The geometry of the unipotent component of the moduli space of Weil-Deligne representations

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Abstract

In this paper, we study the moduli space of unipotent Weil-Deligne representations and characterise which irreducible components are smooth. We also study a certain class of unions of irreducible components, and prove that they are Cohen-Macaulay at points (Φ, N) with Φ regular semisimple. We apply the smoothness results proved earlier to show that a certain space of ordinary automorphic forms is a locally generically free module over the corresponding global deformation ring.

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1 Introduction and overview

Let F be a local p-adic field. and let G be a connected reductive algebraic group over F. The local Langlands conjectures (proven for GL_n by Harris and Taylor in [HT99]) stipulate the existence of a natural map, with finite fibres

$$\frac{\{\text{smooth irreducible representations of } G(F)\}}{\{\text{isomorphism}\}} \to \frac{\{\text{L-parameters of } {}^LG\}}{\{\hat{G}-\text{conjugacy}\}}.$$

Let l be a prime, different to p. Let $L \subset \bar{\mathbb{Q}}_l$ be an l-adic field, and $\mathcal{O}$ its ring of integers, with residue field $\mathbb{F}$. Note, that later, we will be interested in relaxing $\mathcal{O}$ to a slightly more general context. In recent years, by work of [BG19], [Hel21], [DHKM20], [Zhu20] and [FS21], there has been great interest in studying the properties of a moduli space of L-parameters $\mathrm{Loc}_{\hat{G},\mathcal{O}}$ and a closely related space, the moduli space of framed L-parameters, $\mathrm{Loc}^{\square}_{\hat{G},\mathcal{O}}$. That is, an algebraic stack over $\mathcal{O}$, which is the the stackification of the prestack whose R-points (R an $\mathcal{O}$-algebra) are naturally identified with the $\hat{G}$-conjugacy classes of L-parameters, and a scheme whose R-points are the set of L-parameters respectively.

$$\mathrm{Loc}_{\hat{G},\mathcal{O}}(R) = \{\text{L-parameters of } \hat{G}, \text{ with } R\text{-coefficients}\}/\cong$$

$$\mathrm{Loc}^{\square}_{\hat{G},\mathcal{O}}(R) = \{\text{L-parameters of } \hat{G}, \text{ with } R\text{-coefficients}\}$$

These spaces ought to have certain nice properties. Firstly, (and trivially)

$$\mathrm{Loc}_{\hat{G},\mathcal{O}} = [\mathrm{Loc}^{\square}_{\hat{G},\mathcal{O}}/\hat{G}]$$

is a quotient stack. Secondly, the (completions of) local rings of $\mathrm{Loc}^{\square}_{\hat{G},\mathcal{O}}$ are local Galois deformation rings. In this way, it is hoped to better understand Galois deformation rings, which is a crucial ingredient in the Taylor-Wiles-Kisin and Calegari-Geraghty patching methods.

To define an L-parameter, one needs the notion of an L-homomorphism. Let W_F be the Weil group of the field F, and for G a connected reductive group let $\hat{G}$ be the Langlands dual group. An L-homomorphism with R-coefficients is a homomorphism $\rho : W_F \to {}^LG(R) := \hat{G}(R) \rtimes W_F$, such that the projection onto the second factor gives the identity map on W_F. In this paper, we reduce to the case where the action of W_F on $\hat{G}$ is trivial (this occurs, for example, when G is split), and so we may view L-homomorphisms as plain homomorphisms $W_F \to \hat{G}$. Historically, there are multiple definitions of L-parameters, with varying degrees of usefulness. We interest ourselves in the moduli space of Bellovin and Gee [BG19] and make the following definition.

Definition 1.1. *A Langlands parameter is a Weil-Deligne representation* (r, N)*, where* $r : W_F \to {}^L G$ *is an L-homomorphism with open kernel, and* N *is an element of* $Lie(\hat{G})$ *such that for any* $g \in W_F$*,* $Ad(g)N = |g|N$*, where* $|.| : W_F \to F^\times \to \mathbb{R}^{\geqslant 0}$ *is the valuation on* W_F *coming from local class field theory.*

It is known, as in Proposition 2.6 of [DHKM20], that this definition works well for characteristic 0. In positive characteristic l, we can still get a similar result, for $l > h_G$ and l-${}^L G$-banal. In this case, we will have an isomorphism between our moduli space and the unipotent connected component of the moduli space of tame parameters seen in [DHKM20], via the exponential and logarithm maps.

By Lemma 2.1.3 of [BG19], this moduli problem can be represented by an algebraic stack over $\mathbb{Q}_l$, $\mathrm{Loc}^{BG}_{G,\mathbb{Q}_l}$, which is a disjoint union of quotient stacks, indexed by the inertial type of the Weil Deligne representation. The moduli problem of framed L-parameters, $\mathrm{Loc}^{\square}_{G,\mathbb{Q}_l}$, can further be represented by an infinite disjoint union of affine varieties, indexed similarly by the inertial type.

From now on until chapter 4, we will denote by $\mathcal{O}$ a regular local ring of residue characteristic l or 0. In this paper, we seek to understand the geometry of the scheme studied in [Hel21]. This is a reduced affine scheme of finite type $S_{G,\mathcal{O}}$, over the ring $\mathcal{O}$, whose R-points (R an $\mathcal{O}$-algebra) are given by

$$S_{G,\mathcal{O}}(R) = \{(\Phi, N) \in G(R) \times \mathfrak{g}(R) | \mathrm{Ad}(\Phi)N = qN\}.$$

This is naturally the space of framed unipotent Weil-Deligne representations over $\mathcal{O}$, with values in G (following Definition 2.1.2 of [BG19]). We will in particular be interested when $\mathcal{O}$ is the ring of integers in a finite extension of $\mathbb{Q}_l$, because the m_R-adic completion of the local rings, R, of the closed points of this scheme can be interpreted as local Galois deformation rings, for sufficiently large l (In fact, whenever the exponential and logarithm maps of Grothendieck's l-adic monodromy theorem exist). We also note, that via Theorem 4.5 of [DHKM20], it is actually sufficient to study $S_{G,\bar{\mathbb{Q}}_l}$ for various groups G to understand the geometry of any connected component of $\mathrm{Loc}^{\square}_{G,\bar{\mathbb{Q}}_l}$, so by restricting to this unipotent case, we do not lose generality in characteristic 0, or whenever this is the correct object of study when l is well behaved.

In sections 2 and 3, we provide a description given by Proposition 2.1 of [Hel21] of the irreducible components of S_n as follows. Let $\mathcal{N} \subseteq \mathfrak{g}$ be the nilpotent cone inside the lie algebra $\mathfrak{g}$. Let

$$p : S_{G,\mathcal{O}} \to \mathcal{N}$$

be the projection map onto the second factor. Let $C \subset \mathcal{N}$ be a locally closed subscheme such that the base change $C_L \subset \mathcal{N}_L$ is a G-conjugacy class inside $\mathcal{N}_L$. (We note that, in the case of GL_n, these can be characterised by partitions of n and in this situation we will denote the conjugacy class corresponding to λ by C_λ.) We remark, that because $S_{G,\mathcal{O}}$ is flat over $\mathcal{O}$, the irreducible components biject naturally with those of $S_{G,L}$. Then $\overline{p^{-1}(C)} \subseteq S_{G,\mathcal{O}}$ is a union of irreducible components of $S_{G,\mathcal{O}}$ (and in the case of $G = \mathrm{GL}_n$, is itself irreducible). All irreducible components arise in this way. In section 3, I expand on and generalise the results of Bellovin [Bel16] section 7.2 and Proposition 7.10 I prove theorems 3.1 and 3.3 which state:

Theorem 1.2. *1. Let $C_r \subseteq \mathcal{N}$ be the regular adjoint orbit, and $C_0 = \{0\} \subseteq \mathcal{N}$ be the zero conjugacy class, and let $X_0 = \overline{p^{-1}(C_0)}$ and $X_r = \overline{p^{-1}(C_r)}$ be the respective irreducible components of $S_{G,\mathcal{O}}$. Then X_0 is smooth over $\mathcal{O}$, and X_r is a disjoint union of $\pi_0(Z)$ smooth connected components, where Z is the centre of G.*

2. Further, in the case $G = GL_n$, these are the only smooth irreducible components of $S_{G,\mathcal{O}}$

In section 4, we turn our interest to certain unions of the components of $S_{n,\mathcal{O}} = S_{\mathrm{GL}_n,\mathcal{O}}$. We will, for each partition p of n, define $X_{\leqslant p} := p^{-1}(\bar{C}_p)$. These varieties arise naturally as the support of certain patched modules. In this section, we conjecture that such varieties are Cohen-Macaulay, and prove it for the following dense subset of points, noted in the following theorem.

Theorem 1.3. *Let $X_{\leqslant p}^{\Phi\text{-}reg}$ be the open subscheme of $X_{\leqslant p}$ whose points (Φ, N) have Φ regular semisimple. Then $X_{\leqslant p}^{\Phi\text{-}reg}$ is Cohen-Macaulay. Further, the local ring at $P = (\Phi, N) \in X_{\leqslant p}^{\Phi\text{-}reg}$ is Gorenstein if and only if either:*

- *$p = 1 + 1 + ... + 1$, and so $X_{\leqslant p}$ is the unramified component of $S_{n,\mathcal{O}}$, or*
- *the inclusion $X_{\leqslant p} \hookrightarrow S_{n,\mathcal{O}}$ defines an isomorphism on stalks at P.*

In addition, we also prove some partial results towards removing the condition of Φ-regular semisimplicity.

In sections 5 and 6 of this paper, I apply the smoothness result of section 3 via the patching method, in a situation very similar to that studied in [Ger18]. Let l be a prime and K a finite extension of $\mathbb{Q}_l$ with ring of integers $\mathcal{O}$. Let F^+ be a totally real global number field, and consider an imaginary quadratic extension F of F^+. The Galois representations considered will correspond to certain Hida families of ordinary automorphic forms on a unitary algebraic group G_D/F^+, which is a unitary form of a unit group of a division algebra, D/F^+. We will define a certain space of Hida families of ordinary automorphic forms $S^{\mathrm{ord}}(U(l^\infty), L/\mathcal{O})_m$ for G_D with Hecke operators $\mathbb{T}$, and a corresponding deformation ring $R_{\mathcal{S}}^{univ}$. We will then use the Taylor-Wiles patching method to deduce the following theorem:

Theorem 1.4. *The module $S^{ord}(U(l^\infty), L/\mathcal{O})_m^\vee[1/l]$ is a finite locally free $R_{\mathcal{S}}^{univ}[1/l]$-module.*

As a consequence, we can deduce that $R_{\mathcal{S}}^{univ}[1/l] \cong \mathbb{T}[1/l]$, and that the multiplicity of automorphic forms with a given characteristic zero Galois representation is constant along connected components of $R_{\mathcal{S}}^{univ}[1/l]$. In particular, one can extend any such multiplicity results from the classical case to the case of non-classical Hida families.

2 Considerateness and the relation to the stack of L-parameters

Let $\mathcal{O}$ be a regular local ring, with residue field $\mathbb{F}$ of characteristic l or 0 and fraction field L. Let G be a connected reductive algebraic group over $\mathcal{O}$ (note that for most of this paper, we will consider $G = \mathrm{GL}_n$) and $\mathfrak{g}$ its Lie algebra. Throughout the paper, whenever l is in play, we will necessarily assume that $l > h_G$, where h_G is the Coxeter number of G.

Definition 2.1. *Let h_G be the Coxeter number of G. Let $q \in \mathcal{O}^\times$ be an element of $\mathcal{O}$ such that $q^k - 1$ is invertible in $\mathcal{O}$ for all $k \leqslant h_G$. When this occurs, we say that q is considerate towards G over $\mathcal{O}$.*

In applications, $\mathcal{O}$ will either be a field, or will be the ring of integers in some field extension of $\mathbb{Q}_l$. Notice that in this case, q being considerate towards G is equivalent to all $1, q, q^2, ..., q^{h_G}$ being distinct in the residue field k (in a sense, q is 'careful' where it treads around G).

Proposition 2.2. *Suppose that $\mathbb{F}$ is a field of positive characteristic $l > h_G$. Suppose further that G is split and one of $GL_n, SL_n, SP_{2n}, SO_{2n+1}$, or SO_{2n} with $n > 1$. Then, in the terminology of [DHKM20], l is ${}^LG$-banal. Conversely, when $G = GL_n$ or SL_n, then l is ${}^LG$-banal implies q is considerate towards G over $\mathcal{O}$.*

Proof. In the split non-exceptional case, Corollary 5.23 of [DHKM20] applies, and thus, if $l > h_G$, we see that l is ${}^LG$-banal if and only if l is G-banal. By Lemma 5.22 of [DHKM20], l is G-banal if and only if l does not divide the order of $G(k_F)$, where k_F is the residue field of F (recall this is a finite field of order q). The following identities can be found in [Sol65]

$$\begin{aligned}
\#\mathrm{GL}_n(k_F) &= q^{\frac{n(n+1)}{2}}(q^1-1)(q^2-1)(q^3-1)...(q^n-1)\\
\#\mathrm{SL}_n(k_F) &= q^{\frac{n(n+1)}{2}}(q^2-1)(q^3-1)...(q^n-1)\\
\#\mathrm{SP}_{2n}(k_F) &= q^{n^2}(q^2-1)(q^4-1)...(q^{2n}-1)\\
\#\mathrm{SO}_{2n+1}(k_F) &= q^{n^2}(q^2-1)(q^4-1)...(q^{2n}-1)\\
\#\mathrm{SO}_{2n}(k_F) &= (q^n \pm 1)q^{n^2-n}(q^2-1)(q^4-1)...(q^{2n-2}-1)
\end{aligned}$$

Notice that in these cases, $h_G = n, n, 2n, 2n, 2n-2$ for $G = \mathrm{GL}_n, \mathrm{SL}_n, \mathrm{SP}_{2n}, \mathrm{SO}_{2n+1}$ and SO_{2n} respectively. Thus, if l is a prime that divides $\#G(k_F)$, then it necessarily divides one of the factors $q^i - 1$ (or possibly $q^n + 1$ if $G = SO_{2n}$). As in all cases but SO_{2n}, $i \leqslant h_G$, this shows the order of q in $\mathbb{F}_l$ is $\leqslant h_G$. When $G = SO_{2n}$, we have that either that the order of q, $o_l(q)$, is either $o_l(q) \leqslant h_G$, or divides $2n = h_G + 2$. Since any prime factors of $2n$ are $\leqslant 2n - 2$ (provided $n > 1$), this completes the forward direction. When $G = \mathrm{GL}_n$ or SL_n, we also see that the converse holds. ∎

We make the following definition.

Definition 2.3. *We define the affine scheme $S_{G,\mathcal{O}}$ over $\mathcal{O}$ as the scheme whose R-points (R, an $\mathcal{O}$ algebra) are $\{(\Phi, N) \in G(R) \times \mathfrak{g}(R) : Ad(\Phi)N = qN\}$*

Corollary 5.4 of [Bel16] shows that this is a reduced scheme, and hence is a variety when $\mathcal{O}$ is a field. As discussed in the introduction, we may picture $S_{G,\mathcal{O}}$ as the moduli space of unipotent Weil-Deligne representations, (r, N) over $G(\mathcal{O})$. The unipotent condition is equivalent to that of $r(I_F) = 1$.

Proposition 2.4.

1. *Suppose q is considerate towards $G_{/\mathcal{O}}$. Then the natural map $p : S_G \to \mathfrak{g}$ factors through the nilpotent cone $\mathcal{N}_G$.*
2. *When G is split, and $l > h_G$ then $S_{\hat{G},\mathcal{O}}$ is isomorphic to a closed subscheme of the moduli space of tame parameters $Z^1(W_F^0/P_F, \hat{G})_{\mathcal{O}}$ (See section 1.2 of [DHKM20] for a definition of this space).*
3. *In addition, when l is ${}^LG$-banal, this space is a connected component of $Z^1(W_F^0/P_F, \hat{G})_{\mathcal{O}}$.*

Proof. Since any S_G can be embedded into S_n for some n where $G \to \mathrm{GL}_n$ is a faithful embedding, we need only show that N is nilpotent for GL_n. Let $(\Phi, N) \in S_n(R)$ be an R-point where R is an $\mathcal{O}$-algebra. If $M \in \mathfrak{gl}_n$ is a matrix, let $s_i(M)$ be the i-th coefficient of the characteristic polynomial of M. Notice that s_i is conjugate invariant, as the characteristic polynomial is.

Hence, we see that for each i, $s_i(\Phi N \Phi^{-1}) = s_i(qN)$, so $s_i(N) = q^i s_i(N)$. As q is considerate towards $G_{/\mathcal{O}}$, we have that $q^i - 1$ is invertible in $\mathcal{O}$, and hence that $s_i(N) = 0$. We hence see that the characteristic polynomial of N is X^n. This shows that N is strongly nilpotent, and lies in the R-points of the nilpotent cone.

When G is a split group, $Z^1 = Z^1(W_F^0/P_F, \hat{G})_{\mathcal{O}}$ has a model as an affine scheme, flat over $\mathcal{O}$ (since $l \neq p$) with R-points equal to

$$Z^1(W_F^0/P_F, \hat{G})_{\mathcal{O}}(R) = \{(\phi, \sigma) \in \hat{G}(R)^2 : \phi\sigma\phi^{-1} = \sigma^q\}.$$

Since $l > h_G$, and we can invert by all primes $\leqslant h_G$, the exponential and logarithm maps of section 6 of [BDP17] are well defined polynomials, and thus we have an isomorphism between the nilpotent cone in $\mathcal{N}_G$ and unipotent cone $\mathfrak{U}_G$. Hence, we have a map

$$\begin{aligned} S_{\hat{G},\mathcal{O}} &\to Z^1(W_F^0/P_F\hat{G})_{\mathcal{O}} \\ (\phi, N) &\mapsto (\phi, \exp N) \end{aligned}$$

which is an isomorphism onto the closed subscheme of $Z^1(W_F^0/P_F\hat{G})_{\mathcal{O}}$ given by those elements (ϕ, σ) with $\sigma \in \mathcal{U} \subseteq \hat{G}$, where $\mathcal{U}$ is the unipotent cone.

