145. (**). Let $n \in \mathbb{N}$ and $X \sim \text{Po}(\lambda)$. Show that $\mathbb{E}\left(X(X-1)(X-2)\cdots(X-n+1)\right) = \lambda^n$ (this is the n-th cumulant of X). [Hint: $\sum_{x=0}^{\infty} \lambda^x/x! = e^{\lambda}$.]

159. (**). Let $X \sim \text{Po}(\lambda)$. In Exercises 131 and 145, we showed that $\mathbb{E}(X) = \lambda$ and $\mathbb{E}(X(X-1)) = \lambda^2$. Use this, along with linearity of expectation, to show that $\mathbb{V}\text{ar}(X) = \lambda$.

138. (*). Let $X \sim \mathrm{U}(0, 2\pi)$. Find $\mathbb{E}(\sin X)$.

163. **(*).** Suppose $X \sim U(0,2)$ and $Y = X^2$. Find $\mathbb{C}\text{ov}\,(X,Y)$.

168. (*). Suppose $U \sim \mathrm{U}(0,1)$ and let $X = \cos 2\pi U$ and $Y = \sin 2\pi U$. Calculate $\mathbb{E}(X)$, $\mathbb{E}(Y)$, $\mathbb{V}\mathrm{ar}(X)$, $\mathbb{V}\mathrm{ar}(Y)$, and $\mathbb{C}\mathrm{ov}(X,Y)$. Are X and Y independent?

169. (***). In Big Town, there are r people who support the 'Really Great Party' and d people who support the 'Dead Good Party'. The 'Totally Impartial Opinion Poll Company' selects people at random and asks them who they support. (To keep things simple, we suppose that everybody supports one of the two parties and nobody tells lies.)

(a) Let $X = 1$ if the first person selected supports the Really Great Party, 0 if they support the Dead Good Party. Let $Y = 1$ if the second person selected supports the Really Great Party, 0 if they support the Dead Good Party. (The opinion pollsters never ask the same person twice in the same sample.) Show that

$$\mathbb{E}(X) = \mathbb{E}(Y) = \frac{r}{r+d}, \qquad \mathbb{V}\text{ar}(X) = \mathbb{V}\text{ar}(Y) = \frac{rd}{(r+d)^2},$$
$$\mathbb{C}\text{ov}(X,Y) = -\frac{rd}{(r+d)^2(r+d-1)}.$$

(b) Suppose that m different people are selected at random. Let Z be the number of supporters of the Really Great Party. Use the previous part to show that

$$\mathbb{E}(Z) = \frac{mr}{r+d}, \qquad \mathbb{V}\text{ar}(Z) = \frac{mrd(r+d-m)}{(r+d)^2(r+d-1)}.$$

(c) Let the proportion of supporters of the Really Great Party in the sample be S (i.e. $S := Z/m$). Find $\mathbb{E}(S)$ and $\mathbb{V}\text{ar}(S)$. Show that a sample of 250 people ensures that $\mathbb{V}\text{ar}(S) \leq 1/1000$.

(b) Suppose that m different people are selected at random. Let Z be the number of supporters of the Really Great Party. Use the previous part to show that

$$\mathbb{E}(Z) = \frac{mr}{r+d}, \qquad \mathrm{Var}(Z) = \frac{mrd(r+d-m)}{(r+d)^2(r+d-1)}.$$

(c) Let the proportion of supporters of the Really Great Party in the sample be S (i.e. $S := Z/m$). Find $\mathbb{E}(S)$ and $\mathrm{Var}(S)$. Show that a sample of 250 people ensures that $\mathrm{Var}(S) \leq 1/1000$.