Advanced Quantum Theory IV — Assignment 2

Lorentz transformations of fields

Michaelmas 2021/22

1.

(a). Consider the infinitesimal Lorentz transformation

$$\Lambda = \mathbb{I} + \omega_{\rho\sigma} M^{\rho\sigma} + O\left(\omega_{\rho\sigma}^2\right). \tag{1}$$

Under this transformation the coordinate x transforms by an infinitesimal amount δx:

$$\begin{aligned} x \to x' &= \Lambda x & (2)\\ &= x + \delta x + O\left(\omega_{\rho\sigma}^2\right). & (3) \end{aligned}$$

Show that

$$\delta x = \omega_{\rho\sigma} M^{\rho\sigma} x. \tag{4}$$

What is this in index notation?

(b). Now consider a scalar field $\phi(x)$. Keeping the coordinate x fixed, the field also transforms by an infinitesimal amount under the Lorentz transformation (1):

$$\phi(x) \to \phi'(x) = \phi(x) + \delta\phi(x) + O\left(\omega_{\rho\sigma}^2\right). \tag{5}$$

Show that

$$\delta\phi(x) = \omega_{\rho\sigma} L^{\rho\sigma} \phi(x), \tag{6}$$

where

$$L^{\rho\sigma} = -\left(M^{\rho\sigma} x\right) \cdot \frac{\partial}{\partial x}. \tag{7}$$

What is this in index notation?

From your answer to the latter question, using:

$$\left(M^{\rho\sigma}\right)^{\mu}{}_{\nu} = \eta^{\sigma\mu}\delta^{\rho}{}_{\nu} - \eta^{\rho\mu}\delta^{\sigma}{}_{\nu}, \tag{8}$$

show that

$$L^{\rho\sigma} = x^{\sigma}\frac{\partial}{\partial x_{\rho}} - x^{\rho}\frac{\partial}{\partial x_{\sigma}}. \tag{9}$$

Show that $L^{\rho\sigma}$ satisfies the Lorentz algebra:

$$\left[L^{\mu\nu}, L^{\rho\sigma}\right] = -\eta^{\nu\rho}L^{\mu\sigma} + \eta^{\mu\rho}L^{\nu\sigma} + \eta^{\nu\sigma}L^{\mu\rho} - \eta^{\mu\sigma}L^{\nu\rho}. \tag{10}$$

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(c). Consider the infinitesimal translation

$$x \to x' = x - a. \tag{11}$$

This means that the components of the four-vector a are infinitesimal, i.e. $|a^\mu| \ll 1$. This induces an infinitesimal transformation in the field ϕ:

$$\phi(x) \to \phi'(x) = \phi(x) + \delta\phi(x) + O\left(a^2\right). \tag{12}$$

Show that

$$\delta\phi(x) = a^\mu \frac{\partial}{\partial x^\mu}\phi(x).$$

From this we identify the generator of translations to be $P_\mu = \frac{\partial}{\partial x^\mu}$. Show that

$$[P^\mu, L^{\nu\rho}] = \eta^{\mu\rho}P^\nu - \eta^{\mu\nu}P^\rho, \tag{13}$$

$$[P^\mu, P^\nu] = 0. \tag{14}$$

From this we see that, together with (10), $L^{\rho\sigma}$ and P^μ generate the Poincaré algebra!