For part 2, suppose l is ${}^LG$-banal. Let $\mathfrak{U}^+$ be the scheme-theoretic image of $Z^1(W_F^0/P_F\hat{G})_{\mathcal{O}}$ through the second projection onto $\hat{G}$. We note, that $\sigma \in \mathfrak{U}^+$ necessarily has σ conjugate to σ^q. Let $T \subset \hat{G}$ and $W = W_{\hat{G}}$ be a maximal split torus and the Weyl group of $\hat{G}$ respectively. Consider the map $\hat{G} \to \hat{G}//\hat{G} \cong T/W$. The image of $\mathfrak{U}^+$ through this map has image given by the scheme-theoretic union $S := \bigcup_{w \in W} \{\sigma \in T : \sigma^q = {}^w\sigma\}$, which is a finite flat scheme over $\mathcal{O}$. Thus, since the fibres of this map are conjugacy classes, they are connected, and hence, the connected components of $\mathfrak{U}^+$ are in bijection with those of S. If l is ${}^LG$-banal, then $Z^1_{\mathbb{F}}$ is reduced, and thus, so is $S_{\mathbb{F}}$. Hence, since S is finite flat over $\mathcal{O}$, we see that the connected components of the generic fibre are in natural bijection with those of the special fibre, and thus the same is true for Z^1. Hence, as $S_{\hat{G},\mathcal{O}}$ defines a connected component over the generic fibre, it is a connected component of Z^1.

□

We quote some results.

Proposition 2.5. *1. $S_{G,\mathcal{O}}$ is flat over $\mathcal{O}$ if q is considerate towards $G_{/\mathcal{O}}$.*

2. *The algebraic group G acts on S_G via the simultaneous conjugation*

$$g.(\Phi, N) = (g\Phi g^{-1}, Ad(g)N)$$

3. *$S_{G,\mathcal{O}}$ is complete intersection $\mathcal{O}$-scheme of relative dimension* $\dim G$ *over $\mathcal{O}$*

4. *Define the second projection map $p : S_G \to \mathcal{N}_G$ as earlier. If C is a $G_{/L}$ conjugacy class inside $\mathcal{N}_{G,L} \subseteq \mathcal{N}_G$, then the closed subscheme $X_C := \overline{p^{-1}(C)} \subset S_G$ is a union of irreducible components, and we have $S_G = \bigcup_C X_C$.*

5. *If in addition $G = GL_n$, the X_C are irreducible components of $S_{n,\mathcal{O}} := S_{GL_n,\mathcal{O}}$, and these can be naturally identified with partitions of n. We call the component corresponding to the partition p, X_p.*

Proof. 1. In this case, $S_{G,\mathcal{O}}$ is a open subscheme of Z^1 which by [DHKM20] is flat over $\mathcal{O}$.

2. This is clear.

3. As $S_{G,\mathcal{O}}$ is isomorphic to the fibre over 0 of the map $G \times \mathfrak{g} \to \mathfrak{g}$ given by $(g, N) \mapsto \mathrm{Ad}(g)N - qN$, we see that each irreducible component is of dimension at least $\dim(G) + \dim(\mathcal{O})$. To show equality, we note that by the previous proposition $S_{G,\mathcal{O}}$ is a closed subvariety of $Z^1(W^0_F/P_F, \hat{G})_{\mathcal{O}}$, which by corollary 2.4 of [DHKM20] has dimension $\dim(G)+\dim(\mathcal{O})$. This shows that $S_{G,\mathcal{O}}$ is a complete intersection.

4. As $S_{G,\mathcal{O}}$ is flat over $\mathcal{O}$, the irreducible components of $S_{G,\mathcal{O}}$ are exactly those of the open subscheme $S_{G,L}$, so we may without loss of generality work with $\mathcal{O} = L$. The characteristic zero case is covered exactly by Proposition 2.1 of [Hel21]. In the characteristic l case, one must utilise

q-considerateness and part 1 of Proposition 2.4 to show that the map p indeed factors through $\mathcal{N}_G$ before one can apply Proposition 2.1 of [Hel21].

5. For $G = \mathrm{GL}_n$, choose a closed point $J \in C$. Then the fibres of the map $p^{-1}(C) \to C$ over any closed point x are a Torsor over the centraliser $C_{\mathrm{GL}_n}(J)$. We remark that the map $p^{-1}(C) \to C$ is flat with smooth fibres, and thus is smooth, and open. Since centralisers inside GL_n are irreducible, C is irreducible, and p is open , by [Sta23, Lemma 004Z], it follows that $p^-1(C)$ is irreducible, and thus so is X_C. The final claim follows from the theory of Jordan normal forms.

□

2.1 Lemmas in commutative algebra and algebraic geometry

The remaining part of this section proves some lemmas from algebraic geometry and commutative algebra that we will need later

Lemma 2.6. *Let G be a smooth algebraic group over a scheme S, and let X be an S scheme. Suppose we have a morphism $m : G \times_S X \to X$ defining a group action of G on X. Then m is a smooth morphism.*

Proof. First, since G is smooth, we have that $G \to S$ is smooth. Hence the projection $p_X : G \times_S X \to X$ obtained by the base change of this map to X, is a smooth morphism. Now, consider the automorphism, ϕ of $G \times_S X$ given by $(g, x) \mapsto (g, g.x)$. as this is an isomorphism, it is a smooth morphism.

Now, observe that $m = p_X \circ \phi$ is a composite of smooth morphisms, and is hence smooth. □

Lemma 2.7. *Let P be one of the properties of local Noetherian rings: regular, local complete intersection, Gorenstein or Cohen Macaulay. Then for (A, m) a local Noetherian ring with maximal ideal m, A is P if and only if the m-adic completion $\hat{A}$ is P.*

Proof. For the properties Cohen Macaulay and regular, this is [Sta23, Lemma 07NX] and [Sta23, Lemma 07NY] respectively. For a local complete intersection, let $A = R/\langle x_1, ..., x_k \rangle$, with R local regular. Since $\hat{R}/x_1, ..., x_n \cong \hat{A}$, and by [Sta23, Lemma 07NV], it follows easily that A is a local complete intersection ring if and only if $\hat{A}$ is. To prove the statement for the Gorenstein property, notice that A is Cohen-Macaulay if and only if $\hat{A}$ is. Hence, after quotienting by a maximal length regular sequence $(\mathbf{x})$ in A, we see that it is sufficient to prove that $A/(\mathbf{x})$ is Gorenstein if and only if $\hat{A}/(\mathbf{x}) \cong (\widehat{A/(\mathbf{x})})$ is. But since these rings are zero dimensional (and are hence, Artinian), the natural inclusion $A/(\mathbf{x}) \hookrightarrow (\widehat{A/(\mathbf{x})})$ is an isomorphism. This proves the Lemma. □

Lemma 2.8. *Let P be one of the local properties: regular, local complete intersection, Gorenstein or Cohen-Macaulay. Let $f : X \to Y$ be a smooth morphism of schemes. Let $p \in X$. Then Y is P at $f(p)$ if and only if X is P at p.*

Proof. Suppose f has relative dimension n. Then by [Sta23, Lemma 054L] the map f factors locally through

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[dr, "f"'] & \mathbb{A}^n_Y \arrow[d] \\ & Y \end{tikzcd}$$

with g étale. Thus, it suffices to prove the lemma in the case f étale, and in the case $\mathbb{A}^n_Y \to Y$. In the étale case, since étale morphisms induce isomorphisms on the completions of stalks, and by the previous lemma, for a Noetherian local ring, R is P if and only if the completion $\hat{R}$ is P, the result of the lemma follows in the étale case. In the affine case, it suffices to note that a local ring R is P if and only if $R[x]_x$ is P. $\square$

Lemma 2.9. *Suppose $(\mathcal{O}, \mathfrak{p}, \mathbb{F})$ is a regular local ring and R is a Noetherian local flat $\mathcal{O}$-algebra, with $\bar{R} = R/\mathfrak{p}$. Then R is Cohen Macaulay if and only if $\bar{R}$ is Cohen Macaulay.*

Proof. Suppose $\mathcal{O}$ has dimension d, and R has dimension n. Suppose R is Cohen Macaulay. Let $x_1, ..., x_d$ be a regular sequence for $\mathcal{O}$. Then this can be extended to a maximal regular sequence for R, $x_1, ..., x_d, x_{d+1}, ..., x_n$. We see immediately that since $\mathcal{O}$ is regular, that $x_{d+1}, ..., x_n$ is a regular sequence for $\bar{R}$ of length $n-d$, and since the dimension of this is also $n-d$, we see $\bar{R}$ is Cohen Macaulay.

Suppose conversely, that $\bar{R}$ is Cohen Macaulay. Then a maximal regular sequence $\bar{y}_1, ..., \bar{y}_{n-d}$ for $\bar{R}$ can be lifted to a sequence $y_1, ..., y_{n-d}$ in R, such that $x_1, ..., x_d, y_1, ..., y_{n-d}$ is a regular sequence for R. R is then Cohen Macaulay. $\square$

Lemma 2.10. *Let R be a finite local $\mathcal{O}$-algebra, and let x, $\bar{x}$ be prime ideals of R that give rise to the following commutative diagram.*

$$\begin{tikzcd} R \arrow[r, "x"] \arrow[dr, "\bar{x}"'] & \mathcal{O} \arrow[r, hook] \arrow[d, two heads] & L = \mathcal{O}[\frac{1}{l}] \\ & \mathbb{F} & \end{tikzcd}$$

Then

$$R^{\wedge}_{\bar{x}}\left[\frac{1}{l}\right]^{\wedge}_{x} \cong R^{\wedge}_{x}$$

Proof. Notice that since $R\backslash x \supseteq R\backslash \bar{x} \cup \{\frac{1}{l}\}$, that $R_{\bar{x}}\left[\frac{1}{l}\right]_x \cong R_x$. Further, since R is of finite type over $\mathcal{O}$, we have $\bigcap_n \bar{x}^n = 0$, and thus we have an injection $R_{\bar{x}} \hookrightarrow R^{\wedge}_{\bar{x}}$. This gives us a local homomorphism inclusion

$$R_x = R_{\bar{x}}\left[\frac{1}{l}\right]_x \hookrightarrow R^{\wedge}_{\bar{x}}\left[\frac{1}{l}\right]_x$$

We notice that $R_x/x \cong L$, and that

$$\left[R^{\wedge}_{\bar{x}}\left[\frac{1}{l}\right]_x\right]/x \cong \left[\varprojlim_n (R/\bar{x}^n)/x\right][1/l] \cong \varprojlim_n (R/(x, l^n))[1/l] \cong (\varprojlim \mathcal{O}/l^n)[1/l] = L.$$

Thus, by [? , Lemma 0394], we have that $R_{\bar{x}}^{\wedge}\left[\frac{1}{l}\right]_x^{\wedge}$ is generated by the same topology as $R_x^{\wedge}$, and is a finite $R_x^{\wedge}$- module. It is now easy to see from looking at the residue field that the natural map

$$R_x^{\wedge} \to R_{\bar{x}}^{\wedge}\left[\frac{1}{l}\right]_x^{\wedge}$$

is a surjection. It is also an injection, because the two rings have the same topology. In particular, if a sequence inside R_x converges to zero inside $R_{\bar{x}}^{\wedge}\left[\frac{1}{l}\right]_x^{\wedge}$, then it must converge to zero inside $R_x^{\wedge}$. This shows that the kernel is zero, and thus that the map is an isomorphism.

□

Corollary 2.11. *Let Λ be a finite type $\mathcal{O}$-algebras, and let R_1, R_2 be finite type Λ-algebras, and let $R = R_1\widehat{\otimes}_\Lambda R_2$. let $x \in Spec(R[1/l])$ be a maximal ideal. Then $(R_1 \otimes_\Lambda R_2)_x^{\wedge} \cong R[1/l]_x^{\wedge}$. In particular, if $R_i[1/l]$ is smooth for each i, then $R[1/l]$ is smooth.*

Proof. Set $\bar{x}$ as the maximal ideal of $R_1 \otimes_\Lambda R_2$. Then for any x as above, we get a commutative diagram as in the statement of Lemma 2.10. Hence, by Lemma 2.10, we see that

$$(R_1 \otimes_\Lambda R_2)[1/l]_x^{\wedge} \cong ((R_1 \otimes_\Lambda R_2)\bar{x}^{\wedge})[1/l]_x^{\wedge} \cong R[1/l]_x^{\wedge}.$$

To show the last part, it is sufficient to notice that since $R_1[1/l]\otimes_{\Lambda[1/l]}R_2[1/l], R[1/l]$ are finite type over L, they are x-adically separated, and thus are regular at x if and only if $R_1[1/l]\otimes_{\Lambda[1/l]} R_2[1/l]_x^{\wedge}, R[1/l]_x^{\wedge}$ are. Since $R_1[1/l]\otimes_{\Lambda[1/l]} R_2[1/l]$ is regular if and only if both $R_i[1/l]$ are, this completes the corollary. □

3 Smoothness results for X_p

In section 7.2 in [Bel16], Bellovin proves in the case where $\mathcal{O}$ is a field of characteristic 0, that the component X_n of $S_{\mathrm{GL}_n,\mathcal{O}}$ corresponding to the regular nilpotent orbit is smooth. The following theorem generalises this result to general connected reductive groups G, and more general regular local rings. Let $\mathcal{O}$ be a regular local ring with residue characteristic l or 0 as before. For general connected reductive groups G, we can generalise the decomposition of Proposition 2.5, to give $S_{G,\mathcal{O}} = \bigcup_C X_C$ where for an adjoint orbit, C, of the nilpotent cone $\mathfrak{n} \subset \mathfrak{g}$, X_C is the closure $\overline{p^{-1}(C)}$ with $p : S_{G,\mathcal{O}} \to \mathfrak{n}$ the natural G-equivariant projection. Note, that for more general groups G, these may not be irreducible. Indeed, if C is the regular nilpotent adjoint orbit of SL_2, then X_C is the union of two connected components. The following theorem shows that in C is a regular nilpotent conjugacy class, then X_C is smooth, and thus the connected components are the same as the irreducible components.

Theorem 3.1. *Let $G_{/\mathcal{O}}$ be a smooth reductive group with smooth centre, Z, and let $\mathfrak{g}$ be the Lie algebra of G, and suppose $q \in \mathcal{O}$ is considerate towards G over*

$\mathcal{O}$. Suppose that $C \subset \mathcal{N}_L$ is either the 0 or the regular nilpotent adjoint orbit. Then X_C is smooth over $\mathcal{O}$, and when C is the regular nilpotent orbit, X_C has the same number of connected components as Z.

Proof. Consider first the case $C = 0$. Then $X_C = \{(\Phi, 0) \in S_{G,\mathcal{O}}\} \cong G$ via the map projecting to the Φ-coordinate. Since $G_{/\mathcal{O}}$ is smooth, this proves the theorem.

For the regular nilpotent case, note that X_C is flat and finitely generated over $\mathcal{O}$, so by [Sta23, Lemma 01V8] we have that $X_{\mathcal{O}}$ is smooth over $\mathcal{O}$ if and only if it is smooth over every localisation. It is therefore sufficient to prove the theorem after a localisation to a field. Without loss of generality, let $k = k(p)$ be a field for $p \in \mathrm{Spec}(\mathcal{O})$, and assume all subsequent schemes are schemes over k. Consider now, the case $C \subseteq \mathcal{N}$ is regular nilpotent adjoint orbit. Since qJ and J are conjugate, there is an element $\Phi_J \in G$ such that $\mathrm{Ad}(\Phi_J).J = qJ$. We claim that Φ_J is regular semisimple.

Since J is regular nilpotent, there is a unique Borel subgroup, B, such that $J \in \mathrm{Lie}(B)$. Let $\Pi = \{\alpha_1, ..., \alpha_h\}$ be the corresponding set of simple roots of G, and let $\{e_\alpha\} \in \mathfrak{g}$ be the set of eigenvectors of $\mathfrak{g}$ corresponding to the roots of G. We can write $J = \sum_{\alpha \in \Pi} c_\alpha e_\alpha \in \mathfrak{g}$ for $c_\alpha \neq 0$. Hence, we see

$$\sum_{\alpha \in \Pi} qc_\alpha e_\alpha = qJ = \mathrm{Ad}(\Phi_J)J = \sum_{\alpha \in \Pi} c_\alpha \alpha(\Phi_J) e_\alpha$$

and so $\alpha(\Phi_J) = q$ for every simple root α. If β is a positive root of G, we see that β is some positive combination of the α_i. Suppose $\beta = \sum_i m_i \alpha_i$. Then $\beta(\Phi_J) = q^{m_1 + \ldots + m_h}$. As q is considerate towards G over $\mathcal{O}$ (and hence is considerate towards G over k), we see that no $\beta(\Phi_J) = 1$. Hence Φ_J is regular semisimple by Lemma 12.2 of [Bor91].

Since Φ_J is regular semisimple, it is contained in a unique torus $T \subset G$. Consider the k-scheme

$$Y = Z\Phi_J \times \overline{T.J}.$$

We first claim that this is a subscheme of X_C. Let $(s\Phi_J, \mathrm{Ad}(t).J) \in Z\Phi_J \times T.J$. Then

$$\begin{aligned}
\mathrm{Ad}(s\Phi_J)(\mathrm{Ad}(t)J) &= \mathrm{Ad}(s\Phi_J t)J \\
&= \mathrm{Ad}(t\Phi_J s)J && \text{because } T \text{ is abelian} \\
&= \mathrm{Ad}(t)\mathrm{Ad}(\Phi_J)J \\
&= \mathrm{Ad}(t)(qJ) \\
&= q\mathrm{Ad}(t)J.
\end{aligned}$$

Hence, $Z\Phi_J \times T.N \subset X_C$. Since X_C is closed, we then see that the closure $\overline{Z\Phi_J \times T.J} = Z\Phi_J \times \overline{T.N} = Y \subset X_C$.

We now claim that Y is smooth over k. This is clear, because $Z_{/\mathcal{O}}$ is smooth by hypothesis and $\overline{T.J} = \mathrm{Span}(e_{\alpha_1}, ..., e_{\alpha_h})$ is isomorphic to affine space, $\mathbb{A}_k^h$.

