117. (*). A fair coin is tossed three times. Suppose X denotes the number of heads in the first two tosses and Y denotes the number of heads in the last two tosses.

(a) Make a table of the joint probability distribution of X and Y.

(b) Use your table for the joint probability mass function to confirm that the marginal probability mass functions of X and Y are both $\mathrm{Bin}(2, 1/2)$.

(c) Compute $\mathbb{P}(X = Y)$.

[Hint: An elementary way to obtain the joint distribution for this problem is to rely on the equally likely outcomes model of probability, and to count outcomes.]

120. (*). Continuing from Exercise 117, confirm that X given $Y = 2$ is not binomial.

119. (**). A fair die is thrown. The score is divided by two and rounded up giving score X. A fair coin is then thrown X times and the number of heads is Y (so if the die roll is 3, then $X = 2$ and the coin is tossed twice).

(a) Write down the marginal probability mass function p_X for X.

(b) What is the (conditional) distribution of Y given $X = x$? Using this conditional probability mass function, and the marginal from part (a), compute the joint probability mass function $p(x, y)$, and present it in a table.

(c) Calculate the marginal probability mass function p_Y of Y.

122. (*). Continuing from Exercise 119, write in a table the conditional probability mass function $p_{X|Y}(x \mid y)$.

116. (**). Let $U_1, U_2 \sim \mathrm{U}(0,1)$ be independent. By calculating the relevant cumulative distribution functions, show that $\max(U_1, U_2)$ has the same distribution as $\sqrt{U_1}$.

123. (*). Let Y be a random variable with $\mathbb{P}(Y = +1) = \mathbb{P}(Y = -1) = 1/2$. Let X be another random variable, taking values in $\mathbb{Z}$, independent of Y.

Show that the random variables Y and XY are independent if and only if the distribution of X is symmetric, i.e., $\mathbb{P}(X = k) = \mathbb{P}(X = -k)$ for all $k \in \mathbb{Z}$.