{(terms involving } D^n \text{ with } n<2m)\\ &= 4\beta_{m,x}\,D^{2m}+\text{(terms involving } D^n \text{ with } n<2m)\,.\end{aligned}$$

Now $[L(u), B(u)]$ multiplicative implies in particular that the $D^{2m}$ derivative term must vanish and hence $\beta_{m,x} = 0$, so $\beta_m$ must be a constant (as a function of $x$) as required.

57. Consider the $m = 2$ case of the equation from Ex 56 (a). Given the result of that question, you can assume that $\beta_2$ is a constant. Fix a normalization by imposing $\beta_2 = 1/2$, and find the most general form of $\beta_1$ which allows $[L(u), B(u)]$ to be multiplicative. Show that the Lax equation $L(u)_t + [L(u), B(u)] = 0$ is equivalent to the following alternative version of the KdV equation

$$u_t = \frac{1}{4}u_{xxx} + \frac{3}{2}uu_x + 2ku_x \, , \qquad (*)$$

where $k$ is an integration constant. Finally, check that the redefined field

$$\tilde{u}(x,t) = u(x + 8kt, -4t)$$

solves the standard KdV equation $\tilde{u}_t + 6\tilde{u}\tilde{u}_x + \tilde{u}_{xxx} = 0$.

**Solution** The first part of this is as in example (iii) of section 10.1 in notes (Epiphany term lecture 5). For the last part, using the chain rule for the derivatives we have

$$\tilde{u}_t = 8ku_x - 4u_t \, , \quad \tilde{u}_x = u_x \, , \quad \tilde{u}_{xxx} = u_{xxx} \, ,$$

and so

$$\begin{aligned} \tilde{u}_t + 6\tilde{u}\tilde{u}_x + \tilde{u}_{xxx} &= 8ku_x - 4u_t + 6uu_x + u_{xxx} \\ &= 4\Big(-u_t + \frac{1}{4}u_{xxx} + \frac{3}{2}uu_x + 2ku\Big) = 0 \, , \end{aligned}$$

the term in brackets vanishing by (*).

58. Consider the $m = 3$ case of the equation from Ex 56 (a). Given the result of that question, you can assume that $L(u)_t + [L(u), B(u)] = 0$ forces $\beta_3$ to be a constant. Complete the calculation to find the most general form of $\beta_2$ and $\beta_1$ which allow $[L(u), B(u)]$ to be multiplicative. Deduce from a special case of your result that a function $u(x,t)$ evolving according to the fifth-order KdV equation

$$u_t + 30u^2u_x + 20u_xu_{xx} + 10uu_{xxx} + u_{xxxxx} = 0$$

leaves the eigenvalues of $L(u) = D^2 + u$ invariant.

**Solution** Assuming $\beta_3$ is constant as in the question, we can set $\beta_3 = 1/2$ by choice of normalisation. Then

$$B(u) = D^5 + (\beta_2D^3 + D^3\beta_2) + (\beta_1D + D\beta_1) \, .$$

Then $[L(u), B(u)] = [D^2+u, B(u)]$ and (long calculation – it is handy to use the formula from question 52 to move all the $D$'s to the far right in every term) the terms here are

$$\begin{aligned}
[D^2, B(u)] &= [D^2, D^5 + (\beta_2 D^3 + D^3\beta_2) + (\beta_1 D + D\beta_1)] \\
&= [D^2, (\beta_2 D^3 + D^3\beta_2) + (\beta_1 D + D\beta_1)] \\
&= 2\beta_{2,x}D^4 + \beta_{2,xx}D^3 + \beta_{2,xxxxx} + 5\beta_{2,xxxx}D + 9\beta_{2,xxx}D^2 + 7\beta_{2,xx}D^3 + 2\beta_{2,x}D^4 \\
&\quad + \beta_{1,xx}D + 2\beta_{1,x}D^2 + \beta_{1,xxx} + 3\beta_{1,xx}D + 2\beta_{1,x}D^2 \\
&= \beta_{2,xxxxx} + \beta_{1,xxx} + (5\beta_{2,xxxx} + 4\beta_{1,xx})D + (9\beta_{2,xxx} + 4\beta_{1,x})D^2 + 8\beta_{2,xx}D^3 + 4\beta_{2,x}D^4
\end{aligned}$$

