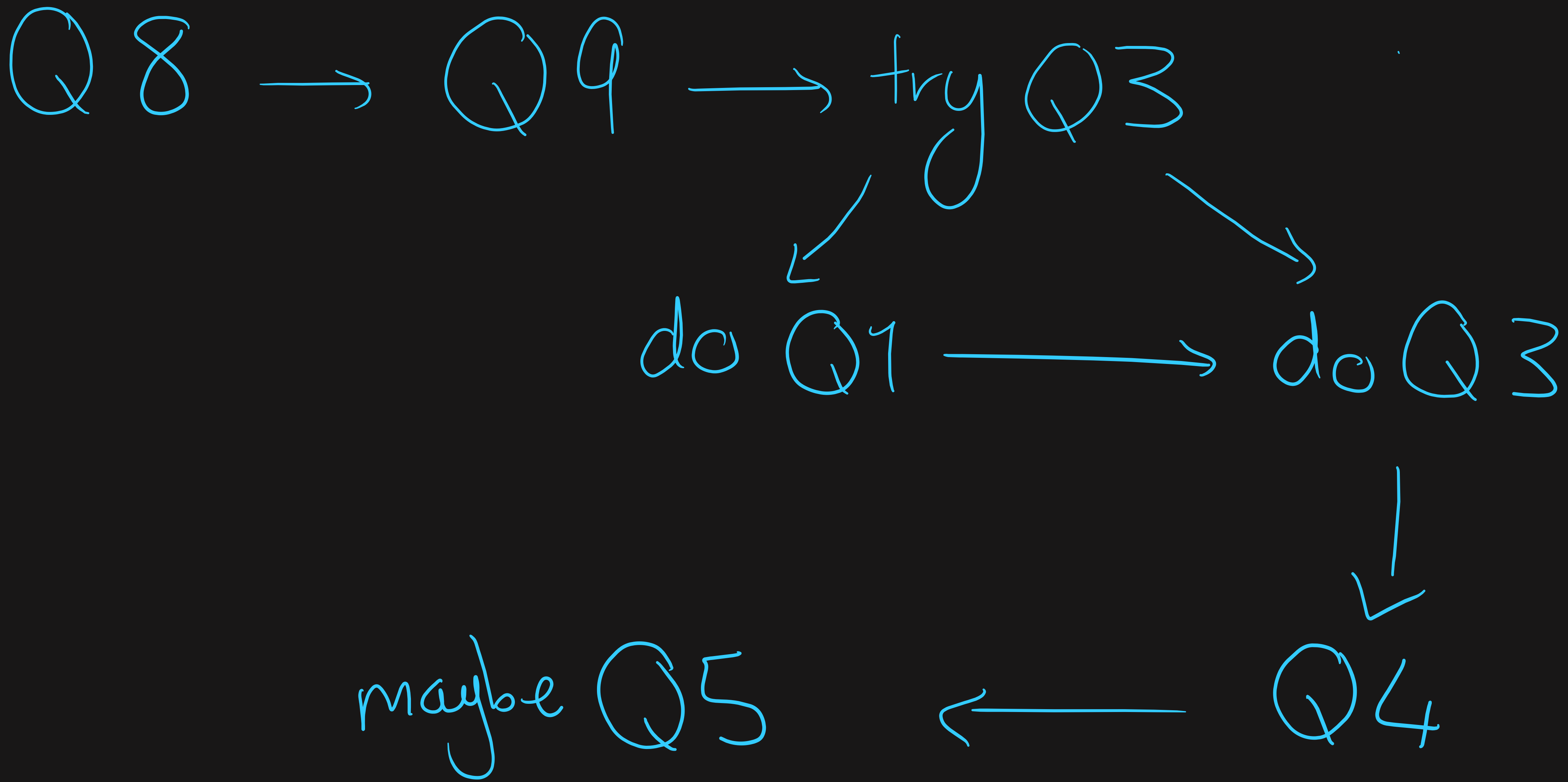
Q8
Q9
try Q3
do Q1
do Q3
Q4
maybe Q5

**Question 8:** Explain how to sample from the bivariate normal distribution for which the mean vector and variance-covariance matrix are respectively

$$\begin{pmatrix} 3 \\ -2 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 25 & 15 \\ 15 & 13 \end{pmatrix}$$

You may assume that you have a supply of randomly sampled numbers from $\mathrm{Normal}(0, 1)$.

**Question 9:** Suppose that I want to generate values for two exponentially distributed random variables, $X \sim \mathrm{Exp}(2)$ and $Y \sim \mathrm{Exp}(5)$, which are not independent. I decide to use the Gaussian copula with $\rho = 0.8$ to model the dependence. Given that 1.2 and 0.2 are two random values from $\mathrm{Normal}(0,1)$, generate a random value for the pair $(X, Y)$.

**Question 1:**

1. Suppose that $R$ and $\Theta$ are independent and that $R$ has the standard Rayleigh distribution with density

$$f(r) = r \exp\left(-\frac{r^2}{2}\right), \quad r \in \mathbb{R}$$

and $\Theta$ is uniformly distributed on $[0, 2\pi)$. Write down the joint probability density function for $(R, \Theta)$.

2. Let $X_1 = R\cos\Theta$ and $X_2 = R\sin\Theta$. Deduce the joint probability distribution for $(X_1, X_2)$ and hence show that $X_1$ and $X_2$ are independent and that each has the standard normal distribution.

3. Based on item 2 and question 7 in Exercise List I, write down a simple algorithm to obtain two independent values from $\text{Normal}(0, 1)$ using two uniform (on $[0, 1]$) random numbers $U_1$ and $U_2$.

**Question 3:**

1. How can I use 3 standard normal values $Z_1$, $Z_2$ and $Z_3$ to sample a random point uniformly distributed on the surface of the unit sphere?

2. Prove that the method works.

Hint: consider the joint probability density function of $Z_1$, $Z_2$ and $Z_3$.

3. Suggest two ways to sample a random point uniformly distributed on the interior of the unit sphere.

**Question 4:** Let $X_1, \ldots, X_n$ be a random sample from the cumulative distribution function $F$. Let $X_{(n)}$ denote the maximum of the sample.

1. $X_{(n)}$ is itself a random variable. Express the cumulative distribution function of $X_{(n)}$ in terms of $F$.

2. If the generalised inverse $F^{-1}$ of $F$ is easily computed, show how to sample a value for $X_{(n)}$ using the inverse transform method.

**Question 5:** The unit simplex in $\mathbb{R}^3$ is the set of points $(x_1, x_2, x_3)$ for which each $x_i \geq 0$ and $x_1 + x_2 + x_3 = 1$.

1. Show that the simplex is a subset of the set of points $(x_1, x_2, x_3)$ for which $x_1 \in [0, 1]$, $x_2 \in [0, 1]$ and $x_3 = 1 - x_2 - x_2$. How could you easily sample a random point from this set? How could you turn that into a method to sample from the unit simplex?

2. A more direct way to sample a point uniformly distributed on the unit simplex is to observe that the \emph{order statistics} for a sample of size 2 from $\mathrm{Uniform}(0, 1)$ divide $[0, 1]$ into three pieces. Can you turn that into an algorithm to sample a point on the unit simplex? Can you prove that your method results in uniformly distributed values on the simplex?

Note: the order statistics for a sample $X_1, \ldots, X_n$ are the same values but sorted in increasing order. A common notation is $X_{(1)} \leq X_{(2)} \leq \cdots \leq X_{(n)}$ where $X_{(1)} = \min(X_1, \ldots, X_n)$ and $X_{(n)} = \max(X_1, \ldots, X_n)$.