Define the morphism

$$\begin{aligned} f : G \times Y &\to X_C \\ (g, (\Phi, N)) &\mapsto (g\Phi g^{-1}, \mathrm{Ad}(g)N). \end{aligned}$$

Consider the following commutative diagram

$$\begin{array}{ccc} G \times Y & \longrightarrow & X_C \\ \downarrow & & \downarrow \\ G \times Z\Phi_J & \longrightarrow & Z.G_{\Phi_J} \end{array}$$

where G_{Φ_J} denotes the conjugacy class of Φ_J in G, the vertical maps come from the "forget N" projections $(g, s\Phi_J, N) \in G \times Y \mapsto (g, s\Phi_J) \in G \times Z\Phi_J$ and $(\Phi, N) \in X_C \mapsto \Phi \in ZG_{\Phi_J}$ respectively and the horizontal maps are defined via the conjugation action of $g \in G$ on Y so that the diagram commutes, and is a pullback square. The bottom map, m, is flat with fibres isomorphic to $\mathrm{Stab}_G(\Phi_J)$, which is simply the Torus T, as Φ_J is regular semisimple. This shows that m is smooth. Hence, since the map f is the base change of m to X_C, by Proposition 10.1 of [Har77] we see that f is smooth.

Then by Lemma 2.8, since every point on $G \times Y$ is regular, this implies that its image in X_C is a smooth variety. To finish the proof, it is enough to show that this map is surjective. This is the same as saying that every pair $(\Phi, N) \in X_C$ is conjugate to something in Y.

Let $(\Phi', N) \in |X_C|$. Then there exists a regular nilpotent J' such that $\mathrm{Ad}(\Phi')J' = qJ'$. Then J' is conjugate to J by some element $g \in G_{/\mathcal{O}}$ (i.e. $\mathrm{Ad}(g)J' = J$). Then if $\Phi = g\Phi' g^{-1}$, we see $\mathrm{Ad}(\Phi)J = qJ$. By conjugating by an element of $\mathrm{Stab}_G(J)$, we can assume without loss of generality that Φ lies in T. Hence, $s = \Phi\Phi_J^{-1}$ is an element of $\mathrm{Stab}_T(J)$. We claim that $\mathrm{Stab}_T(J) = Z$. It is clear that there is a closed immersion $Z \subseteq \mathrm{Stab}_T(J)$, so we need only show this is surjective (as Z is smooth). Since $s \in \mathrm{Stab}_T(J)$ commutes with J, we see that $\mathrm{Ad}(s)J = J$, and thus $\sum_{\alpha \in \Pi} c_\alpha \alpha(s) e_\alpha = \sum_{\alpha \in \Pi} c_\alpha e_\alpha$. Since e_α form a basis of $\mathfrak{g}$, we see that $\alpha(s) = 1$ for each $\alpha \in \Pi$. Since this is a base, we see that $\beta(s) = 1$ for all roots β of G. Hence, s acts as the identity on the adjoint representation, and so lies in the centre $s \in Z$. Since $\mathrm{Ad}(g)N$ conjugates with Φ in the correct way, we see that N is a span of simple roots of G, and thus lies in $\overline{T.J}$. This shows that (Φ', N) is the image of $(g^{-1}, (A\Phi_J, \mathrm{Ad}(g)N)) \in G \times Y$. This proves the smoothness statement.

For the statement about the connected components, it suffices to notice that since G is connected, that the connected components of $G \times Y$ biject with those of Y, which in turn biject with the connected components of Z. Hence it suffices to show that there is a bijection between the connected components of $G \times Y$ and X_C. It is sufficient to show that the fibres of the G equivariant map $f : G \times Y \to X_C$ are connected. Since the action of G gives an isomorphism on fibres, it is sufficient to show that the fibres of $Y \subseteq X_C$ are connected. Let $P = (\Phi, N) \in Y$. Then $f^{-1}(P) = \{(g, \Phi', N') \in G \times Y : g\Phi' g^{-1} = \Phi \text{ and } \mathrm{Ad}(g)(N') = N\}$. Since

$\Phi, \Phi' \in Z\Phi_J \subset T$ are regular semisimple, any $g \in G$ such that $g\Phi g^{-1} = \Phi'$ lies in the normaliser $N_G(T)$. Notice that for any simple root α of G, $\alpha(g\Phi g^{-1}) = \alpha(\Phi') = q = \alpha(\Phi)$. This implies that g must actually lie in $Z_G(T) = T$, and thus we get a well defined isomorphism

$$\begin{aligned} f^{-1}(P) &\leftrightarrow T \\ (g, \Phi', N') &\mapsto g \\ (g, \Phi, \mathrm{Ad}(g)^{-1}(N)) &\leftarrow g \end{aligned}$$

Thus, since T is connected, so is $f^{-1}(P)$. This proves the final part of the theorem. $\square$

The conditions that G has smooth centre and that $q \in \mathcal{O}$ is considerate towards $G/\mathcal{O}$ are quite mild conditions. For example, if $\mathcal{O}$ is a field of characteristic 0 and q isn't a root of unity, q is automatically considerate. Further, when $q \in \mathbb{Z}$ is a prime power, if the residue characteristic, l, of $\mathcal{O}$ is larger than $q^{t(G)}$, then q is considerate. Since the centre of a reductive group G is smooth in large enough characteristic, this also shows that X_C is smooth over $\mathcal{O}$ with sufficiently large residue characteristic.

One may hope that the previous result holds for all components of S_G. i.e. that all components of S_G are smooth. When $G = \mathrm{GL}_2$, this is true since the only two components are those arising from the nilpotent conjugacy classes of $N = 0$, and $N = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$, and both cases studied in the previous theorem, (see also proposition 4.8.1 of [Pil08]). In [Bel16], Bellovin proves that this fails for GL_3, demonstrating that the component X_{21} is not smooth, and gives a description of all the points where singularities exist. Theorem 3.3 generalises these results, and shows us that, for $G = \mathrm{GL}_n$ and any partition $p \neq 1^n, n$, the component X_p is always singular.

We define some notation. For a an element of an $\mathcal{O}$-algebra R, and k a positive integer, define the $k \times k$ matrix,

$$M_k(a) = \begin{pmatrix} aq^{k-1} & \dots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & aq & 0 \\ 0 & \dots & 0 & a \end{pmatrix}.$$

If k is a positive integer, and $\underline{b} = (b_1, ..., b_{k-1}) \in R^{k-1}$ are a $k-1$-tuple of elements of R, then set the $k \times k$ matrix

$$J_k(\underline{b}) = \begin{pmatrix} 0 & b_1 & & \\ & 0 & b_2 & \\ & & 0 & \ddots \\ & & & \ddots \end{pmatrix}.$$

Lemma 3.2. *Let R be a finitely generated $\mathcal{O}$-algebra. Let $p = k_1 + k_2 + ... + k_m$ be a partition of n. For $a_i \in R^\times$, and $\underline{b}_i \in R^{k_i - 1}$ the pair*

$$\left(\begin{pmatrix} M_{k_1}(a_1) & \dots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & M_{k_{m-1}}(a_{m-1}) & 0 \\ 0 & \dots & 0 & M_{k_m}(a_m) \end{pmatrix}, \begin{pmatrix} J_{k_1}(\underline{b}_1) & \dots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & J_{k_{m-1}}(\underline{b}_{m-1}) & 0 \\ 0 & \dots & 0 & J_{k_m}(\underline{b}_m) \end{pmatrix} \right) \in X_p(R).$$

Proof. When each of the vectors $\underline{b}_i$ lie in $R^\times$, the pair

$$(\Phi, N_\lambda) = \left(\begin{pmatrix} M(k_1,a_1) & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & M(k_m,a_m) \end{pmatrix}, \begin{pmatrix} \lambda J_{k_1} & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \lambda J_{k_m} \end{pmatrix} \right) \in p^{-1}(C_p)(R)$$

is inside $X_p(R)$. Hence, we obtain a morphism of schemes over R:

$$\pi' : \mathbb{G}_{m,R}^{n-m} \to p^{-1}(C_p)_R$$
$$(\underline{b}_1, \dots, \underline{b}_m) \mapsto \left(\begin{pmatrix} M(k_1,a_1) & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & M(k_m,a_m) \end{pmatrix}, \begin{pmatrix} J_{k_1}(\underline{b}_1) & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & J_{k_m}(\underline{b}_m) \end{pmatrix} \right)$$

which is an isomorphism onto it's scheme theoretic image and which extends naturally to a map $\pi : \mathbb{A}_R^{n-m} \to S_{n,R}$. Since the Zariski closure of $\mathbb{G}_{m,R}^{n-m}$ inside $\mathbb{A}_R^{n-m}$ is $\mathbb{A}_R^{n-m}$, we see that the Zariski closure of the image of π' inside $S_{n,R}$ is the image of π. Since $X_{p,R}$ is the Zariski closure of $p^{-1}(C_p)_R$, it follows that $X_{p,R}$ contains the image of π. The lemma then follows by looking at the R points of the image of π and $S_{n,R}$. □

Theorem 3.3. *Let $G = GL_n$, and let p be a partition of n with $p \neq 1^n, n$. Then X_p is singular.*

Proof. Let $\mathbb{F}$ be the residue field of $\mathcal{O}$. Consider the following Cartesian diagram

$$\begin{array}{ccc} X_{p,\mathbb{F}} & \longrightarrow & \operatorname{Spec}(\mathbb{F}) \\ \downarrow & & \downarrow \\ X_{p,\mathcal{O}} & \longrightarrow & \operatorname{Spec}(\mathcal{O}) \end{array}$$

If the map $X_{p,\mathcal{O}} \to \operatorname{Spec}(\mathcal{O})$ were smooth, then by Proposition 10.1b) of [Har77] the map $X_{p,\mathbb{F}} \to \operatorname{Spec}(\mathbb{F})$ would also be smooth. Hence, without loss of generality, it suffices to show that $X_{p,\mathcal{O}}$ is singular when $\mathcal{O} = \mathbb{F}$ a field.

Choose any point $P = (\Phi_0, 0) \in X_p$, with Φ_0 semisimple. Define three subvarieties of S_n that contain P as follows.

1. Let $C = \mathrm{GL}_n.P$, be the GL_n-orbit of P.

2. Let D be the variety of diagonal matrices inside GL_n, seen as a subvariety of S_n via the inclusion $\Phi \mapsto (\Phi, 0)$.

3. Let $\mathcal{N}_0 = \{N \in \mathfrak{gl}_n : \Phi_0 N \Phi_0^{-1} = qN\}$ viewed as a closed subvariety of S_n via the inclusion $N \mapsto (\Phi_0, N)$.

Let $\mathbb{F}[\epsilon]$ be the ring of dual numbers. The first claim we make, is that the tangent space $T_P C$ can be identified with the elements of $X_p(k[\epsilon])$ that are $\mathrm{GL}_n(\mathbb{F}[\epsilon])$-conjugate to P, and have image P under the base change of the natural map $\operatorname{Spec}(\mathbb{F}) \to \operatorname{Spec}(\mathbb{F}[\epsilon])$ which sends $\epsilon \mapsto 0$. Note that we have a smooth surjective morphism $\mathrm{GL}_n \to C$, given by the conjugation action $g \mapsto g.P$, and so we have a surjection on the level of tangent spaces and a surjection $\mathrm{GL}_n(\mathbb{F}[\epsilon]) \to C(\mathbb{F}[\epsilon])$. This shows that any element of $C(\mathbb{F}[\epsilon])$ is conjugate to P via some element of $\mathrm{GL}_n(\mathbb{F})$. The rest of the claim is obvious.

Consider the tangent spaces of these varieties at P, T_PC,T_PD and $T_P\mathcal{N}_0$. We claim that they form a direct sum inside T_PS_n. Let $P' = (\Phi', 0) \in T_PC \cap T_PD$. Then Φ' is a diagonal matrix in $\mathrm{GL}_n(\mathbb{F}[\epsilon])$, and is conjugate to Φ_0. Since diagonal matrices are only conjugate to each other if they share the same entries, this means that Φ' lies inside $\mathrm{GL}_n(\mathbb{F})$, and thus, $P' = P$. To show that $T_P\mathcal{N}_0$ intersects at the origin with T_PC or T_PD, it suffices to notice that in either case, an element of T_PC or T_PD takes the form $P' = (\Phi', 0)$, while an element $P' \in T_P\mathcal{N}_0$ takes the form $P' = (\Phi_0, N) \in S_n(\mathbb{F}[\epsilon])$. For these to be equal, we must have $\Phi' = \Phi_0$ and $N = 0$, so $P' = P$. This proves the claim.

We split the proof of this theorem into two cases: the case where the parts of p are not all the same and the case where $p = k^m$ for integers $k, m > 1$ such that $km = n$. In both cases, the following strategy will be to count the number of linearly independent deformations in each of the subspaces of T_PX_p, T_PC, $T_PD \cap T_PX_p$ and $T_P\mathcal{N}_0 \cap T_PX_p$ and combine to give a lower bound on the dimension of T_PX_p, showing that $\dim_{\mathbb{F}} T_P > n^2 = \dim X_p$. This will prove the theorem.

Consider the case $p = (k_1, ..., k_m)$ with $k_1 \geqslant k_2 \geqslant ... \geqslant k_m$, not all equal. Consider the $n \times n$ diagonal matrix, $\Phi_0 = \mathrm{Diag}(q^{n-1}, ..., q, 1)$. Notice that Φ_0 has distinct eigenvalues, so that the stabiliser of $P = (\Phi_0, 0)$ is the n dimensional torus T_n. By orbit-stabiliser, we then note that the orbit space must be $n^2 - n$ dimensional, and thus $\dim_{\mathbb{F}}(T_PC) \geqslant n^2 - n$. Consider now the deformations in $T_P\mathcal{N}_0$. Let $(\Phi_0, M\epsilon) \in X_p(\mathbb{F}[\epsilon]) \subseteq S_n(\mathbb{F}[\epsilon])$. The defining equation of $S_{n,\mathbb{F}}$ shows that all non-zero entries of M must lie on the off-diagonal. Further, to ensure $(\Phi_0, M\epsilon)$ lies on the component defined by p, one may choose, in accordance with Lemma 2.7, M as a block diagonal matrix, with blocks of size $k_1, k_2, ..., k_m$, each of the form

$$\begin{pmatrix} 0 & * & & & \\ & 0 & * & & \\ & & \ddots & & \\ & & & 0 & * \\ & & & & 0 \end{pmatrix}$$

This leaves us with $\sum_i (k_i - 1) = n - m$ different non-zero entries of M, each of which defines a deformation, all of which are linearly independent, because they are inside $T_P(\mathrm{GL}_n \times \mathfrak{gl}_n) = \mathfrak{gl}_n^2$. Finally, consider the blocks of Φ defined by the partition p. For each $1 \leqslant i \leqslant m$, consider the matrix

$$E_i = \begin{pmatrix} I_{k_1} & & & & & \\ & I_{k_2} & & & & \\ & & \ddots & & & \\ & & & (1+\epsilon)I_{k_i} & & \\ & & & & \ddots & \\ & & & & & I_{k_m} \end{pmatrix} \in M_n(\mathbb{F}[\epsilon])$$

where I_k denotes the $k \times k$ identity matrix.

We consider the deformation $(\Phi E_i, 0)$ and note that this is contained in $X_p(\mathbb{F}[\epsilon])$ via Lemma 2.7, because we can split ΦE_i into block diagonal parts of sizes $k_1, ..., k_m$. This gives us m further deformations, which are similarly linearly independent because they are linearly independent inside $T_P(\mathrm{GL}_n \times \mathfrak{gl}_n)$. Finally, we note that we may reorder the blocks of the partition p, to give us the deformation $(\Phi E_{m+1}, 0)$ where

$$E_{m+1} = \begin{pmatrix} (1+\epsilon)I_{k_m} & \\ & I_{n-k_m} \end{pmatrix} \in M_n(R[\epsilon])$$

By the same reasoning, this deformation also lies on $X_p(\mathbb{F}[\epsilon])$, and since $k_m < k_1$, we see this adds a genuinely new deformation inside $T_P D$, because the deformations $\{(\Phi E_i) : 1 \leqslant i \leqslant m+1\}$ are all linearly independent in $T_P(\mathrm{GL}_n \times \mathfrak{gl}_n)$.

Piecing everything together, we have at least $(n^2 - n) + (n - m) + m + 1 = n^2 + 1 > \dim(X_p)$ linearly independent deformations, which exceeds the dimension of the variety X_p. We conclude that $\dim_{\mathbb{F}} T_P X_p \geqslant \dim X_p$ and that P is a singular point.

Now, in the case $p = k^m$, we instead choose a point

$$(\Phi, 0) = \left(\begin{pmatrix} M(k, q^{k(m-1)-1}) & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & M(k, q^{k-1}) & 0 \\ 0 & \cdots & 0 & M(k, 1) \end{pmatrix}, 0 \right) \in X_p(R).$$

so that Φ is a diagonal matrix, with increasing powers of q going up the diagonal, with a single power of q repeated, that being q^{k-1}. Then the conjugation orbit is $n^2 - (n - 2 + 4) = n^2 - n - 2$ dimensional. The $T_P \mathcal{N}_0$-space deformations give us again, $(k-1)m$ deformations on the off-diagonal, and an additional two in the entries marked with a $\square$ below, appearing because of the repeated power of q in Φ

$$\left(\begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \\ \cdots & q^k & & & & \cdots \\ \cdots & & q^{k-1} & & & \cdots \\ \cdots & & & q^{k-1} & & \cdots \\ \cdots & & & & q^{k-2} & \cdots \\ & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \\ \cdots & 0 & * & \square & & \cdots \\ \cdots & & 0 & 0 & \square & \cdots \\ \cdots & & & 0 & * & \cdots \\ \cdots & & & & 0 & \cdots \\ & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \right) \in X_p(R).$$

Each of these deformations lie inside $T_P X_p$ because they are conjugate inside $\mathrm{GL}_{n,\mathbb{F}[\epsilon]}$ to pairs in the form of Lemma 2.7.

Now if we define E_i as before, for $i \leqslant m$, we see by the lemma that $(\Phi E_i, 0) \in X_p(R)$ for each i, and this gives us another m deformations. Finally, let E_{m+1} be as follows:

$$E_{m+1} = \begin{pmatrix} I_{k(m-1)} & & & \\ & 1+\epsilon & & \\ & & 1 & \\ & & & (1+\epsilon)I_{k-1} \end{pmatrix} \in M_n(\mathbb{F}[\epsilon]).$$

Then, because ΦE_{m+1} is conjugate to something of the form in Lemma 2.7, it lies inside $X_p(\mathbb{F}[\epsilon])$. Notice that the deformations ΦE_i for $i = 1, ..., m+1$ are linearly independent, because they are linearly independent inside $T_P(\mathrm{GL}_n) \supseteq T_P D$. This gives a total of $(n^2 - n - 2) + ((k-1)m + 2) + (m+1) = n^2 - n + mk + 1 = n^2 + 1 > n^2 = \dim(X_p)$ deformations, and shows that X_p is singular at $(\Phi, 0)$. $\square$

4 Φ-Regular points of $X_{\leqslant p}$ are Cohen-Macaulay

In this section, we take a closer study of certain unions of irreducible components of $S_{n,\mathcal{O}}$ which appear as the support of certain maximal Cohen-Macaulay sheaves that appear as the outputs of patching functors.