and

$$\begin{aligned}
[u, B(u)] &= [u, D^5 + (\beta_2 D^3 + D^3\beta_2) + (\beta_1 D + D\beta_1)] \\
&= -u_{xxxxx} - 5u_{xxxx}D - 10u_{xxx}D^2 - 10u_{xx}D^3 - 5u_x D^4 \\
&\quad - \beta_2 u_{xxx} - 3\beta_2 u_{xx}D - 3\beta_2 u_x D^2 \\
&\quad - \beta_2 u_{xxx} - 3\beta_{2,x}u_{xx} - 3\beta_{2,xx}u_x \\
&\quad - 3(\beta_2 u_{xx} + 2\beta_{2,x}u_x)D - 3\beta_2 u_x D^2 \\
&\quad - 2\beta_1 u_x \\
&= -u_{xxxxx} - 2\beta_2 u_{xxx} - 3\beta_{2,x}u_{xx} - (3\beta_{2,xx} + 2\beta_1)u_x \\
&\quad - (5u_{xxxx} + 6\beta_2 u_{xx} + 6\beta_{2,x}u_x)D \\
&\quad - (10u_{xxx} + 6\beta_2 u_x)D^2 \\
&\quad - 10u_{xx}D^3 - 5u_x D^4\,.
\end{aligned}$$

Collecting the pieces:

$$\begin{aligned}
[L(u), B(u)] = {}& (4\beta_{2,x} - 5u_x)D^4 + (8\beta_{2,xx} - 10u_{xx})D^3 \\
&+ (9\beta_{2,xxx} + 4\beta_{1,x} - 10u_{xxx} - 6\beta_2 u_x)D^2 \\
&+ (5\beta_{2,xxxx} + 4\beta_{1,xx} - 5u_{xxxx} - 6\beta_2 u_{xx} - 6\beta_{2,x}u_x)D \\
&+ \beta_{2,xxxxx} + \beta_{1,xxx} - u_{xxxxx} - 2\beta_2 u_{xxx} - 3\beta_{2,x}u_{xx} - (3\beta_{2,xx} + 2\beta_1)u_x\,.
\end{aligned}$$

Next we must set the coefficients of the derivative terms to zero.

$D^4$: $(4\beta_2 - 5u)_x = 0 \;\Rightarrow\; \beta_2 = \frac{5}{4}(u+k)$ where $k$ is a constant with respect to $x$.

$D^3$: now automatic.

$D^2$: $9\beta_{2,xxx} + 4\beta_{1,x} - 10u_{xxx} - 6\beta_2 u_x = 0$
$\Rightarrow\; \frac{45}{4}u_{xxx} + 4\beta_{1,x} - 10u_{xxx} - \frac{15}{2}(u+k)u_x = \frac{5}{4}u_{xxx} + 4\beta_{1,x} - \frac{15}{2}(u+k)u_x = 0\,,$
$\Rightarrow\; (\frac{5}{4}u_{xx} + 4\beta_1 - \frac{15}{4}u^2 - \frac{15}{2}ku)_x = 0$ and hence $\beta_1 = -\frac{5}{16}\,(u_{xx} - 3u^2 - 6ku + h)$
where $h$ is another constant.

$D^1$: $(5\beta_{2,xxxx} + 4\beta_{1,xx} - 5u_{xxxx} - 6\beta_2 u_{xx} - 6\beta_{2,x}u_x) = 0$.
It can be checked (bonus exercise!) that this is now automatic.

