Advanced Quantum Theory IV — Assignment 1

Lorentz transformations

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1. *(can be skipped)* The symmetries of the three-dimensional Euclidean space $\mathbb{R}^3$ are those which preserve the distance between any two points. Obviously this includes the translations,

$$\vec{x}' = \vec{x} + \vec{a}, \tag{1}$$

for some constant vector $\vec{a}$. We can also consider linear transformations,

$$\vec{x}' = Q\vec{x}, \tag{2}$$

where Q is a 3×3 matrix. This preserves the distance between two points if it preserves the dot product,

$$\vec{x}' \cdot \vec{y}' = \vec{x} \cdot \vec{y}. \tag{3}$$

Show that this implies

$$Q^{\mathsf{T}} \mathbb{I}_{3\times3} Q = \mathbb{I}_{3\times3}, \tag{4}$$

where $\mathbb{I}_{3\times3}$ is the 3×3 identity matrix.
Show that this condition implies:

$$\det Q = \pm 1. \tag{5}$$

For the matrices Q with $\det Q = 1$, we can make the ansatz

$$Q = e^{\theta J}, \tag{6}$$

where $\theta \in \mathbb{R}$ is a continuous parameter and J is a 3×3 matrix known as the generator of the transformation. By considering infinitesimal $\theta \ll 1$, show that the constraint (4) implies

$$J + J^{\mathsf{T}} = 0, \tag{7}$$

i.e. that the J must be anti-symmetric 3×3 matrices.

There are three independent anti-symmetric 3×3 matrices, so we can package the independent solutions to this constraint into a three-component vector:

$$\vec{J} = (J_1, J_2, J_3)\,, \tag{8}$$

where a useful basis is given by

$$J_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \qquad J_2 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \qquad J_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{9}$$

Show that these generate, respectively, rotations about the x^1, x^2 and x^3 axes.

2. Minkowski space-time is $\mathbb{R}^4$ endowed with the Minkowski metric:

$$\eta = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{10}$$

The symmetries of Minkowski space-time include space-time translations:

$$x' = x + a, \tag{11}$$

for a constant four-vector a. In components $x = \left(x^0, x^1, x^2, x^3\right)$ and $a = \left(a^0, a^1, a^2, a^3\right)$. The other symmetries are linear transformations Λ, the Lorentz transformations,

$$x' = \Lambda\, x, \tag{12}$$

which preserve the scalar product of four-vectors,

$$\left(x'\right)^{\mathsf{T}} \eta \left(y'\right) = x^{\mathsf{T}} \eta\, y. \tag{13}$$

Show that this constraint implies

$$\Lambda^{\mathsf{T}} \eta \Lambda = \eta. \tag{14}$$

Show also that $\det\Lambda = \pm 1$.

Lorentz transformations with $\det\Lambda = 1$ can be expressed in the form:

$$\Lambda = e^{\omega M}, \tag{15}$$

where is a continuous real parameter $\omega \in \mathbb{R}$. The 4×4 matrices M are the generators of the transformation. By considering infinitesimal transformations $\omega \ll 1$, show that the constraint (14) implies

$$M^{\mathsf{T}} = -\eta M \eta. \tag{16}$$

Show that there are six independent solutions to this equation.

Why can each independent solution be labelled with a pair of anti-symmetric indices $M^{\rho\sigma} = -M^{\sigma\rho}$ with $\rho, \sigma = 0, 1, 2, 3$? Deduce that the most general solution to this equation is given by:

$$M = \begin{pmatrix} 0 & \omega_{01} & \omega_{02} & \omega_{03} \\ \omega_{01} & 0 & \omega_{12} & \omega_{13} \\ \omega_{02} & -\omega_{12} & 0 & \omega_{23} \\ \omega_{03} & -\omega_{13} & -\omega_{23} & 0 \end{pmatrix} = \sum_{\rho,\sigma=0}^{3} \omega_{\rho\sigma} M^{\rho\sigma}, \tag{17}$$

for six arbitrary constants $\omega_{\rho\sigma} = -\omega_{\sigma\rho}$, where

$$M^{01} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad M^{02} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad M^{03} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \tag{18}$$

$$M^{12} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad M^{13} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \quad M^{23} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}. \tag{19}$$

Show that the solutions parameterised by ω_{12}, ω_{13} and ω_{23} respectively generate rotations around the x^3, x^2 and x^1 axes by angle ω_{12}, ω_{13} and ω_{23}. Show that solutions parameterised by ω_{01}, ω_{02} and ω_{03} correspond to Lorentz Boosts in the positive x^1, x^2 and x^3 directions.

The anti-symmetric indices ρ, σ labelling the different matrices $M^{\rho\sigma}$ are *not* to be confused with index notation which instead labels matrix elements. In index notation, the elements of each matrix $M^{\rho\sigma}$ are $(M^{\rho\sigma})^{\mu}{}_{\nu}$ with $\mu, \nu = 0, 1, 2, 3$. In this way, the pseudo-orthogonality constraint (16) reads:

$$\left(M^{\mathsf{T}}\right)_{\mu}{}^{\nu} = -\eta_{\mu\mu'} M^{\mu'}{}_{\nu'} \eta^{\nu'\nu}, \qquad \mu, \nu, \mu', \nu' = 0, 1, 2, 3. \tag{20}$$

Lower the upper index in $\left(M^{\mathsf{T}}\right)_{\mu}{}^{\nu}$ and $M^{\mu'}{}_{\nu'}$ using the Minkowski metric. Show that the pseudo-orthogonality constraint is equivalent to the anti-symmetry constraint – i.e. that for each independent solution $M^{\rho\sigma}$, we must have:

$$\left(M^{\rho\sigma}\right)_{\mu\nu} = -\left(M^{\rho\sigma}\right)_{\nu\mu}. \tag{21}$$

Confirm that the matrices (18) and (19) indeed become anti-symmetric if we multiply them with the Minkowski metric η.