4.1 Motivation

Let $F/\mathbb{Q}_l$ be a finite field extension as before. Let W_F be the Weil group of F and let I_F be the inertia subgroup.

Recall the dominance partial order on the set of partitions of n, which can be defined as follows: For p and q two partitions of n, we say $q \leqslant p$ if their corresponding nilpotent conjugacy classes C_q and C_p inside the nilpotent cone, $\mathcal{N}$, satisfy $C_q \subseteq \bar{C}_p$. Equivalently, if $p = (p_1, ..., p_k)$ and $q = (q_1, ..., q_m)$ and we adopt the conventions that $q_i = 0$ if $i > m$ and $p_i = 0$ if $i > k$, then $q \leqslant p$ if and only if for every $j \in \mathbb{N}$, $\sum_{i=1}^{j} q_i \leqslant \sum_{i=1}^{j} p_i$.We can make the following definition.

Definition 4.1. *For a given partition p of n, let $X_{\leqslant p} := \bigcup_{q \leqslant p} X_q \subseteq S_n$.*

We present a little motivation why these varieties are interesting to study.

Definition 4.2. *An inertial type is an isomorphism class of continuous representations $\tau : I_F \to GL(V)$ where V is a finite dimensional $E = \bar{\mathbb{Q}}_l$-vector space, that extends to a representation of W_F. A basic inertial type is an inertial type, that extends to an irreducible representation of W_F. Let $\mathcal{I}_0$ be the set of all basic inertial types.*

Let Part_n be the set of all partitions of n, and $\mathrm{Part} = \bigcup_n \mathrm{Part}_n$. In [Sho18] it is shown that there is a bijection between inertial types and the set $\mathcal{I}$ of all functions

$$\mathcal{P} : \mathcal{I}_0 \to \mathrm{Part}$$

of finite support, where and Part is the set of all partitions. We will denote the partition corresponding to $\tau \in \mathcal{I}_0$ by $\mathcal{P}_\tau$. For a partition $p \in \mathrm{Part}$, we say that the degree $\deg(p)$ is the number n that p partitions. We can extend deg to the set $\mathcal{I}$ by

$$\deg(\mathcal{P})(\tau_0) = \deg(\mathcal{P}(\tau_0))$$

and we can extend the dominance ordering on Part by saying that two inertial types $\mathcal{P}$ and $\mathcal{Q}$ have $\mathcal{P} \geqslant \mathcal{Q}$ if and only if they have the same degree, and if $\mathcal{P}(\tau_0) \geqslant \mathcal{Q}(\tau_0)$ for each $\tau_0 \in \mathcal{I}_0$.

Let $\bar{\rho} : G_F \to \mathrm{GL}_n(\bar{\mathbb{F}})$ be a representation with inertial type τ. Let $R^\square(\bar{\rho})$ be its framed deformation ring. and let $R^\square(\bar{\rho}, \tau)$ be the framed fixed inertial type deformation ring.

In chapter 6 of [EGS14] (see section 6.1 for full details), the notion of a patching functor (at least in the GL_2 case, though this notion can be generalised to more general connected reductive groups) is defined as an exact covariant functor M_∞ from the category of $K = \mathrm{GL}_n(\mathcal{O})$ representations on finite free $\mathcal{O}$-modules to the category of coherent sheaves on a certain space

$X_\infty = \mathrm{Spec}(\widehat{\bigotimes}_v R_v^\square[[x_1, ..., x_h]])$ a finite product of local deformation rings, with certain properties. One of the properties we expect is that a certain K-representation $\sigma(\tau)$ (arising naturally from an inertial type τ) has the coherent sheaf $M_\infty(\sigma(\tau))$ supported on the closed subscheme $X_\infty(\tau)$, of points in X_∞ with inertial type $\leqslant \tau$. Further, $M_\infty(\sigma(\tau))$ is maximal Cohen-Macaulay over $X_\infty(\tau)$. We may hope then, since spaces arise as the supports of these patching functors, that the spaces $X_\infty(\tau)$ may be Cohen-Macaulay. This would happen if we can prove that each $X_{\leqslant p}$ is Cohen-Macaulay.

4.2 The main theorem

Let L be the fraction field of $\mathcal{O}$ as before. Let $\mathcal{N}_n \subseteq \mathfrak{gl}_n$ be the nilpotent cone. Recall there is a GL_n-equivariant morphism, given by the second projection, $p_2 : S_{\mathrm{GL}_n} \to \mathcal{N}_n$. For each partition p, we can find the locally closed subspace $C_p \subseteq \mathcal{N}_n$ given by the preimage of the conjugacy class given by p inside $(\mathcal{N}_n)_L$ through the flat morphism $\mathcal{N}_n \hookrightarrow (\mathcal{N}_n)_L$. Then $\bar{C}_p$ is a union of conjugacy classes in $\mathcal{N}_n$, and $\bar{C}_p = \bigcup_{q \leqslant p} C_q$. We may henceforth view $X_{\leqslant p} = p_2^{-1}(\bar{C}_p)$ as the preimage of $\bar{C}_p$ under the projection p_2. This is advantageous, as it shows us that any additional equations specifying $X_{\leqslant p}$ as a subspace of S_n need only have equations in the variables of N (namely, those equations that define the subvariety $\bar{C}_p$).

Definition 4.3. *We define $X_{\leqslant p}^{\Phi\text{-}reg} \subseteq X_{\leqslant p}$ to be the open subscheme over $\mathcal{O}$ defined as the complement of the equation $Disc(\chi_\Phi(X)) = 0$.*

Remark. Let $P \in |X_{\leqslant p}|$ lie in the fibre of a prime $\mathfrak{p} \in \mathrm{Spec}\mathcal{O}$ with residue field $K = k(\mathfrak{p})$ and separable closure K^{sep}, and suppose P corresponds to a $(\mathrm{Gal}_K$-equivalence class of) pair of matrices $(\Phi, N) \in X_{\leqslant p}(K^{\mathrm{sep}})$. We notice that $P \in |X_{\leqslant p}^{\Phi\text{-reg}}|$ if and only if $\mathrm{Disc}(\chi_\Phi(X)) \notin \mathfrak{p}$, which occurs if and only if $\mathrm{Disc}(\chi_\Phi(X)) \neq 0$ inside the field $k(P)$, which is equivalent to the eigenvalues of Φ being distinct inside a separable closure $k(P)^{\mathrm{sep}}$, by virtue of $\mathrm{char}(k(P)) = 0$ or $l > n$.

Theorem 4.4. *Suppose that q is considerate towards GL_n over $\mathcal{O}$. Let p be a partition of n. Then $X_{\leqslant p}^{\Phi\text{-}reg}$ is Cohen-Macaulay.*

To approach this problem, we start by reducing the question to a ring R_P (to be defined) with which we can make explicit calculations.

Let $\mathfrak{p} \in \mathrm{Spec}\mathcal{O}$, and let $K = k(\mathfrak{p})$. Choose a separable closure K^{sep} as before, and let $P \in |(X_{\leqslant p})|$ lie above $\mathfrak{p}$ correspond to a pair of matrices $(\Phi, N) \in X_{\leqslant p}^{\Phi\text{-reg}}(K^{\mathrm{sep}})$. We may assume without loss of generality that $P = (\Phi, 0)$ with Φ semisimple. This is because the set of non-Cohen-Macaulay points is a closed subspace of $X_{\leqslant p}$. If $P = (\Phi, N) \in X_{\leqslant p}$ is a non-Cohen-Macaulay point, then the action of GL_n on $X_{\leqslant p}$ provides an isomorphism of local rings of any two points in the orbit of P. Thus, any point in the orbit of P is non-Cohen-Macaulay. Further, the semisimplification $(\Phi^{\mathrm{s.s.}}, 0)$ is contained inside the closure of the orbit of P, and thus, $(\Phi^{\mathrm{s.s.}}, 0)$ is also a non-Cohen-Macaulay point. As a consequence, if we show that every point $(\Phi, 0)$ with Φ semisimple is Cohen-Macaulay, we can deduce that $X_{\leqslant p}$ is Cohen-Macaulay, and thus we can reduce our attention to points of this form.

Let M be the stabiliser of Φ (necessarily M is of the form $M = \prod_{i=1}^{m} \mathrm{GL}_{k_i}$). We may assume that Φ has the form of a block diagonal matrix $\Phi = \mathrm{Diag}(a_1 I_{k_1}, a_2 I_{k_2}, ..., a_m I_{k_m})$ where I_k are $k \times k$ identity matrices, and all the a_i are distinct with an ordering chosen such that $a_i/a_j = q$ inside K^{sep} implies that $j = i + 1$.

We set V_M to be the subscheme of $X_{\leqslant p}$, flat over $\mathcal{O}$ defined as $\{(\Phi, N) \in M \times \mathfrak{gl}_n : \Phi N \Phi^{-1} = qN$ and N has conjugacy class $\leqslant p\}$. We now set $R_P := \mathcal{O}_{V_M, P}$ to be the local ring at P of this space.

Lemma 4.5. *Let $\mathcal{P}$ be one of the properties of local rings: smooth/ a local complete intersection/ Gorenstein/ Cohen-Macaulay. The scheme $X_{\leqslant p}$ is $\mathcal{P}$ at P if and only if R_P is $\mathcal{P}$ at P.*

Proof. We have a pullback diagram of $\mathcal{O}$-schemes

$$\begin{array}{ccc} \mathrm{GL}_n \times V_M & \longrightarrow & X_{\leqslant p} \\ \big\downarrow & & \big\downarrow \\ \mathrm{GL}_n \times M & \longrightarrow & \mathrm{GL}_n \end{array}$$

where the map horizontal maps are given by conjugation $(g, x) \mapsto gxg^{-1}$, and the vertical maps are 'forget the second coordinate'. Localising and completing along maximal ideals gives us a pushout diagram of complete local rings as follows:

$$\begin{array}{ccc} k[\mathrm{GL}_n]^{\wedge}_I \widehat{\otimes} R^{\wedge}_P & \longleftarrow & R^{\wedge} \\ \big\uparrow & & \big\uparrow \\ k[\mathrm{GL}_n]^{\wedge}_I \widehat{\otimes} k[M]^{\wedge}_P & \longleftarrow & k[\mathrm{GL}_n]^{\wedge}_P \end{array}$$

with R the local ring of P on $X_{\leqslant p}$ Since this is a pushout diagram, the top map is smooth if the bottom map is smooth. We claim that the bottom map is smooth. Let $\mathcal{C}_{\mathcal{O}}$ be the category of complete Noetherian local $\mathcal{O}$-algebras with residue field k. We have $T := k[\mathrm{GL}_n]^{\wedge}_P \cong \mathcal{O}[[X_1, ..., X_{n^2}]]$ represents the functor on $\mathcal{C}_{\mathcal{O}}$ given by $A \in \mathcal{C}_{\mathcal{O}}$ maps to those elements of $\mathrm{GL}_n(A)$ which map to P in $\mathrm{GL}_n(k)$. This is the same as the set $P + \mathfrak{gl}_n(m_A)$, where m_A is the maximal ideal of A. likewise, $k[M]^{\wedge}_P \cong \mathcal{O}[[Y_1, ..., Y_{\dim M}]]$ represents the functor $A \mapsto P + \mathrm{Lie}(M)(m_A)$.

Consider $A = k[t]/t^2 \in \mathcal{C}_{\mathcal{O}}$, then the map of Zariski tangent spaces

$$\begin{aligned} [I + \mathfrak{gl}_n(m_A)] \times [P + \mathrm{Lie}(M)(m_A)] &\to P + \mathfrak{gl}_n(m_A) \\ (I + m, P + x) &\mapsto (I + x)(P + m)(I + x)^{-1} \\ &= P + [x, P] + m) \end{aligned}$$

is a surjection because $M = \mathrm{Stab}(P)$.

This provides us with an injection $m_T/m_T^2 \to m/m^2$ where m_R is the maximal ideal of $T = k[\mathrm{GL}_n]^\wedge_P$ and m is the maximal ideal of $k[\mathrm{GL}_n]^\wedge_I \widehat{\otimes} k[M]^\wedge_P$. Let $T_1, ..., T_r$ be a set of elements of m such that they form a basis of $(m/m^2)/(m_R/m_R^2)$. Then, since T and $k[\mathrm{GL}_n]^\wedge_I \widehat{\otimes} k[M]^\wedge_P$ are both power series rings, we see that $k[\mathrm{GL}_n]^\wedge_I \widehat{\otimes} k[M]^\wedge_P = R[[T_1, ..., T_r]]$. This shows that the bottom map is smooth, and hence that the top map is smooth.

As a result,

$$R^\wedge_P[[X_1, ..., X_{n^2}]] \cong k[\mathrm{GL}_n]^\wedge_I \widehat{\otimes} R^\wedge_P$$

is a power series ring in $R^\wedge$. Thus, if $\mathcal{P}$ is one of the properties in the lemma, we see that R is $\mathcal{P}$ if and only if $R^\wedge$ is $\mathcal{P}$ via lemma 2.7, if and only if $R^\wedge_P[[X_1, ..., X_{n^2}]]$ is $\mathcal{P}$ if and only if R_P is $\mathcal{P}$. This completes the lemma.

□

Thus, to show that $X^{\Phi\text{-reg}}_{\leqslant p}$ is Cohen Macaulay at $P \in X^{\Phi\text{-reg}}_{\leqslant p}$ it suffices to show that R_P is Cohen-Macaulay. We now give an explicit description of R_P.

Consider the universal coordinates of V_M which (in block matrix form blocks of size $k_1, ..., k_m$:

$$\left(\begin{pmatrix} a_1(I_{k_1}+M_1) & 0 & \dots & 0 \\ 0 & a_2(I_{k_2}+M_2) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_m(I_{k_m}+M_m) \end{pmatrix}, \begin{pmatrix} b_{1,1} & b_{1,2} & \dots & b_{1,n} \\ b_{2,1} & b_{2,2} & \dots & b_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n,1} & b_{n,2} & \dots & b_{n,n} \end{pmatrix} \right)$$

Where each M_i is a $k_i \times k_i$ matrix, and each $b_{i,j}$ is a $k_i \times k_j$ matrix.

The equation $\Phi N = qN\Phi$ gives us the following for each (i, j)

$$a_i(I_{k_i} + M_i)b_{i,j} = qa_j b_{i,j}(I_{k_j} + M_j) = 0$$

which in turn give us

$$(a_i - qa_j)b_{i,j} + a_i M_i b_{i,j} - qa_j b_{i,j} M_j = 0$$

when $a_i - qa_j$ is non-zero in K^{sep}, it is invertible inside $\mathcal{O}_\mathfrak{p}$, Hence

$$b_{i,j} = -(a_i - qa_j)^{-1} a_i M_i b_{i,j} + (a_i - qa_j)^{-1} qa_j b_{i,j} M_j.$$

Let I be the ideal of R_P generated by the coordinates of $b_{i,j}$. Then we see from the above equation that $I = mI$ where m is the maximal ideal of R_P. Consequently by Nakayama's lemma, we see that $I = 0$.

Thus, $b_{i,j} = 0$ unless $j = i+1$ and $a_i = qa_{i+1}$ in K^{sep}. When $a_i - qa_{i+1} \in \mathfrak{p}$, set $\pi = a_i^{-1}(a_i - qa_{i+1}) \in \mathfrak{p}$, then we get that the equations given by $\Phi N = qN\Phi$ give us exactly

$$(M_i b_{i,i+1} - b_{i,i+1} M_{i+1}) + \pi b_{i,j}(I + M_{i+1}) = 0$$

inside V_M. We will, from now on, write $N_i := b_{i,i+1}$

As a result, we see that

$$R_P = \frac{\mathcal{O}_\mathfrak{p}[M_1, ..., M_m, N_1, ..., N_{m-1}]}{\langle \{M_i b_{i,i+1} - b_{i,i+1} M_{i+1} + \pi N_i(I + M_{i+1}) : i < m\} \cup \{\text{some equations only in } N_i\} \rangle}$$

Where the equations in the coordinates of N_i are those that describe the conjugacy classes inside $\bar{C}_p$. As $\mathcal{O}_\mathfrak{p}$ is regular, and R_P is a Noetherian flat local $\mathcal{O}_\mathfrak{p}$-algebra, by Lemma 2.9 we see that R_P is Cohen Macaulay if and only if the ring

$$\bar{R}_P = \frac{K[M_1, ..., M_m, N_1, ..., N_{m-1}]}{\langle\{M_i b_{i,i+1} - b_{i,i+1} M_{i+1} : i < m\} \cup \{\text{some equations only in } N_i\}\rangle}$$

is Cohen Macaulay. Hence we reduce the problem to showing that $\bar{R}_P$ is Cohen Macaulay.

When $P = (\Phi, 0)$ is Φ -regular, the M_i and N_i are 1×1-matrices and thus commute, and so we can simplify even further. By setting $\lambda_i = M_i - M_{i+1}$, we see that that $\lambda_i N_i = 0$. We hence have reduced the problem to proving that this explicit $\bar{R}_P$ is Cohen-Macaulay, and have proven most of the following lemma

Lemma 4.6. *For $S \subseteq \{1, ..., n-1\}$, define $N_S := \prod_{i \in S} N_i$. Let P be Φ-regular, and let $\bar{R}_P$ be as above. Then there exists a family $\mathcal{F}$ of subsets of $\{1, ..., n-1\}$ such that the local ring $\bar{R}_P$ has the following form:*

$$\bar{R}_P = \left(\frac{K[\lambda_1, ..., \lambda_n, N_1, ..., N_{n-1}]}{I_P}\right)_m,$$

where

$$I_P := \langle\{\lambda_i N_i | 1 \leqslant i < n\} \cup \{N_i | a_i/a_{i+1} \neq q\} \cup \{N_S | S \in \mathcal{F}\}\rangle,$$

and m is the maximal ideal $\langle \lambda_1, ..., \lambda_n, N_1, ..., N_{n-1}\rangle$. Furthermore, every set $S \in \mathcal{F}$ has empty intersection with the set $\{i | b_i \neq 0\}$.