Finally(!) the $D^0$ term gives us the general form of $[L(u), B(u)]$, given that it has to be multiplicative:

$$
\begin{aligned}
[L(u),B(u)] &= \beta_{2,xxxxx}+\beta_{1,xxx}-u_{xxxxx}-2\beta_2 u_{xxx}-3\beta_{2,x}u_{xx}-3\beta_{2,xx}u_x-2\beta_1 u_x\\
&= \tfrac{5}{4}u_{xxxxx}-\tfrac{5}{16}(u_{xx}-3u^2-6ku)_{xxx}-u_{xxxxx}\\
&\quad -\tfrac{5}{2}(u+k)u_{xxx}-\tfrac{15}{4}u_xu_{xx}-\tfrac{15}{4}u_{xx}u_x+\tfrac{5}{8}(u_{xx}-3u^2-6ku+h)u_x\\
&= -\tfrac{1}{16}u_{xxxxx}+\tfrac{15}{8}(3u_xu_{xx}+uu_{xxx})+\tfrac{15}{8}ku_{xxx}\\
&\quad -\tfrac{5}{2}(u+k)u_{xxx}-\tfrac{15}{4}u_xu_{xx}-\tfrac{15}{4}u_{xx}u_x+\tfrac{5}{8}(u_{xx}-3u^2-6ku+h)u_x\\
&= -\tfrac{1}{16}u_{xxxxx}-\tfrac{5}{4}u_xu_{xx}-\tfrac{5}{8}(u+k)u_{xxx}-\tfrac{15}{8}u^2u_x-\tfrac{15}{4}kuu_x+\tfrac{5}{8}hu_x\,.
\end{aligned}
$$

Rescaling $B(u)\to -16B(u)$ and setting $k=h=0$, we have, applying the general theorem about Lax pairs, that if

$$0 = L(u)_t + [L(u),B(u)] = u_t + u_{xxxxx} + 20u_xu_{xx} + 10uu_{xxx} + 30u^2u_x$$

then the spectrum of $L(u)$ is independent of $t$, as required.

59. The functional derivative $\delta F/\delta u$ of $F[u]$ is defined by the equation

$$F[u+\delta u] = F[u] + \int_{-\infty}^{+\infty} dx\, \frac{\delta F[u]}{\delta u(x)}\delta u(x) + \mathcal{O}((\delta u)^2)\,,$$

where the infinitesimal variation $\delta u(x)$ is small everywhere and goes to zero at the boundaries of the integration range (the same applies to its derivatives $\delta u_x,\ \delta u_{xx},\ \dots$). If

$$F[u] = \int_{-\infty}^{+\infty} dx\ f(u,u_x,u_{xx},u_{xxx},\dots)\,,$$

show that

$$\frac{\delta F[u]}{\delta u} = \frac{\partial f}{\partial u} - \frac{\partial}{\partial x}\frac{\partial f}{\partial u_x} + \frac{\partial^2}{\partial x^2}\frac{\partial f}{\partial u_{xx}} - \frac{\partial^3}{\partial x^3}\frac{\partial f}{\partial u_{xxx}} + \dots$$

**Solution** We have

$$
\begin{aligned}
F[u+\delta u] &= \int_{-\infty}^{+\infty} dx\ f(u+\delta u, u_x+\delta u_x, u_{xx}+\delta u_{xx},\dots)\\
&= \int_{-\infty}^{+\infty} dx\ f(u) + \frac{\partial f}{\partial u}\delta u + \frac{\partial f}{\partial u_x}\delta u_x + \frac{\partial f}{\partial u_{xx}}\delta u_{xx} + \dots + \mathcal{O}((\delta u)^2)\\
&= F[u] + \int_{-\infty}^{+\infty} dx\ \frac{\partial f}{\partial u}\,\delta u - \frac{\partial}{\partial x}\Big(\frac{\partial f}{\partial u_x}\Big)\,\delta u + \frac{\partial^2}{\partial x^2}\Big(\frac{\partial f}{\partial u_{xx}}\Big)\,\delta u + \dots + \mathcal{O}((\delta u)^2)\\
&= F[u] + \int_{-\infty}^{+\infty} dx\ \left(\frac{\partial f}{\partial u} - \frac{\partial}{\partial x}\Big(\frac{\partial f}{\partial u_x}\Big) + \frac{\partial^2}{\partial x^2}\Big(\frac{\partial f}{\partial u_{xx}}\Big) + \dots\right)\delta u + \mathcal{O}((\delta u)^2)
\end{aligned}
$$