Proof. We note that the only part left to prove is the statement about the remaining equations in the N_i that describe the conjugacy class of nilpotent matrix

$$\begin{pmatrix} 0 & N_1 & 0 & \dots & 0 \\ 0 & 0 & N_2 & \dots & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & \dots & N_{n-1} \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix} \in \bar{C}_p$$

in $\mathcal{N}_n$. By Lemma 4.7 in the next section, the equations that cut out W_p, defined as the closed subscheme of $\overline{C_p}$ with all non-zero entries on the off-diagonal, are given by products of the form

$$0 = \prod_{i \in S} N_i$$

for some set $S \subseteq \{1, ..., n-1\}$. The Lemma follows.

□

4.3 Calculations of the families $\mathcal{F}$ that appear for a given partition p

In this section, we study and calculate the equations that specify the union $X_{\leqslant p}$. We start off with a lemma.

Lemma 4.7. *Let $W^+ \cong \mathbb{A}^{n-1}_{\mathcal{O}}$ be the subscheme of the scheme M_n of $n \times n$ matrices over $\mathcal{O}$, consisting of matrices with entries only on the off-diagonal, so that*

$$W = \{\begin{pmatrix} 0 & N_1 & & \\ & \ddots & \ddots & \\ & & \ddots & N_{n-1} \\ & & & 0 \end{pmatrix} : (N_1, ..., N_{n-1}) \in \mathbb{A}^{n-1}\}$$

Let W_p be the subscheme $W_p = (\overline{C_p} \cap W^+)^{red}$. Then W_p is cut out by squarefree products of the N_i.

Proof. Let $f = f(N_1, ..., N_{n-1})$ be a polynomial in the N_i such that $f = 0$. Since W_p is invariant under conjugation by the maximal torus T of GL_n, This action defines an action on f via $\lambda.f(N_1, ..., N_{n-1}) = f(\lambda_1\lambda_2^{-1}N_1, ..., \lambda_{n-1}\lambda_n^{-1}N_{n-1})$, where $\lambda = (\lambda_1, ..., \lambda_n) \in T$, and we must have that $f(N) = 0$ implies $\lambda.f(N) = 0$. View f as a polynomial in N_i, and coefficients in the ring of polynomials $k[N_1, ..., N_{i-1}, N_{i+1}, ..., N_{n-1}]$. Consider the action of $\lambda = (\lambda_1, ..., \lambda_n)$ where $\lambda_j = \alpha \in k^\times$ for all $j \leqslant i$ and $\lambda_j = 1$ for all $j > i$. Then this action preserves the coefficients of f, and multiplies the N_i^k term by α^k. We hence see that all the N_i-graded parts of f lie in the ideal. Since this is true for each i, we see that there are generating equations, $\{f_s : s \in I\}$ such that each f_s is a product of N_i's, up to a constant coefficient, which we may forget without loss of generality. To prove that the generators are squarefree, it is sufficient to note that W_p is a reduced scheme. □

We now give a complete description of the families $\mathcal{F}$ that occur. They depend only on the partition p. We will denote the family obtained from R_P by $\mathcal{F}_p$, as this only depends on p.

Remark. Notice that as written in Lemma 4.6, $\mathcal{F}$ has no dependence on (a_i). If we wanted to we could change this, and include $\{i\} \in \mathcal{F}$ for each i such that $\{i | a_i/a_{i+1} \neq q\}$.

Let $T \subseteq S \subseteq \{1, ..., n-1\}$. Then $N_T | N_S$, so that we can enlarge $\mathcal{F}_p$ to make it an order ideal of $\mathcal{P}(\{1, .., n-1\})$. With this, we can observe that we have an order reversal, in that if q, p are partitions of n, and $q \leqslant p$, then $\mathcal{F}_q \supseteq \mathcal{F}_p$ (this happens, precisely because $C_q \subseteq \bar{C}_p$).

Proposition 4.8. *There is an algorithm to calculate $\mathcal{F}_p$ given a partition p of n.*

Proof. The algorithm consists of the following steps.

Step 1 Form the set $\mathcal{Q}$ of all 'minimal breaking' partitions $q = (q_1, ..., q_r)$ defined to be partitions of n such that there exists some integer s such that:

a) for every $j < s$, $\sum_{i=1}^{j} q_i \leqslant \sum_{i=1}^{j} p_i$.
b) $\sum_{i=1}^{s} q_i = \sum_{i=1}^{s} p_i + 1$
c) for each $i \in (s, r]$, $q_i = 1$.

Note that the minimal referred to here does not mean that q is minimal in the dominance order.

Step 2 For each minimal breaking partition $q = (q_1, ..., q_r)$, form the family $\mathcal{S}_q$ of all subsets of $\{1, ..., n-1\}$ that are a union of runs of length $q_1 - 1, q_2 - 1, ..., q_s - 1$ of the following form. To clarify, let $|a, q|$ be the set $\{a, a+1, a+2, ..., a+q-1\}$ (we call this a run of length q). The sets inside $\mathcal{S}_q$ are exactly those of the form $|a_1, q_{\sigma(1)}-1| \cup |a_2, q_{\sigma(2)}-1| \cup ... \cup |a_s, q_{\sigma(s)}-1|$ with $a_{i+1} \geqslant a_i + q_{\sigma(i)}$ for every i, and some permutation $\sigma \in \mathrm{Sym}_s$.

Step 3 Take $\mathcal{F}_p$ to be the order ideal generated by the set $\bigcup_{q \in \mathcal{Q}} \mathcal{S}_q$.

It can be seen that this produces $\mathcal{F}_p$, since the equations $\mathcal{S}_q$ exclude, on the level of points, any nilpotent matrix in the conjugacy class defined by q. Since any partition $q' = (q'_1, ..., q'_{r'})$ such that q' is not dominated by p has some minimal breaking partition q such that $q \leqslant q'$, namely, if q' has s the smallest integer such that $\sum_{i=1}^{s} q'_i > \sum_{i=1}^{s} p_i$, then $q = (q'_1, ..., q'_{s-1}, \sum_{i=1}^{s} p_i + 1 - \sum_{i=1}^{s-1} q'_i, 1, ..., 1)$ does the job, we also see that any nilpotent matrix in the conjugacy class defined by q' is also excluded. Since each q is not dominated by p, this shows that any matrix in the conjugacy class defined by p is not excluded, and nor is any partition dominated by p. This shows that, at the level of points, these equations determine W_p. ☐

We now present an example of this calculation in the case of $n = 6$ and $p = (4, 1, 1)$, and a diagram that shows $\mathcal{F}_p$ for each partition p of $n = 6$. On the left of the diagram are the partitions of 6, ordered according to the dominance order and on the right are the families $\mathcal{F}_p$, that correspond to p.

A brief remark about notation For clarity's sake, instead of usual set notation, I will denote the set containing the numbers 1, 3 and 5 by the triple 135. Further, given sets 12,134,234, I will denote the order ideal $\mathcal{F} \subseteq \mathcal{P}(1, ..., 4)$ generated by 12, 134 and 234 by angled bracket notation $\langle 12, 134, 234 \rangle$. We note that $\langle \varnothing \rangle = \varnothing$.

Example 1. *Let $n = 6$, and $p = (4, 1, 1)$. Then the minimal breaking partitions of p are $(5, 1), (4, 2)$ and $(3, 3)$. Form $\mathcal{S}_{(5,1)} = \{1234, 2345\}$, the set of all runs of length 4. The set $\mathcal{S}_{(4,2)} = \{1235, 1345\}$ is the set of all sets containing a run of length 3 and a run of length 1, and the set $\mathcal{S}_{(3,3)} = \{1245\}$ is the only set that contains two runs of length 2. Thus, we see that $\mathcal{F}_p = \langle 1234, 2345, 1235, 1345, 1245 \rangle$.*

$$
\begin{tikzcd}
& 6 \arrow[d] & \\
& 51 \arrow[d] & \\
& 42 \arrow[dl] \arrow[dr] & \\
3^2 \arrow[dr] & & 41^2 \arrow[dl] \\
& 321 \arrow[dl] \arrow[dr] & \\
31^3 \arrow[dr] & & 2^3 \arrow[dl] \\
& 2^2 1^2 \arrow[d] & \\
& 21^4 \arrow[d] & \\
& 1^6 &
\end{tikzcd}
\qquad
\begin{tikzcd}
& \langle \varnothing \rangle \arrow[d] & \\
& \langle 12345 \rangle \arrow[d] & \\
& \langle 1234, 2345 \rangle \arrow[dl] \arrow[dr] & \\
\langle 123, 234, 345 \rangle \arrow[dr] & & \langle \text{all quadruples} \rangle \arrow[dl] \\
& \langle 123, 234, 345, 1245 \rangle \arrow[dl] \arrow[dr] & \\
\langle \text{all triples} \rangle \arrow[dr] & & \langle 12, 23, 34, 45 \rangle \arrow[dl] \\
& \langle 12, 23, 34, 45, 135 \rangle \arrow[d] & \\
& \langle \text{all pairs} \rangle \arrow[d] & \\
& \langle 1, 2, 3, 4, 5 \rangle &
\end{tikzcd}
$$

4.4 Proof of Theorem 4.4

We prove a slight generalisation of Theorem 4.4:

Theorem 4.9. *Suppose K is a field, let $n \in \mathbb{N}$ and let $\mathcal{F} \subseteq \mathcal{P}(1, ..., n)$. Let*

$$R := K[\lambda_1, ..., \lambda_n, N_1, ..., N_n]/I$$

where I is the ideal generated by the set

$$\{\lambda_i N_i | 1 \leqslant i \leqslant n\} \cup \{N_S | S \in \mathcal{F}\}$$

where $N_S = \prod_{i \in S} N_i$ as before. Suppose $m \trianglelefteq R$ is the maximal homogeneous ideal. Then depth$(m, R) \geqslant n$.

Lemma 4.10. *Suppose R is a ring and $x \in R$ is a non-unit, such that for any $a \in R$, we have*

$$x^2 b = 0 \implies xb = 0.$$

Define

$$T := \frac{R[y]}{\langle xy \rangle}$$

Then $y - x$ is a non-unit and not a zero divisor of T.

Proof. There is a grading on T defined by $T_n = Ry^n$ for each $n \in \mathbb{N}$. Let $f \in T$. Since x is not a unit, the degree 0 part of $(y-x)f$ cannot be 1. Thus, $y-x$ is not a unit.

To show that it is a non-zero divisor, let $f \in T$ be such that $(y-x)f = 0$. Write $f = \sum_{i=0}^{n} a_i y^i$ for some $n \in \mathbb{N}$, and $a_i \in R$. Then:

$$\begin{aligned}
0 = (y-x)f &= a_n y^{n+1} + \sum_{i=0}^{n-1} (a_i - x a_{i+1}) y^{i+1} - x a_0 && (1)\\
&= a_n y^{n+1} + \sum_{i=0}^{n-1} a_i y^{i+1} - x a_0 && (2)\\
&= fy - x a_0. && (3)
\end{aligned}$$

Hence, $fy = xa_0 \in T_0$, and so $fy \in (\bigoplus_{i=1}^{n+1} T_i) \cap T_0 = 0$ and so $fy = 0$. So each of the constituents of the sum are zero too. Hence $a_0 y = 0$.

So we have an element $a_0 \in R$ such that $ya_0 = xa_0 = 0$ As $ya_0 = 0$ in T, we must have that $ya_0 \in \langle xy \rangle \trianglelefteq R[y]$. So $ya_0 = xyb$ for some $b \in R[y]$, and since $\deg(ya_0) = 1$, have $\deg(xb) = 0$, so we can choose $b \in R$. Hence, $(a_0 - xb)y = 0$ in $R[y]$, and so $a_0 = xb$ in R. Hence $0 = xa_0 = x^2 b$. So by hypothesis, $a_0 = xb = 0$. and so $f = \sum_{i=1}^{n} a_i y^i$.

Recall that $fy = 0$. So $fy = \sum_{i=1}^{n} a_i y^{i+1} = 0$. Then each of the terms $a_i = 0$ in S, so $a_i \in \langle xy \rangle$ in $R[y]$. So $f = \sum_{i=1}^{n} a_i y^i = 0$. This shows that $y - x$ is not a zero divisor. ☐

Lemma 4.11. *Let R be a ring, and J some finite indexing set, and $a_j \in R$ for $j \in J$. Let $T = R[x]/I$ where $I = \langle \{xa_j | j \in J\} \rangle$. Then x has the property that, for any $a \in T$*

$$x^2 a = 0 \implies xa = 0$$

Proof. First, we see that $R[x]$ is a graded ring, and I is a homogeneous ideal of degree 1, so T is also graded. Suppose $a \in T$ is such that $x^2 a = 0$. in S. We may lift a to $a' \in R[x]$, so that $x^2 a' \in I$. Then, for some $b_j \in R[x]$,

$$x^2 a' = \sum_{j \in J} x a_j b_j,$$

and so

$$xa' = \sum_{j \in J} a_j b_j.$$

Consider the degree zero part of xa'. Then

$$0 = \sum_{j \in J} a_j b_j(0),$$

where $b_j(d)$ denotes the degree d part of b_j. Therefore

$$\begin{aligned} xa' &= \sum_{d\geqslant 1}\sum_{j\in J} a_j b_j(d)x^d \\ &= \sum_{j\in J} x a_j c_j \in I \end{aligned}$$

with $c_j := \sum_{d\geqslant 1} b_j x^{d-1} \in R[x]$. Hence, $xa = 0$ in S.

$\square$

Proof of Theorem 4.9. We show explicitly that the sequence $\{\lambda_i - N_i : i = 1, ..., n\}$ is a regular sequence. Let $J_i = \langle \lambda_1 - N_1, ..., \lambda_{i-1} - N_{i-1}\rangle$, and let

$$\begin{aligned} R_i &:= R/J_i \\ &\cong \frac{K[\lambda_1, ..., \lambda_n, N_1, ..., N_n]}{\langle \{N_S | S \in \mathcal{F}\} \cup \{\lambda_1 - N_1, ..., \lambda_{i-1} - N_{i-1}\}\rangle} \\ &\cong \frac{A[\lambda_i, N_i]}{\langle \{N_S | S \in \mathcal{F} \text{ and } i \in S\}, \lambda_i N_i\rangle} \\ &\cong \frac{(\frac{A[N_i]}{\langle N_i a_t | t \in T\rangle})[\lambda_i]}{\langle \lambda_i N_i\rangle} \end{aligned}$$

where T is some finite indexing set, $a_t \in A$ are some explicit elements of A, and

$$A = \frac{K[\lambda_1, N_1, ..., \lambda_{i-1}, N_{i-1}, \lambda_{i+1}, N_{i+1}, ..., \lambda_n, N_n]}{\langle \{\lambda_j N_j | j \neq i\} \cup \{\lambda_1 - N_1, ..., \lambda_{i-1} - N_{i-1}\} \cup \{N_S | S \in \mathcal{F} \text{ and } i \notin S\}\rangle}.$$

Now, since $B := \frac{A[N_i]}{\langle N_i a_t | t \in T\rangle}$ is of the form in Lemma 4.11, we know N_i is an element of B such that $N_i^2 a = 0 \implies N_i a = 0$, for $a \in B$. Hence, by Lemma 4.10, $\lambda_i - N_i$ is a non-unit, non-zero divisor in R_i. It then follows that $\lambda_1 - N_1, ..., \lambda_n - N_n$ is a regular sequence of length n. $\square$

We now prove Theorem 4.4.

Proof. Proof of Theorem 4.4 Recall from Lemma 4.6 that the local ring of a $P \in X^{\Phi\text{-reg}}_{\leqslant p}$ is of the following form:

$$\bar{R}_P = \frac{K[\lambda_1, ..., \lambda_n, N_1, ..., N_{n-1}]}{\langle \{\lambda_i N_i | 1 \leqslant i < n\} \cup \{N_i | a_i/a_{i+1} \neq q\} \cup \{\lambda_i | b_i \neq 0\} \cup \{N_S | S \in \mathcal{F}\}\rangle_m}$$

with $\mathcal{F}$ a family of subsets of $\{1, ..., n-1\}$.

We first can make a simplification. Notice that, by expanding $\mathcal{F}$ to include the sets $\{\{i\} | a_i/a_{i+1} \neq q\}$, we may assume without loss of generality that the second set of generators is empty. Reorder the i, so that $\{i | b_i \neq 0\} = \{k+1, k+2, ..., n-1\}$ for some k. Now, since for any $S \in \mathcal{F}$, $S \cap \{i | b_i \neq 0\} = \varnothing$, we can view $\mathcal{F}$ as a family of subsets of $\{1, ..., k\}$.Hence we see that

$$\bar{R}_P \cong \frac{K[\lambda_1, ..., \lambda_k, N_1, ..., N_k]}{\langle \{\lambda_i N_i | 1 \leqslant i \leqslant k\} \cup \{N_S | S \in \mathcal{F} \subseteq \mathcal{P}(\{1, ..., k\})\}\rangle}[N_{k+1}, ..., N_{n-1}, \lambda_n].$$

By Theorem 4.9, $\frac{K[\lambda_1,\ldots,\lambda_k,N_1,\ldots,N_k]}{\langle\{\lambda_i N_i|1\leqslant i\leqslant k\}\cup\{N_S|S\in\mathcal{F}\subseteq\mathcal{P}(\{1,\ldots,k\})\}\rangle}$ has a regular sequence of length k given by $\lambda_1 - N_1, ..., \lambda_k - N_k$. We can now extend this regular sequence by $N_{k+1}, ..., N_{n-1}, \lambda_n$ to get a regular sequence of length n in $m_P \trianglelefteq \bar{R}_P$. This shows that $\text{depth}(m_P, \bar{R}_P) \geqslant n$. Further, since $\bar{R}_P$ is a local ring of a subvariety V of the affine variety $\text{Spec}\left(\frac{K[\lambda_1,\ldots,\lambda_n,N_1,\ldots,N_{n-1}]}{\langle\lambda_i N_i|1\leqslant i<n\rangle}\right)$ which has dimension n, we see

$$n \leqslant \text{depth}(\bar{R}_P) \leqslant \dim(\bar{R}_P) \leqslant n$$

which implies equality throughout, Therefore $\bar{R}_P$ is Cohen-Macaulay of dimension n.

By the previous reductions, it follows that $X_{\leqslant p}^{\Phi\text{-reg}}$ is Cohen Macaulay. $\square$

4.5 The Gorenstein condition

Once we know that our rings are Cohen-Macaulay, and we have a regular sequence for each of the rings, we can answer the question about when exactly the ring R_P is Gorenstein.

Theorem 4.12. *Suppose $P \in X_{\leqslant p}^{\Phi\text{-reg}}$. Then the local ring R_P is Gorenstein if and only if either:*

1. *$p = 1^n$; or*
2. *Every component X_q that contains P, has $q \leqslant p$.*

Proof. We prove that the rings in these two cases are Gorenstein first. In case 1, $X_{\leqslant p} \cong \text{GL}_n$ is smooth, therefore is Gorenstein. In case 2, we notice that the natural inclusion map $X_{\leqslant p} \hookrightarrow S_n$ induces an isomorphism of local rings at P. Because S_n is a complete intersection, this implies that the local ring R_P is a complete intersection too, and thus is Gorenstein.

For the converse, suppose R_P is Gorenstein. Then R_P has type 1, ie, that

$$\dim(\text{Ext}^{\dim R_P}(R_P/m, R_P)) = 1.$$

Consider the maximal regular sequence

$$(\mathbf{x}') = (\lambda_1 - N_1, ..., \lambda_k - N_k, N_{k+1}, ..., N_{n-1}, \lambda_n)$$

of $\bar{R}_P$ given in the previous section. Extend it by a regular sequence of $\mathcal{O}$ to a maximal regular sequence of R_P,

$$(\mathbf{x}) = (y_1, ..., y_{\dim\mathcal{O}}, \lambda_1 - N_1, ..., \lambda_k - N_k, N_{k+1}, ..., N_{n-1}, \lambda_n).$$

Consider the Artinian ring $R_0 := R_P/(\mathbf{x}) \cong \frac{K[N_1,\ldots,N_{n-1}]}{\langle\{N_i^2:1\leqslant i<n\}\cup\{N_S|S\in\mathcal{F}\}\rangle}$ with $\mathcal{F}$ as before. Let $\tilde{m}$ be the maximal ideal of R_0. By Lemma 3.1.16 of [BH93], we note that $\text{Ext}^{\dim R_P}(R_P/m, R_P) \cong \text{Hom}(R_0/\tilde{m}, R_0) \cong \text{Soc}(R_0)$. We can describe the socle of R_0 as the span of those monomials corresponding to the maximal sets in the partially ordered set $\mathcal{T} = \{S \subseteq \{1, ..., n-1\} | N_S \neq 0\}$ (ordered by inclusion).

So since R_0 has one-dimensional socle, we see that $\mathcal{T}$ has a unique maximal element.

Assume we are not in the case $p = 1^n$. Then each singleton $\{i\} \in \mathcal{T}$. And thus, since $\mathcal{T}$ has a unique maximal element, the union $\{1, 2, ..., n-1\} \in \mathcal{T}$. This shows that the family $\mathcal{F} = \mathcal{P}(1, ..., n-1) \backslash \mathcal{T}$ is empty, and thus, that R_P is isomorphic to the local ring of P in S_n. This shows the second condition.

□

4.6 The Cohen Macaulay-ness of non-Φ-regular points

When $P = (\Phi, N)$ is a non-Φ-regular point, we make the following conjecture.

Conjecture 4.13. *Let p be a partition of n. Then $X_{\leqslant p}$ is Cohen-Macaulay.*

In other words, we conjecture that Theorem 4.4 should be true, without the extra condition of Φ-regularity. One can prove this in a special case strong enough to prove the conjecture in the case $n = 3$.

Definition 4.14. *Let $\Phi \in GL_n$ be an $n \times n$ matrix, Let $\lambda = (\lambda_1, ..., \lambda_k)$ be a list of non-negative integers that add up to n. We say that Φ has signature λ, if Φ has k distinct eigenvalues $a_1, ..., a_k$ where we require without loss of generality that these eigenvalues are ordered in such a way, that whenever $a_i/a_j = q$, then $j = i + 1$, and the generalised a_i-eigenspace is λ_i-dimensional.*

Note that Φ may not have a unique signature, because we only specify one property the ordering of the a_i should satisfy, which is not strong enough to specify uniqueness.

It should also be noted that Φ has signature $(1, 1, ..., 1)$ if and only if it is regular. Thus we have shown already that points $P = (\Phi, N)$ such that Φ has signature $(1, 1, ..., 1)$ are Cohen-Macaulay.

For the following result, we need a tool from commutative algebra called 'graded Hodge algebras'. We recall the definition and main result of these objects, and I refer the interested reader to [BH93].

Let H be a finite set. Set $\mathbb{N}^H$ as the set of monomials in the variables H. Notice that $\mathbb{N}^H$ naturally has a partial order on it defined by divisibility in the R-algebra $R[\mathbb{N}^H]$. An ideal of monomials is an order ideal $\Sigma \subseteq \mathbb{N}^H$ of the set of monomials, as ordered by divisibility. A generator of Σ is a minimal element, in the divisibility partial order. We call the set of monomials outside Σ the standard monomials.

Definition 4.15. *Let R be a ring and A an R-algebra. Let H be a partially ordered finite set, with an inclusion into A.*

We call A a graded Hodge algebra governed by Σ if the following axioms hold:

1. *A is a free R-module, which admits the set of standard monomials $\mathbb{N}^H \backslash \Sigma$ as a basis.*

2. *For any generator of $t \in \Sigma$, we can write t as a finite R-linear combination of standard monomials*
$$t = \sum_{s \in \mathbb{N}^H \setminus \Sigma} r_s s,$$
such that for any divisor $y \in H$ of t, and for any s that appears in the above sum, there is a divisor $z \in H$ of s for which $z < y$ in the partial order of H.

The equations found in axiom 2 are called the *straightening laws*. When all straightening laws are trivial (ie, the right hand side is 0) we call this a discrete graded Hodge algebra.

Let $\mathrm{Ind}(A) \subseteq H$ be the subset of H consisting of elements that appear on the right hand side in one of the straightening law equations. Let $h \in \mathrm{Ind}(A)$ be a minimal element under the ordering of H. Give A the filtration defined by $\mathrm{Fil}_n = \langle h^n \rangle$, and form the graded algebra

$$\mathrm{Gr}_h A := \bigoplus_n (\mathrm{Fil}_n / \mathrm{Fil}_{n+1}).$$

This is a new graded Hodge algebra, governed by the same data as A, but with every instance of h removed from the straightening laws (so $\mathrm{Ind}(\mathrm{Gr}_h A) \subseteq \mathrm{Ind}(A) \backslash \{h\}$).

Theorem 4.16. *Let H be a partial order, and Σ an order ideal in $\mathbb{N}^H$. Let A be a graded Hodge algebra with data (H, Σ).*

If $Gr_h A$ is Cohen-Macaulay, then so is A.

Proof. See the proof of Corollary 7.1.6 of [BH93]. □

Corollary 4.17. *If the discrete Hodge algebra with data (H, Σ) is Cohen Macaulay, then so is any graded Hodge algebra with data (H, Σ).*

We can now continue with the following theorem.

Theorem 4.18. *Suppose that k_1, k_2, m are all non-negative integers, and that $m > 0$. Suppose that Φ is of signature $(k_2, 1^m, k_1)$. Then the local ring at a point $(\Phi, N) \in X_{\leqslant p}$ is Cohen-Macaulay.*

Proof. Let N_i, λ_i, $\nu_{i,j}$ and $\epsilon_{i,j}$ all be formal variables with appropriate indices. The local deformations at (Φ, N) take the form

$$\left(\begin{pmatrix} q^{m+1}(I_{k_2}+M_2) & & & & \\ & q^m(1+\lambda_{k_1+m}) & & & \\ & & \ddots & & \\ & & & q(1+\lambda_{k_1+1}) & \\ & & & & (I_{k_1}+M_1) \end{pmatrix}, \begin{pmatrix} 0_{k_2} & \underline{v}_2 & & & \\ & 0 & N_{k_1+m-1} & & \\ & & \ddots & \ddots & \\ & & & 0 & \underline{v}_1 \\ & & & & 0_{k_1} \end{pmatrix} \right)$$

where

$$M_1 = \begin{pmatrix} \lambda_{k_1} & \epsilon_{k_1,k_1-1} & \cdots & \epsilon_{k_1,2} & \epsilon_{k_1,1} \\ \epsilon_{k_1-1,k_1} & \lambda_{k_1-1} & \cdots & \epsilon_{k_1-1,2} & \epsilon_{k_1-1,1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \epsilon_{2,k_1} & \epsilon_{2,k_1-1} & \cdots & \lambda_2 & \epsilon_{2,1} \\ \epsilon_{1,k_1} & \epsilon_{1,k_1-1} & \cdots & \epsilon_{1,2} & \lambda_1 \end{pmatrix}$$

is a $k_1 \times k_1$ matrix,

$$M_2 = \begin{pmatrix} \lambda_{k_1+m+k_2} & \nu_{k_2,k_2-1} & \cdots & \nu_{k_2,2} & \nu_{k_2,1} \\ \nu_{k_2-1,k_2} & \lambda_{k_1+m+k_2-1} & \cdots & \nu_{k_2-1,2} & \nu_{k_2-1,1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \nu_{2,k_2} & \nu_{2,k_2-1} & \cdots & \lambda_{k_1+m+2} & \nu_{2,1} \\ \nu_{1,k_2} & \nu_{1,k_2-1} & \cdots & \nu_{1,2} & \lambda_{k_1+m+1} \end{pmatrix}$$

is a $k_2 \times k_2$ matrix, $\underline{v}_2 = \begin{pmatrix} N_{n-1} \\ \vdots \\ N_{k_1+m} \end{pmatrix}$ is a k_2-dimensional column vector and $\underline{v}_1 = \begin{pmatrix} N_{k_1} & \cdots & N_1 \end{pmatrix}$ is a k_1-dimensional row vector.

Notice that because the N_i's are located on the block off-diagonal, there are $k_1 + (m-1) + k_2 = n-1$ in total.

In this case, the equations take the form:

1. $M_2 \underline{v}_2 = \underline{v}_2 \lambda_{k_1+m}$
2. $\lambda_{k_1+1} \underline{v}_1 = \underline{v}_1 M_1$
3. $\lambda_{i+1} N_{k_1+i} = N_{k_1+i} \lambda_i$ for $1 < i < m$
4. Some other equations in the variables N_i, $\underline{N}_1$ and $\underline{N}_m$ which depend only on the equations defining $\overline{C_p}$, the closure of the nilpotent conjugacy class of p. From section 4 of [Wey89], these equations are polynomials which are simply sums of square-free monomials.

We give our ring the structure of a graded Hodge algebra. Consider the generator set $H = \{\lambda_i, \nu_{i,j}, \epsilon_{i,j}, N_i\}$ and give H any partial order such that

- for any i, j, a, b, $N_i > \phi_j > \epsilon_a > \nu_b$
- $\phi_n > \phi_{n-1} > \ldots > \phi_{k_1+m+1} > \ldots > \phi_{k_1+2} > \phi_1 > \phi_2 > \ldots > \phi_{k_1} > \phi_{k_1+1}$

Now take $\Sigma \subset \mathbb{N}^H$ to be the order ideal generated by $\{\lambda_i + 1 N_i : i > k_1\} \cup \{\lambda_i N_i : i \leqslant k_1\}$ and finally, we consider the straightening laws, for each generator in the above generating set:

$$\text{for } i \leqslant k_1;\ N_i \lambda_i = N_i \lambda_{k_1+1} - \sum_{j=1, j \neq i}^{k_1} N_j \nu_{j,i}$$

$$\text{for } k_1 < i < k_1+m;\ N_i \lambda_{i+1} = N_i \lambda_i$$

$$\text{for } i \geqslant k_1+m;\ N_i \lambda_{i+1} = N_i \lambda_{k_1+m} - \sum_{j=k_1+m+1, j \neq i}^{n} \epsilon_{i,j} N_j$$

It is readily checked that these equations do form a straightening law, due to our choice of order on the generating set, H.

Utilising Corollary 4.17, it can be seen that this ring is Cohen-Macaulay if the corresponding discrete graded Hodge algebra (with the same data) is. However, since the discrete graded Hodge algebra $R_0 = \mathcal{O}[\lambda_1, ..., \lambda_n, N_1, ..., N_n]/I$ with $I = \langle\{\lambda_i Ni : i \leqslant k_1\} \cup \{\lambda_i + 1N_i : i > k_1\}\rangle + J$ with J an ideal generated by squarefree monomials in the N_i is of the form in Theorem 4.9, it follows that R is Cohen-Macaulay.

□

Corollary 4.19. *Let p be a partition of 3. Then $X_{\leqslant p}$ is a Cohen Macaulay variety.*

Proof. The cases $p = 3$ and $p = 1^3$ are a complete intersection and a smooth variety respectively. This leaves only $p = 21$. Let $P = (\Phi, N) \in X_{\leqslant 21}$. Then Φ can have signature $(1, 1, 1)$, $(2, 1)$, $(1, 2)$ or 3. The case $(1, 1, 1)$ is the Φ-regular case, so is CM by Corollary 12. The signature (3) case also follows because P is only on the component X_{1^3}, which is smooth, ergo Cohen-Macaulay. The cases $(2, 1)$ and $(1, 2)$ are covered by Theorem 4.18. □

5 Automorphic forms for unitary groups

We now turn to an application of the smoothness result found in section 3. In this section, we define the space of ordinary automorphic forms, and the Hecke algebra attached to it. We then state a freeness result, and prove it in the final section of this paper.

Let l be a prime. Suppose F^+ is a totally real number field with an imaginary quadratic extension F, such that for any prime v of F^+ that lies above l, then v splits in F. We will also make the rather strong assumption that $F : F^+$ is an unramified extension. Let S_l be the set of all primes of F^+ that lie above l. Let G_{F^+} and G_F be the absolute Galois groups of F^+ and F respectively. Let L be a finite extension of $\mathbb{Q}_l$ with ring of integers $\mathcal{O}$, and residue field k. Let $\bar{L}$ be a choice of algebraic closure. We will assume that L is large enough that it contains all of the embeddings $F \hookrightarrow \bar{L}$ lie inside L. Let $c \in \mathrm{Gal}(F : F^+) = G_{F^+}/G_F$ be the unique non-trivial element, given by complex conjugation. For $a \in F$, we will denote $c(a)$ by $\bar{a}$ when convenient.

5.1 Unitary groups

Consider D/F a central simple algebra of F-dimension n^2, and let S_D be a finite set of primes of F^+ that split in F. Suppose that

- D splits at places w of F that do not lie above some place in S_D;
- There is an isomorphism $D^{op} \cong D \otimes_{F,c} F$ of F-algebras;
- The intersection $S_D \cap S_l = \varnothing$;
- For all places w of F above some place in S_D, D_w is a division algebra;
- Either n is odd, or n is even and $\frac{n}{2}[F^+ : \mathbb{Q}] + \#S_D \equiv 0 \pmod 2$.

By [HT99] section 3.3 we can find an involution of the second kind on D, that is, because of the condition that either n is odd, or n is even with $\frac{n}{2}[F^+ : \mathbb{Q}] + \#S_D \equiv 0 \pmod 2$, we may construct a map

$$* : D \to D$$

such that:

- $*$ is an F^+ linear anti-automorphism of D;
- $(a^*)^* = a$ for all $a \in D$;
- When restricted to F, $*$ coincides with complex conjugation.

In addition, we assume that this involution of the second kind is positive, that is, for any $\gamma \in D\backslash\{0\}$,

$$\mathrm{tr}_{F:\mathbb{Q}}[\mathrm{tr}_{D/F}(\gamma\gamma^*)] > 0.$$

Such an involution gives rise to a Hermitian form $\langle , \rangle : D \times D \to D$ given by $\langle x, y\rangle = x^* y$, and by [HT99] we may find such an involution such that the Hermitian form is non-degenerate. We make the assumption that the involution has this property.

Let $\mathcal{O}_D$ be an order in D, such that $\mathcal{O}_D^* = \mathcal{O}_D$, and such that for any split prime v of F^+, $\mathcal{O}_{D,v}$ is a maximal order of D_v. Such an order exists by section 3.3 of [CHT08]. Define the unitary group over $\mathcal{O}_{F^+}$, whose R-points (R an $\mathcal{O}_{F^+}$-algebra) are given by $G_D = \{g \in (\mathcal{O}_D \otimes_{\mathcal{O}_{F^+}} R)^\times : g^* g = 1\}$. Then G_D is an algebraic group over $\mathcal{O}_{F^+}$. By the positivity condition, we have that at each infinite place v of F^+, that $G_{D,v} \cong U(n)$.

For each prime v of F^+ that splits in F, choose a prime $\tilde{v}$ of F lying above v. This choice allows us to give an isomorphism $i_{\tilde{v}} : G_D(F_v^+) \longrightarrow D \otimes_F F_{\tilde{v}}$, which restricts to an isomorphism $G_D(\mathcal{O}_{F^+,v}) \cong \mathcal{O}_{D,\tilde{v}}$ as in section 3.3 of [CHT08]. Note that when $v \notin S_D$ is split in F with w lying above v, G_D is split, so that $G_D(F_v^+) \cong (D \otimes_F F_{\tilde{v}})^\times \cong \mathrm{GL}_n(F_w)$. If T is a set of primes of F^+ that splits in F, set $\tilde{T} = \{\tilde{v} | v \in T\}$.

5.2 Automorphic forms of G_D

We define the automorphic forms for G_D as in [Gro99] and [CHT08].

Recall from the classification of representations of algebraic groups that finite dimensional simple modules for a reductive group G over a field L are uniquely determined by the highest weight in the character group of a maximal torus $T_G \subseteq G$ $X(T_G) := \mathrm{Hom}(T_G, \mathbb{G}_m)$. Recall further, that there is a unique simple module with highest weight λ if and only if λ is dominant.

In the case of GL_n, the weights are naturally in correspondence with $\mathbb{Z}^n$, and the dominant weights are $\mathbb{Z}^n_+ := \{\lambda = (\lambda_1, ..., \lambda_n) \in \mathbb{Z}^n : \lambda_i \geqslant \lambda_{i+1} \forall i\}$. We set the L-vector space W_λ to be the irreducible representation of weight λ. We will need to choose a $\mathcal{O}$ lattice of W_λ. For λ a dominant weight, we do this as in [Ger18] by setting ξ_λ the representation $\mathrm{Ind}_{B_n}^{\mathrm{GL}_n}(w_0\lambda)_{/\mathcal{O}}$, for B_n a choice of Borel with maximal torus $T_n \subset \mathrm{GL}_n$, and w_0 the longest element of the Weyl group. We denote by M_λ the representation given by the $\mathcal{O}$-points of ξ_λ, so that $W_\lambda \cong M_\lambda \otimes_{\mathcal{O}} L$.