where to get from the second line to the third we integrated by parts once for the $\delta u_x$ term, twice for $\delta u_{xx}$, and so on, each time using the fact that the variation and its dervitives go to zero at the boundaries of the integration range. Comparing the last line with the formula in the question gives the desired result.

60. (a) Find a function $f(u, u_x, u_{xx})$ and a functional

$$F[u] = \int_{-\infty}^{+\infty} dx\, f(u, u_x, u_{xx})$$

such that the equation

$$u_t = \frac{\partial}{\partial x}\frac{\delta F}{\delta u}$$

is the same as the fifth-order KdV equation from question 58.

(b) Show that your $F[u]$ is a conserved quantity if $u$ evolves according to the standard third order KdV equation.

(c) Show that $\int_{-\infty}^{+\infty} dx\, u$ is a conserved quantity if $u$ evolves according to the fifth-order KdV equation ($\gamma$).

**Solution** See the handwritten solutions in "lecture plan" folder.

61. The *Wronskian* $W[f, g](x)$ of two differentiable functions $f(x)$ and $g(x)$ is defined as

$$W[f, g](x) = f'(x)g(x) - f(x)g'(x) .$$

If the functions $f$ and $g$ are linearly dependent, then their Wronskian vanishes identically: $W[f, g](x) = 0$. (Equivalently, if $W[f, g](x) \neq 0$, the functions $f$ and $g$ are linearly independent.) Conversely, if the Wronskian vanishes identically for two *analytic* functions $f$ and $g$, then $f$ and $g$ are linearly dependent.

(a) Write down the Wronskian $W[\psi_1^*, \psi_2](x)$ of two eigenfunctions $\psi_{1,2}(x)$ of the time-independent Schrödinger equation with the same potential $V(x)$ and possibly different eigenvalues $k_i^2$:

$$\psi_i''(x) - V(x)\psi_i(x) = -k_i^2\psi_i(x) \qquad (i = 1, 2) . \qquad (**)$$

(This is just preparation for what follows, no computation is needed.)

(b) Show that the Wronskian is constant if the two eigenfunctions correspond to the same eigenvalue.

(c) Show that two eigenfunctions with different eigenvalues are orthogonal with respect to the (hermitian) inner product

$$(\psi_1, \psi_2) := \int_{-\infty}^{+\infty} dx\, \psi_1^*(x)\psi_2(x)$$

if at least one of the two eigenfunctions describes a bound state.

(d) Show that the Wronskian vanishes for two eigenfunctions with the same eigenvalue in the discrete spectrum. (This implies the linear dependence of the two eigenfunctions, provided that they are analytic.) [**Hint**: consider the limit $x \to \pm\infty$.]