Let $L : \mathbb{Q}_l$ be the finite field extension defined before, with ring of integers $\mathcal{O}$. The finite dimensional algebraic representations in L vector spaces of $\mathrm{Res}_{\mathbb{Q}}^{F^+} G_D \otimes \mathbb{Q}_l \cong \prod_{w \in \tilde{S}_l} \mathrm{Res}_{\mathbb{Q}_l}^{F_{\tilde{v}}}(\mathrm{GL}_n)$ are characterised by the sequence of dominant weights, one for each embedding corresponding to $w \in \tilde{S}_l$. We define the set as $W = (\mathbb{Z}_+^n)^{\mathrm{Hom}(F^+,L)}$. For each $\mu \in W$, we can now define the algebraic representation of $G_D/\mathcal{O}_{F^+}$ with highest weight μ by $M_\mu = \bigotimes_{\tau \in \mathrm{Hom}(F^+,L),\mathcal{O}} M_{\lambda_\tau}$, and $W_\mu = M_\mu \otimes_{\mathcal{O}} L$.

For each $v \in S_D$, choose a finite-free $\mathcal{O}$-module representation $\rho_v : G_D(\mathcal{O}_{F^+,v}) \to \mathrm{GL}(M_v)$. Set $M_{\{\rho_v\}} = \bigotimes_{v \in S_D} M_v$. We set $M_{\mu,\{\rho_v\}} = M_\lambda \otimes M_{\{\rho_v\}}$.

Definition 5.1. *Let $\lambda = (\mu, \{\rho_v\})$ be as above. We define the space of automorphic forms for G_D of weight λ with A-coefficients $S_\lambda(A)$, where A is an $\mathcal{O}$-algebra or $\mathcal{O}$-module, as the space of functions*

$$f : G_D(F^+)\backslash G_D(\mathbb{A}_{F^+}^\infty) \to M_\lambda \otimes_{\mathcal{O}} A$$

such that there is an open compact subgroup

$$U \subset G_D(\mathbb{A}_{F^+}^{\infty, S_l}) \times G_D(\mathcal{O}_{F^+,l})$$

with

$$u \cdot f(gu) = f(g)$$

for all $g \in G_D(\mathbb{A}_{F^+}^\infty)$ and $u \in U$ where $u\cdot$ denotes the action of u on M_λ factoring through $\prod_{v \in S} G_D(F_v^+)$.

Notice that $S_\lambda(A)$ is a smooth representation of $G_D(\mathbb{A}_{F^+}^\infty)$, under the action $(hf)(g) = h \cdot f(gh^{-1})$ (again, the $\cdot$ action acting through the representation of $G_D(F_l^+) \times \prod_{v \in S_D} G_D(F_v^+)$ on M_λ). We denote by $S_\lambda(U, A) = S_\lambda(A)^U$ the invariants under this action.

5.3 Hecke Operators

For much of the next two sections, the argument will be a slight adaptation on that in [Ger18]. As such, the details can be found in sections 2 and 4 of [Ger18], so this will just highlight the definitions and results needed, and refer to [Ger18] for the proofs, which we will adapt into this case. Let T be a finite set of places of F^+ containing $S_D \cup S_l$ such that every place in T splits in F, and let $\tilde{T}$ be a set of primes of F above those in T as defined before. Fix an open compact subgroup $U = \prod_v U_v$ of $G_D(\mathbb{A}_{F^+}^\infty)$, such that for any split place v outside T, $U_v \cong \mathrm{GL}_n(\mathcal{O}_{F,\tilde{v}})$ via the map i_v, and such that for any place of F^+, v, inert in F, suppose U_v is hyperspecial. Suppose further that U is sufficiently small, that is, there is a place v such that U_v contains no non-identity roots of unity. We define the Hecke operators on the subspace $S_\lambda(U, A)$.

Hecke operators at unramified places Let v be a place of F^+ split in F and $w = \tilde{v}$ be a place in F. Let ϖ_w be a uniformiser. We can define the Hecke operators as the double coset operators:

$$T_p^{(i)} = \left[i_v^{-1} \left(\mathrm{GL}_n(\mathcal{O}_{F,w}) \begin{pmatrix} \varpi_w I_i & 0 \\ 0 & I_{n-i} \end{pmatrix} \mathrm{GL}_n(\mathcal{O}_{F,w}) \right) \times U^v \right]$$

Hecke operators at places dividing l At places dividing the residual characteristic of $\mathcal{O}$, we set $\alpha_{\tilde{v}}^{(i)} = \begin{pmatrix} \varpi_{\tilde{v}} I_i & 0 \\ 0 & I_{n-i} \end{pmatrix}$, and define

$$U_{\mu,\tilde{v}}^{(i)} = (w_0 \mu_v)(\alpha_{\tilde{v}}^{(i)})^{-1} [U \alpha_{\tilde{v}}^{(i)} U]$$

where w_0 is the longest element of the Weyl group of GL_n, and $\mu \in W$, with μ_v the dominant weight for the embedding $F^+ \hookrightarrow L$.

We make the following adjustment to the group U.

Definition 5.2. *For v a place of F^+ above l, and b a positive integer, let $I^b(\tilde{v})$ be the set of matrices in $GL_n(F^{\tilde{v}})$ which are upper triangular unipotent mod $\tilde{v}^b$. Define $U(l^b) = \prod_{v \in S_l} I^{b,c}(\tilde{v}) \times U^l$.*

In the case with the group $U(l^b)$, further define the following diamond operators:

Definition 5.3. *Let T_n be the maximal torus inside GL_n as before. For $v \in S_l$, and $u \in T_n(\mathcal{O}_{F_{\tilde{v}}})$, define $\langle u \rangle$ as the operator*

$$[U(l^b) u U(l^b)]$$

on $S_\lambda(U(l^b), A)$. For $u \in T_n(\mathcal{O}_{F^+,l}) = \prod_{v \in S_l} T_n(\mathcal{O}_{F_v}) \cong \prod_{v \in S_l} T_n(\mathcal{O}_{F_{\tilde{v}}})$, define $\langle u \rangle = \prod_{v \in S_l} \langle u_{\tilde{v}} \rangle$.

Define the Hecke algebra $\mathbb{T}^T = \mathbb{T}^T(U(l^b), A)$ as the A-subalgebra of $\mathrm{End}(S_\lambda(U(l^b), A))$ generated by all the operators $\{T_{\tilde{v}}^{(i)}, (T_{\tilde{v}}^{(n)})^{-1}) | v \text{ split in } F \text{ outside of } T\}$, $\{U_{\mu,\tilde{v}}^{(i)} | v \in S_l\}$ and $\{\langle u \rangle | u \in T_n(\mathcal{O}_{F^+,l})\}$.

Notice that the map $u \mapsto \langle u \rangle$ defines a group homomorphism

$$T_n(\mathcal{O}_{F^+,l}) \to \mathbb{T}^T(U(l^b), A)^\times$$

which factors through $T_n(\mathcal{O}_{F^+,l}/l^b) = \prod_{v \in S_l} T_n(\mathcal{O}_{F^+,v}/v^b)$.

5.4 Big ordinary Hecke algebras and the action of Λ

From this point on, we wish to focus on the cases where $A = \mathcal{O}, L/\mathcal{O}$, or is a finite module $\mathcal{O}/\pi^n\mathcal{O}$.

Recall from Hida theory, as explained fully in section 2.4 of [Ger18], that for any place $v \in S_l$, and any i, the operator $e_v^{(i)} := \lim_{n \to \infty} (U_{\mu,\tilde{v}}^{(i)})^{n!}$ is a projection on $S_\lambda(U, A)$. We can further define the projection $e = \prod_{v,i} e_v^{(i)}$. We define the ordinary submodule $S^{\mathrm{ord}}_\lambda(U,A) := e.S_\lambda(U,A)$ as the image of this projection. Notice, since all the Hecke operators commute, that this is a Hecke invariant submodule. We also define $\mathbb{T}^{T,\mathrm{ord}}(U(l^b),A) = e\mathbb{T}^T(U(l^b),A)$.

Definition 5.4. *Let T_n be the maximal torus of GL_n as before. For $b \geqslant 1$, let $T_n(l^b)$ be the kernel of $T_n(\mathcal{O}_{F^+,l}) \to T_n(\mathcal{O}/l^b)$.*

We define the following algebras,

$$\Lambda_b = \mathcal{O}[[T_n(l^b)]] = \varprojlim_{b' \geqslant b} \mathcal{O}[T_n(l^b)/T_n(l^{b'})]$$

$$\Lambda = \mathcal{O}[[T_n(l)]] = \mathcal{O}[[T_n(l^1)]]$$

$$\Lambda^+ = \mathcal{O}[[T_n(\mathcal{O}_{F^+,l})]] = \varprojlim_{b' \geqslant b} \mathcal{O}[T_n(\mathcal{O}_{F^+,l})/T_n(l^{b'})].$$

We denote by a_N the kernel of the map $\Lambda \to \mathcal{O}[T_n(l)/T_n(l^N)]$. Notice that, since U is sufficiently small, $S^{\mathrm{ord}}_\lambda(U(l^{b,c}),A)$ is a free Λ/a_b-module, through the action of $T_n(\mathcal{O}_{F^+,l})$, and hence we have an inclusion $\Lambda/a_b \hookrightarrow \mathbb{T}(U(l^b),L/\mathcal{O})$ by Proposition 2.5.3 of [Ger18].

5.4.1 Infinite level

We need to consider the big ordinary Hecke algebra. Set

$$\mathbb{T}^{T,\mathrm{ord}}(U(l^\infty),A) = \varprojlim_{b>0} \mathbb{T}^{T,\mathrm{ord}}(U(l^{b,b}),A)$$

and

$$S^{\mathrm{ord}}(U(l^\infty),A) = \varinjlim_{b>0} S^{\mathrm{ord}}(U(l^{b,b}),A).$$

Notice that because of the inclusions $\Lambda/a_b \hookrightarrow \mathbb{T}^{T,\mathrm{ord}}(U(l^{b,c}),L/\mathcal{O})$, we get an inclusion $\Lambda \hookrightarrow \mathbb{T}^{T,\mathrm{ord}}(U(l^\infty),L/\mathcal{O})$, and we see that $S^{\mathrm{ord}}(U(l^\infty),L/\mathcal{O})$ is a discrete Λ-module, so its Pontryagin dual is a compact Λ-module. (and in fact is finite free, by Proposition 2.5.3 of [Ger18] since we assume $U(l)$ is sufficiently small.)

We can now give a statement of a theorem that can be proved by the application Theorem 3.1. Under certain hypotheses (to be determined in section 6) we have Theorem 6.9, which states: The $\mathbb{T}^{T,\mathrm{ord}}(U(l^\infty),L/\mathcal{O})$-module $S^{\mathrm{ord}}(U(l^\infty),L/\mathcal{O})^\vee$ is locally free over the generic fibre $\mathbb{T}^{T,\mathrm{ord}}(U(l^\infty),L/\mathcal{O})[1/l]$.

As a consequence, the multiplicity of $S^{\mathrm{ord}}(U(l^\infty),L/\mathcal{O})^\vee$ is the same at every characteristic zero point of $\mathbb{T}^{T,\mathrm{ord}}(U(l^\infty),L/\mathcal{O})$, and thus, we expect the multiplicity of non-classical points (those corresponding to Hida families of ordinary automorphic forms) is the same as at classical modular forms.

6 Galois representations and deformation rings

6.1 Local deformation rings

We now define a deformation problem. Let $v \in S_D$ with residue field of size q_v. We say that an n-dimensional representation $\rho : G_{F_v^+} \to \mathrm{GL}_n(A)$ is Steinberg if the map $R_{\bar{\rho}}^{\square} \to A$ determined by ρ factors through $\hat{\mathcal{O}}_{X_n}$.

We note that this is equivalent to the statement, that the representation ρ lies on the irreducible component $X_n(A)$ of S_{GL_n}, which in the case when $A = L$ is a characteristic 0 field, the Weil-Deligne representation obtained from ρ, $WD(\rho) = (r, N)$, then r is unramified and the eigenvalues of $r(\mathrm{Frob}_{q_v})$ are in the ratio $q_v^{n-1} : q_v^{n-2} : ... : q_v : 1$. Note that this definition puts ρ on the irreducible component X_n of S_n.

Let $C_{\mathcal{O}}$ be the category of Artinian local $\mathcal{O}$-algebras with residue field $\mathbb{F}$, as in Mazur. For each $v \in S_D$, and Steinberg representation $\bar{\rho}_v : G_{F,\tilde{v}} \to \mathrm{GL}_n(\mathbb{F})$ define a functor

$$\begin{aligned} D_{\bar{\rho}_v}^{n,\square} : \mathcal{C}_{\mathcal{O}} &\to \mathfrak{Set} \\ A &\mapsto \{\text{Steinberg liftings of } \bar{\rho}_v \text{ to } A\} \end{aligned}$$

This functor is pro-representable by the complete Noetherian local ring $R_v^{\square,\mathrm{st}} := \mathcal{O}_{X_n,\bar{\rho}}$. We notice that when we view X_n as a scheme over L, Theorem 3.1 tells us, since q is not a root of unity in L, that any localisation of $R_v^{\square,\mathrm{st}}[1/l]$ is a regular ring. This shows us that $R_v^{\square,\mathrm{st}}[1/l]$ is regular.

For ρ a deformation of $\bar{\rho}_v$ to A, we say that ρ is of type X_n if the map $R_{\bar{\rho}}^{\square} \to A$ defined by ρ factors through $R_v^{\square,st}$.

We recall the definition of $\tilde{r}$-discrete series found in section 2.4.5 in [CHT08].

Let $\tilde{r}_v : G_{F,\tilde{v}} \to \mathrm{GL}_d(\mathcal{O})$ be a representation such that:

1. $\tilde{r}_v \otimes k$ is absolutely irreducible (k the residue field of $\mathcal{O}$;
2. Every irreducible subquotient of $\tilde{r}_v|_{I_{\tilde{v}}}$ is absolutely irreducible;
3. For each $i = 0, ..., m$, $\tilde{r} \otimes k \ncong \tilde{r} \otimes k(i)$.

For R an $\mathcal{O}$ algebra, we say a representation $\rho : G_{F,\tilde{v}} \to \mathrm{GL}_{md}(R)$ is $\tilde{r}$-discrete series if there is an decreasing filtration $\{\mathrm{Fil}^i\}$ of ρ by R-direct summands such that

1. $\mathrm{gr}^i \rho \cong \mathrm{gr}^0 \rho(i)$ for $i = 0, ..., m-1$
2. $\mathrm{gr}^0 \rho|_{I,\tilde{v}} \cong \tilde{r}|_{I,\tilde{v}} \otimes_{\mathcal{O}} R$.

Proposition 6.1. *Suppose $l > h_G$. Let $\tilde{r}$ be a rank d representation as above, and let n be an integer with $d|n$. Let $X_{\tilde{r},n}$ be the moduli space, defined over $\mathcal{O}$, of framed $\tilde{r}$-discrete series representations of rank n. Then the base change, $X_{\tilde{r},nL}$, to L is smooth over L.*

Proof. Let $S_{\tilde{r}}$ be the moduli stack over $\mathcal{O}$ of n-dimensional $\tilde{r}$-discrete representations, so that $S_{\tilde{r}} \cong [X_{\tilde{r}}/\mathrm{GL}_n]$ and let $S_{\mathbb{1}}$ be the stack of $m := n/d$-dimensional $\mathbb{1}$-discrete series representations. Let $S_{\tilde{r}}^{\mathrm{WD}}$ be the stack over L whose groupoid over R consists of objects (ρ', N) where ρ' is a rank $n = dm$ $\tilde{r}$-discrete series representation with open kernel, and N is an element of $\mathrm{End}_R(R^n)$ such that $\rho' N \rho'^{-1} = q^{\nu} N$. Define $S_{\mathbb{1}}^{\mathrm{WD}}$ analogously. Recall that there is a morphism $S_{\tilde{r}}^{\mathrm{WD}} \to S_{\tilde{r}}$ given by (ρ', N) is sent to the unique representation ρ given by $g \mapsto \rho(g)\exp(t_l(g))$ for $g \in I$ and $\rho(\mathrm{Frob}) = \rho'(\mathrm{Frob})$. Recall that this is an isomorphism on the base change to L.

Then we have an morphism of algebraic stacks $S_{\mathbb{1}}^{\mathrm{WD}} \to S_{\tilde{r}}^{\mathrm{WD}}$ given by the morphism $(\rho', N) \mapsto (\rho', N) \otimes \tilde{r}$. We claim that this is an isomorphism. By an exercise in Clifford theory and by assumptions on $\tilde{r}$, $\tilde{r}|_I$ can be written as a direct sum of pairwise non-isomorphic absolutely irreducible I-representations $\tau \oplus \tau^{\mathrm{Frob}} \oplus, ..., \oplus \tau^{\mathrm{Frob}^{k-1}}$ for some $k \in \mathbb{N}$. As ρ' is $\tilde{r}$-discrete series in characteristic zero, we see that $\rho'|_I \cong m(\tau \oplus \tau^{\mathrm{Frob}} \oplus, ..., \oplus \tau^{\mathrm{Frob}^{k-1}})$. Let $V_{\tilde{r}}(R) = \mathrm{End}_{R[I]}(\tilde{r}^m)$ be the the space of I-equivariant maps of any representation in $S^{\mathrm{WD}_{\tilde{r}}}(R)$, and define $V_{\mathbb{1}}(R) = \mathrm{End}_{R[I]}(\mathbb{1}^m)$ similarly. Note that the map

$$V_{\mathbb{1}}(R) \to V_{\tilde{r}}(R) \tag{4}$$
$$N \mapsto N \otimes \mathrm{id}_{\tilde{r}} \tag{5}$$

is injective, and hence is isomorphic onto its image. We claim that if $(\rho', N) \in S_{\tilde{r}}^{\mathrm{WD}}(R)$, then N is in the image of this map.

First, note that N is I-equivariant. We calculate using Schur's lemma that $V_{\tilde{r}}(R) \cong k M_m(R)^k$, since each τ^{Frob^i} is absolutely irreducible, and we see the above map corresponds to the diagonal map $\Delta : M_m(R) \to M_m(R)^k$.