(e) The $x \to \pm\infty$ asymptotics of a scattering solution $\psi(x)$ with eigenvalue $k^2 > 0$ is

$$\psi(x) \approx \begin{cases} e^{ikx} + R(k)\, e^{-ikx}\,, & x \to -\infty \\ T(k)\, e^{ikx}\,, & x \to +\infty \end{cases}$$

By evaluating the Wronskian $W[\psi^*, \psi]$ at $x \to \pm\infty$, show that the reflection and transmission coefficients $R(k)$ and $T(k)$ satisfy

$$|R(k)|^2 + |T(k)|^2 = 1\,.$$

**Solution**

(a) $W[\psi_1^*, \psi_2] = \psi_1^{*\prime}(x)\psi_2(x) - \psi_1^*(x)\psi_2'(x)$

(b) We have

$$\frac{d}{dx}W[\psi_1^*, \psi_2](x) = \psi_1^{*\prime\prime}\psi_2 + \psi_1^{*\prime}\psi_2' - \psi_1^{*\prime}\psi_2' - \psi_1^*\psi_2'' = \psi_1^{*\prime\prime}\psi_2 - \psi_1^*\psi_2''\,.$$

Then use the differential equation (details should be given) to substitute for $\psi_1^{*\prime\prime}$ and $\psi_2''$ to see that the two terms on the RHS cancel when the eigenvalues are the same, so the Wronskian is indeed constant.

(c) Multiply the complex conjugate of (**) with $i = 1$ by $\psi_2(x)$ and subtract $\psi_1(x)^*$ times (**) with $i = 2$ and integrate from $-\infty$ to $\infty$ to find

$$\int_{-\infty}^{\infty} (\psi_1^{*\prime\prime}\psi_2 - \psi_1^*\psi_2'')\, dx = (k_2^2 - k_1^2)\,(\psi_1, \psi_2)\,.$$

As in the solution to part 2, the integrand on the LHS of this equation is equal to $\frac{d}{dx}W[\psi_1^*, \psi_2](x)$, and integrates to zero since at least one of $\psi_1$ and $\psi_2$ is a bound state and vanishes at $\pm\infty$ (while the other, even if not in the discrete spectrum, must be bounded at infinity). Hence for $k_1^2 \neq k_2^2$, $(\psi_1, \psi_2) = 0$.

(d) From part 2, the Wronskian of the two eigenfunctions, sharing the same eigenvalue, is constant. Since this eigenvalue is in the discrete spectrum these eigenfunctions vanish at $\pm\infty$, and so their Wronskian vanishes there. It therefore vanishes for all $x$, as required.

(e) As $x \to +\infty$,

$$W[\psi^*, \psi] \to (T(k)^* e^{-ikx})'\, T(k)e^{ikx} - T(k)^* e^{-ikx}\,(T(k)e^{ikx})' = -2ik|T(k)|^2\,.$$

Likewise, as $x \to -\infty$

$$\begin{aligned} W[\psi^*, \psi] &\to (e^{-ikx} + R(k)^* e^{ikx})'\,(e^{ikx} + R(k)e^{-ikx}) - (e^{-ikx} + R(k)^* e^{ikx})\,(e^{ikx} + R(k)e^{-ikx})' \\ &= -2ik + 2ik|R(k)|^2\,. \end{aligned}$$

Since by part 2 this Wronskian is constant, these two limits must agree, and with a little rearrangement the desired result follows.

62. Consider the time independent Schrödinger equation

$$\left(-\frac{d^2}{dx^2} + V(x)\right)\psi(x) = k^2\psi(x)$$

with energy $E = k^2$ for the square barrier/well potential

$$V(x) = \begin{cases} 0\,, & x < 0 \\ V_0\,, & 0 < x < a \\ 0\,, & x > a \end{cases}$$

where $a > 0$ and $V_0$ are constants.

(a) Show that the matching conditions to be imposed at $x = 0$ and $a$, where the square well potential is discontinuous (but finite), are that $\psi(x)$ and $\psi'(x)$ are continuous.

(b) Solve the Schrödinger equation for this potential in the three given regions and impose the matching conditions to find the scattering solutions associated to energy eigenvalues $k^2 > 0$ in the continuous spectrum, and determine the reflection and transmission coefficients $R(k)$ and $T(k)$ in terms of $a$ and $l = \sqrt{k^2 - V_0}$.

(c) For which values of the wavenumber $k$ is the square well potential transparent, that is $R(k) = 0$?

(d) Write down the bound state solutions corresponding to the discrete spectrum $k^2 = -\mu^2 < 0$. Find the equations that determine implicitly the allowed values of $\mu$ in terms of $a$ and $l$ (or $V_0$).

(e) Do bound state solutions exist for $V_0 > 0$? And for $V_0 < 0$? In the latter case, use a graphical argument to show that a new bound state solution appears every time that $\sqrt{-V_0}$ crosses a non-negative integer multiple of $\pi/a$.

(f) Show that in the limit $a \to 0$, $V_0 \to +\infty$ with $b = aV_0$ fixed, the reflection and transmission coefficients reduce to those of the delta-function potential $V(x) = b\delta(x)$.

**Solution** Check back later.

63. Consider the time independent Schrödinger equation

$$-\psi''(x) + V(x)\psi(x) = k^2\psi(x)\ ,$$

where the potential $V(x)$ is the sum of two delta functions:

$$V(x) = -a\delta(x) - b\delta(x - r)\ .$$

Taking $r > 0$, the solution $\psi(x)$ can be split into three pieces, $\psi_1(x)$, $\psi_2(x)$ and $\psi_3(x)$, defined on $(-\infty, 0)$, $(0, r)$, and $(r, +\infty)$ respectively.

(a) Write down the four matching conditions relating $\psi_1$, $\psi_2$ and $\psi_3$, and their derivatives, at $x = 0$ and $x = r$.

(b) For a scattering solution describing waves incident from the left, $\psi_1$ and $\psi_3$ are given by

$$\psi_1(x) = e^{ikx} + R(k)\, e^{-ikx}, \quad \psi_3(x) = T(k)\, e^{ikx}.$$

Write down the general form of $\psi_2$, and then use the matching conditions found in part 1 to eliminate the unknowns and determine $R(k)$ and $T(k)$.

(c) Show from the answer to part 2 that, for there to be a bound state pole at $k = i\mu$, $\mu$ must satisfy

$$e^{-2\mu r} = (1 - 2\mu/a)(1 - 2\mu/b)\ . \qquad (***)$$

(d) The solutions to (***) can be analysed using a graphical method. Show that:

i. if both $a$ and $b$ are negative, then there are no bound states;

ii. if $a$ and $b$ have opposite signs, then there is at most one bound state, occurring when $a + b > rab$ (note: since $a$ and $b$ have opposite signs, $rab$ is negative);

iii. if $a$ and $b$ are positive, then the number of bound states is one if $rab \leq a + b$, and two otherwise.

Sketch on the $ab$-plane the regions which correspond to zero, one and two bound states, and indicate the form of $\psi(x)$ for each of the two bound states found when $ab/(a + b) > r^{-1}$.

**Solution** See module webpage.

64. The time independent Schrödinger equation

$$-\psi''(x) + V(x)\psi(x) = k^2\psi(x)$$

is conjectured to have solutions in the form

$$\psi(x) = e^{ikx}(2k + iw(x))\ ,$$

where $w(x)$ is real, non-singular for all $x$, independent of $k$, and has finite limits as $x \to \pm\infty$. Substituting in, deduce the equation

$$w'(x) + \frac{1}{2}w^2(x) = 2\mu^2\ ,$$

where $\mu$ is an integration constant. [**Hint**: take real and imaginary parts of an intermediate equation.] Solve this via the substitution $w(x) = 2f'(x)/f(x)$, and deduce that $V(x)$ must have the form

$$V(x) = -2\mu^2\,\mathrm{sech}^2(\mu(x - x_0))\ .$$

Show also that $u = -V$ is a solution of the KdV equation provided that $x_0$ depends on $t$ in a certain way that you should determine.

**Solution** Substituting in, we need

$$\begin{aligned}
0 &= \left(-\frac{d^2}{dx^2} + V(x) - k^2\right)\left(e^{ikx}(2k + iw(x))\right) \\
&= e^{ikx}\left(2k^3 + ik^2w(x) + 2kw'(x) - iw''(x) + 2kV(x) + iw(x)V(x) - 2k^3 - ik^2w(x)\right) \\
&= e^{ikx}\left(2kw'(x) - iw''(x) + 2kV(x) + iw(x)V(x)\right) \\
&= e^{ikx}\left(2k(V(x) + w'(x)) + i(w(x)V(x) - w''(x))\right).
\end{aligned}$$

Setting real and imaginary parts of the term in big brackets on the last line equal to zero (and noting that $w$, $V(x)$ and $k$ are all real) implies

$$\begin{cases} V(x) = -w'(x) \\ w''(x) = w(x)V(x)\,. \end{cases}$$

Substituting the first of these into the second,

$$0 = w''(x) + w(x)w'(x) = \frac{d}{dx}\Big(w'(x) + \frac{1}{2}w^2(x)\Big)\,.$$

Integrating once and setting the constant of integration equal to $2\mu^2$ gives us the claimed result.

Substituting $w(x) = 2f'(x)/f(x)$ and cancelling some terms,

$$f''(x) = \mu^2 f(x)$$

and hence $f(x) = Ae^{\mu x} + Be^{-\mu x}$ for some $A$ and $B$, and

$$w(x) = 2\,\frac{f'(x)}{f(x)} = 2\mu\,\frac{Ae^{\mu x} - Be^{-\mu x}}{Ae^{\mu x} + Be^{-\mu x}}\,.$$

Since $(A, B)$ and $(\lambda A, \lambda B)$ give the same $w(x)$, we can take $AB = 1$ without loss of generality, and set $A = e^{-\mu x_0}$, $B = e^{\mu x_0}$ for some $x_0$. Hence

$$w(x) = 2\mu\tanh(\mu(x - x_0))\,, \qquad V(x) = -w'(x) = -2\mu^2\,\mathrm{sech}^2(\mu(x - x_0))\,.$$

As seen earlier in the course (so it won't be repeated here), substituting $u = -V$ into the KdV equation leads to a solution provided that $x_0(t) = x_0(0) + 4\mu^2 t$.

65. Using the results of the last question, show that $V(x) = -2\mu^2\,\mathrm{sech}^2(\mu(x - x_0))$ is an example of a reflectionless potential, for which $R(k) = 0$. By adjusting the normalisation of the wavefunction $\psi(x)$ correctly, find out what the transmission coefficient $T(k)$ is for this potential. Verify that $|T(k)|^2 = 1$, consistent with the idea that for such a potential an incident particle must certainly be transmitted.

**Solution** Substituting $w = 2\mu\tanh(\mu(x - x_0))$ into the given equation we have

$$\psi(x) = 2e^{ikx}(k + i\mu\tanh(\mu(x - x_0))) \sim \begin{cases} 2e^{ikx}(k - i\mu) & x \to -\infty \\ 2e^{ikx}(k + i\mu) & x \to +\infty\,. \end{cases}$$

Dividing through by $2(k - i\mu)$ gives us the correctly-normalised scattering solution:

$$\psi_{\text{scattering}}(x) = e^{ikx}\,\frac{k + i\mu\tanh(\mu(x - x_0))}{k - i\mu} \sim \begin{cases} e^{ikx} & x \to -\infty \\ \frac{k+i\mu}{k-i\mu}\,e^{ikx} & x \to +\infty \end{cases}$$

from which we can read off that $R(k) = 0$ (so the potential is indeed reflectionless) and

$$T(k) = \frac{k + i\mu}{k - i\mu}\,.$$

Furthermore

$$|T(k)|^2 = \frac{|k + i\mu|^2}{|k - i\mu|^2} = \frac{k^2 + \mu^2}{k^2 + \mu^2} = 1$$

as expected.