The space $V_{\tilde{r}}(R)$ has a natural action of Frobenius on it, and under this action $N = (N_1, ..., N_k) \in M_m(R)^k$ has $Frob.(N_1, ..., N_k) = q(N_1, ..., N_k)$. Notice that Frob induces an isomorphism of the underlying spaces $\tau^m \to (\tau^{\mathrm{Frob}})^m$, which gives us a commutative diagram

$$\begin{tikzcd} \tau^m \arrow[r, "\mathrm{Frob}"] \arrow[d, "N_1"] & (\tau^{\mathrm{Frob}})^m \arrow[d, "qN_2"] \\ \tau^m \arrow[r, "\mathrm{Frob}"] & (\tau^{\mathrm{Frob}})^m \end{tikzcd}$$

Hence, we see $(qN_2, ..., qN_k, qN_1) = q(N_1, ..., N_{k-1}, N_k)$, and thus N lies in the image of the diagonal map. This proves the claim.

Let $\chi_{\tilde{r}} = \hom_I(\tau, \tilde{r})$. Notice that this is an unramified character. We claim that $(\mathrm{Hom}_I(\tau, \_) \otimes \chi_{\tilde{r}}^{-1}, \Delta^{-1}) : S_{\tilde{r}}^{\mathrm{WD}} \to S_{\mathbb{1}}^{\mathrm{WD}}$ is an inverse defining the equivalence.

For $(\Theta, N) \in S_{\tilde{r}}^{\mathrm{WD}}(R)$, the previous claim gives us an isomorphism on the N-part of the stacks $S_{\tilde{r}}^{\mathrm{WD}}(R)$, so we focus on the representation part. Since $\theta|_I$ acts through a finite quotient, and R is an algebra over a characteristic 0-field, we have that Θ is semisimple and hence we get a decomposition of

I-representations:

$$M \cong \bigoplus_{i=0}^{k-1} \mathrm{Hom}_I(\tau^{\mathrm{Frob}^i}, \Theta) \otimes \chi_{\tilde{r}}^{-1} \otimes \tau^{\mathrm{Frob}^i}$$

for some positive integer k. Since each τ^{Frob^i} occurs in Θ with equal multiplicity, we see that each $\mathrm{Hom}_I(\tau^{\mathrm{Frob}^i}, \Theta) \cong \mathrm{Hom}_I(\tau, \Theta)$, and thus,

$$\Theta \cong \mathrm{Hom}_I(\tau, \Theta) \otimes \chi_{\tilde{r}}^{-1} \otimes \bigoplus_{i=0}^{m-1} \tau^{\mathrm{Frob}^i} \cong \mathrm{Hom}_I(\tau, \Theta) \otimes \chi_{\tilde{r}}^{-1} \otimes \tilde{r}$$

As I representations. To see an isomorphism on the level of W_F-representations, notice that we have an unramified character χ defined over an algebraic closure $\bar{L}$ such that for each i $\mathrm{gr}^i\Theta \cong \tilde{r} \otimes \chi(i)$. Then

$$\mathrm{Hom}_{\bar{L}[I]}(\tau, \mathrm{gr}^m(\Theta)) \cong \mathrm{Hom}_{\bar{L}[I]}(\tau, \tilde{r} \otimes \chi) \cong \chi_{\tilde{r}} \otimes \chi(i).$$

Since $\tilde{r}(i) \ncong \tilde{r}$ for each $1 \leqslant i \leqslant m$, $\Theta = \bigoplus_i \mathrm{gr}^i(\tilde{r})$,so we get a $\bar{L}[W_F]$ isomorphism

$$\Theta \otimes \bar{L} \cong (\mathrm{Hom}_I(\tau, \Theta) \otimes \chi_{\tilde{r}}^{-1} \otimes \tilde{r}) \otimes_L \bar{L}$$

Finally, since $\tilde{r}$ is absolutely irreducible, this can be upgraded to an isomorphism L-vector spaces. Hence, the composite $S_{\tilde{r}}^{\mathrm{WD}}(R) \to S_{\mathbb{1}}^{\mathrm{WD}}(R) \to S_{\tilde{r}}^{\mathrm{WD}}(R)$ is the identity.

To show $S_{\mathbb{1}}^{\mathrm{WD}}(R) \to S_{\tilde{r}}^{\mathrm{WD}}(R) \to S_{\mathbb{1}}^{\mathrm{WD}}(R)$ is the identity, let $\rho \in S_{\mathbb{1}}(R)$. Then the natural map

$$\rho \to \mathrm{Hom}_I(\tau, \rho \otimes \tilde{r}) \tag{6}$$
$$v \mapsto \{w \mapsto v \otimes w\} \tag{7}$$

defines an I isomorphism. So we need only check that $\rho \otimes \chi_{\tilde{r}}$ and $\mathrm{Hom}_I(\tau, \rho \otimes \tilde{r})$ have the same action of Frobenius. This can be checked again, by looking at the character $\mathrm{gr}_i(\rho)$. Hence, we have exhibited an equivalence of categories $S_{\mathbb{1}} \leftrightarrow S_{\tilde{r}}$.

Given a choice of Frobenius, Frob, and a topological generator of the tame inertia group, s, we can explicitly write an isomorphism of stacks

$$\begin{aligned} S_{\mathbb{1}} &\cong [X_m/\mathrm{GL}_m] \\ \rho &\mapsto (\rho(\mathrm{Frob}), \log(\rho(s))) \\ \rho_\Phi(\mathrm{Frob}^n x) = \Phi^n \exp(N t_l(x)) &\leftarrow\!\shortmid (\Phi, N) \end{aligned}$$

As $(X_m)_L$ is a smooth scheme, it shows that $S_{\mathbb{1}}[1/l]$ is a smooth stack, and thus that $S_{\tilde{r}}[1/l]$ and $X_{\tilde{r},nL}$ are smooth. $\square$

In light of this proposition, if $\bar{\rho} : G_{F,\tilde{v}} \to \mathrm{GL}_n(\mathbb{F})$ is $\tilde{r}$-discrete series, we let $R_v^{\square,\tilde{r}}$ be the universal lifting ring of $\tilde{r}$-discrete series representations. By the proposition, $R_v^{\square,\tilde{r}}[1/l]$ is regular at every maximal ideal.

For $v \in S_l$, Let $\bar{I}_{\tilde{v}}$ be the inertia subgroup of $G^{\mathrm{ab}}_{F,\tilde{v}}$, and let $\bar{I}_{\tilde{v}}(l)$ be the pro-l part. As in chapter 3 of [Ger18] we can define a lifting $\Lambda_{\tilde{v}} : \mathcal{O}[[\bar{I}_{\tilde{v}}(l)]]$-algebra $R_v^{\triangle}$. This is the quotient of the universal lifting ring $R_v^{\square}$ of pairs $(\rho, \{\chi_i\})$, such that a morphism $r : R_v^{\square} \to A$ corresponding to representation $\rho : G_v \to \mathrm{GL}_n(A)$ and a sequence of characters $\chi_i : I_{\tilde{v}}$ factors through $R_v^{\triangle}$ if and only if ρ is $\mathrm{GL}_n(\mathcal{O})$-conjugate to an upper triangular representation with diagonal characters equal to $\chi_1, ..., \chi_n$ when restricted to inertia.

Lemma 6.2. *Suppose that $\bar{\rho}_v : G_{F,\tilde{v}} \to GL_n(\mathbb{F})$ is an ordinary Galois representation with diagonal characters $\bar{\chi}_1, \bar{\chi}_2, ..., \bar{\chi}_n$, such that for no pair $i < j$ is $\chi_i = \varepsilon\chi_j$, with ε the cyclotomic character, then $R_v^{\triangle}[1/l]$ is formally smooth.*

Proof. We see that the dimension of $R_v^{\triangle}[1/l]$ is $n^2 + [F_v : \mathbb{Q}_l]\frac{n(n+1)}{2}$. For any choice of closed point x of $\mathrm{Spec}R_v^{\triangle}[1/l]$, part 1 of Lemma 3.2.3 of [Ger18] tells us that the dimension of the tangent space of $R_w^{\triangle,ar}[1/l]$ is $n^2 + [F_w : \mathbb{Q}_l] + \dim H^2(G_{F_w}, \mathrm{Fil}^0\mathrm{ad}(V_x))$. From part 3 of Lemma 3.2.3, we also see that if the diagonal characters of $\bar{\rho}$, $(\bar{\chi}_i)$ have $\chi_i/\chi_j \neq \epsilon$ for every pair $i < j$, then $\dim H^2(G_{F_w}, \mathrm{Fil}^0\mathrm{ad}(V_x)) = 0$. Hence, the ring $R_v^{\triangle}[1/l]$ is regular. $\square$

6.2 Local-Global compatibility

We start by introducing the group $\mathcal{G}_n$ from [CHT08], defined as the group scheme that is the semi-direct product of $\mathrm{GL}_n \times \mathrm{GL}_1$ with $C_2 = \{1, j\}$ where j acts as

$$j(g, \mu)j^{-1} = (\mu(g^{-1})^{\mathrm{T}}, \mu).$$

By Lemma 2.1.1 of [CHT08], we have that representations $r : G_{F^+} \to \mathcal{G}_n(R)$ such that $r^{-1}(\mathrm{GL}_n(R) \times \mathrm{GL}_1(R)) = G_F$ correspond with pairs (ρ, χ), where ρ is an n-dimensional representation of G_F, and χ is a character of G_{F^+}, such that $\rho^c \cong \chi\rho^\vee$, and $c \in G_{F^+}$ is sent to j.

For brevity, whenever we have a homomorphism $r : G_{F^+} \to \mathcal{G}_n(R)$, and a subgroup $H \subset G_{F^+}$, we use $r|_{G_F}$ to mean the restriction, followed by the projection to GL_n. Typically, H will be the subgroup G_F or its localisations G_{F_w}.

Proposition 6.3. *Suppose that $\mathfrak{m} \trianglelefteq \mathbb{T}^{T,ord}(U(l^\infty), \mathcal{O})$ is a maximal ideal, with residue field $\mathbb{F}$. Then there is a unique continuous semisimple representation*

$$\bar{r}_{\mathfrak{m}} : G_F \to GL_n(\mathbb{F})$$

such that:

1. $$\bar{r}^c \cong \bar{r}_{\mathfrak{m}}^\vee(1-n);$$

2. *For any place v of F^+, outside T, $\bar{r}_{\mathfrak{m}}|_w$ is unramified;*

3. *If further, v splits as $v = ww^c$ in F, then the characteristic polynomial of $\bar{r}_{\mathfrak{m}}(Frob_w)$ is*

$$X^n - T_w^{(1)} X^{n-1} + ... + (-1)^j N(w)^{\frac{j(j-1)}{2}} T_w^{(j)} X^{n-j} + ... + (-1)^n N(w)^{\frac{n(n-1)}{2}} T_w^{(n)}$$

   modulo $\mathfrak{m}$;

4. *Let $\tilde{r}_{\tilde{v}} : G_F \to GL_{m_v}(\mathcal{O})$ be as in section 3.2 of [CHT08] (note: this is constructed from the smooth representation $\rho_v : G_D(F_v^+) \to GL(M_v)$ via the Jacquet-Langlands and local Langlands correspondences). If $v \in S_D$ and $U_v = G_D(\mathcal{O}_{F^+,v})$, then $\bar{r}_{\mathfrak{m}}|_{G_{F,v}}$ is $\tilde{r}_{\tilde{v}}$-discrete series.*

Proof. Apart from statement 4, this is Propositions 2.7.3 in [Ger18], so we prove only this part. By the argument of Proposition 2.7.3 in [Ger18], the maximal ideals of $\mathbb{T}$ are in bijection with those of $\mathbb{T}/m_\Lambda$. Hence, this proposition follows immediately from the classical situation. The proof of this can be found in Proposition 3.4.2 of [CHT08], which proves the proposition. □

Proposition 6.4. *If $\mathfrak{m}$ is non-Eisenstein, that is, $\bar{r}_{\mathfrak{m}}$ is irreducible, then $\bar{r}_{\mathfrak{m}}$ can be extended to a representation $\bar{r}_{\mathfrak{m}} : G_{F^+} \to \mathcal{G}_n(\mathbb{F})$, and this representation can be lifted to a representation*

$$r_{\mathfrak{m}} : G_{F^+} \to \mathcal{G}_n(\mathbb{T}^{T,ord}(U(l^\infty), \mathcal{O})_{\mathfrak{m}})$$

1. *For $\nu : \mathcal{G}_n \to GL_1$, the second projection, $\nu \circ r_{\mathfrak{m}} = \epsilon^{1-n}\delta_{F/F^+}^{\mu_{\mathfrak{m}}}$. where ϵ is the cyclotomic character, δ_{F/F^+} is the non-trivial character of G_{F^+}/G_F, and $\mu_m \in \mathbb{Z}/2$;*

2. *For any place $v \notin T$ of F^+, $\bar{r}_{\mathfrak{m}}|_{\tilde{v}}$ is unramified;*

3. *If further, v splits as $v = ww^c$ in F, then the characteristic polynomial of $\bar{r}_{\mathfrak{m}}(Frob_w)$ is*

$$X^n - T_w^{(1)} X^{n-1} + ... + (-1)^j N(w)^{\frac{j(j-1)}{2}} T_w^{(j)} X^{n-j} + ... + (-1)^n N(w)^{\frac{n(n-1)}{2}} T_w^{(n)};$$

4. *If $v \in S_D$, then $r_{\mathfrak{m}}|_{G_{F,\tilde{v}}}$ is $\tilde{r}_{\tilde{v}}$-discrete series.*

Proof. As the previous proposition, apart from statement 4, this is Proposition 2.7.4 in [Ger18], so we prove only this final statement. By the proof of Proposition 2.7.4 of [Ger18], we may find a sequence of maximal ideals $\mathfrak{m}_b \subset \mathbb{T}^{T,\mathrm{ord}}(U(l^{b,b}), \mathcal{O})$ such that $\mathbb{T}_{\mathfrak{m}} = \varprojlim_b \mathbb{T}^{T,\mathrm{ord}}(U(l^{b,b}), \mathcal{O})_{\mathfrak{m}_b}$, and we define $r_{\mathfrak{m}} = \varprojlim_b r_{\mathfrak{m}_b}$. By Lemma 3.4.4 of [CHT08], each $r_{\mathfrak{m}_b}|_{G_{F,\tilde{v}}}$ is $\tilde{r}_{\tilde{v}}$-discrete series, and so now it remains to show that $r_m|_{G_{F,\tilde{v}}}$ is too. Since for each $b > c$ each $r_{\mathfrak{m}_b} \otimes \mathbb{T}^{T,\mathrm{ord}}(U(l^{c,c}), \mathcal{O})_{\mathfrak{m}_c} = r_{\mathfrak{m}_c}$, if follows that the filtration, Fil_b^i on $r_{\mathfrak{m}_b}$ descends to a filtration $\mathrm{Fil}_b^i \otimes \mathbb{T}^{T,\mathrm{ord}}(U(l^{c,c}), \mathcal{O})_{\mathfrak{m}_c}$ on $r_{\mathfrak{m}_c}$, and that the graded parts have $[\mathrm{gr}^i(r_{\mathfrak{m}b})] \otimes \mathbb{T}^{T,\mathrm{ord}}(U(l^{c,c}), \mathcal{O})_{\mathfrak{m}_c} \cong \mathrm{gr}^i[r_{\mathfrak{m}b}) \otimes \mathbb{T}^{T,\mathrm{ord}}(U(l^{c,c}), \mathcal{O})_{\mathfrak{m}_c}]$. It follows that $\mathrm{Fil}_b^i \otimes \mathbb{T}^{T,\mathrm{ord}}(U(l^{c,c}), \mathcal{O})_{\mathfrak{m}_c}$ is a defining filtration on $r_{\mathfrak{m}_c}$. From Lemma 2.4.25 of [CHT08], such a filtration is unique, so we have a compatible system of filtrations on the $r_{\mathfrak{m}_b}$ which lift to a filtration on $r_{\mathfrak{m}}|_{G_{F,\tilde{v}}}$. We see from compatibility that $\mathrm{gr}_i(r_{\mathfrak{m}}) = \varprojlim_b \mathrm{gr}_i(r_{\mathfrak{m}_b})$, and so it is easy to check that $r_{\mathfrak{m}}|_{G_{F,\tilde{v}}}$ is $\tilde{r}_{\tilde{v}}$-discrete series.

□

6.3 Global deformation rings

Let $F : F^+$, $T = S_l \coprod S_D \coprod R$, $\tilde{T}$ all be as before. Let $\bar{\rho} : G_F \to \mathrm{GL}(\mathbb{F})$ be a representation with local representations $\rho_w = \bar{\rho}|_{G_{F,w}}$, where w is a place of F. Assume that:

- the representation $\bar{\rho}$ is a irreducible automorphic representation, I.E., there is a non-Eisenstein maximal ideal $\mathfrak{m} \trianglelefteq \mathbb{T}^{T,\mathrm{ord}}(U(l^\infty, \mathcal{O}))$ so that $\bar{\rho} \cong \bar{r}_{\mathfrak{m}}$;
- the subgroup $\rho(G_{F^+(\zeta_l))}) \subset \mathcal{G}_n(\mathbb{F})$ is adequate in the sense of Definition 2.3 of [Tho12];
- the Level structure is minimal for $\bar{\rho}$;
- the representation $\bar{\rho}$ is unramified outside $\tilde{T}$;
- For each $v \in S_l$, have $\mathrm{Hom}_{G_{F,\tilde{v}}}(\bar{\rho}_{\tilde{v}}, \bar{\rho}_{\tilde{v}}\varepsilon) = 0$ for ε the cyclotomic character.

As $\bar{\rho} \cong \bar{r}_{\mathfrak{m}}$ is irreducible, via Proposition 6.4, ρ can be extended to a representation $\bar{\rho} : G_{F^+} \to \mathcal{G}_n(\mathbb{F})$ such that $\nu \circ \bar{\rho} = \epsilon^{1-n}\delta_{F:F^+}^{\mu_{\mathfrak{m}}}$, and we fix such an extension.

For each $v \in T$, define $R_v^\square$ as the framed deformation ring for $\bar{\rho}_{\tilde{v}}$. Set

$$R^{\mathrm{loc}} := (\widehat{\bigotimes}_{\mathcal{O},v\in S_l} R_v^\triangle)\widehat{\otimes}_{\mathcal{O}}(\widehat{\bigotimes}_{\mathcal{O},v\in S_D} R_v^{\square,\tilde{r}_{\tilde{v}}})\widehat{\otimes}_{\mathcal{O}}(\widehat{\bigotimes}_{\mathcal{O},v\in R} R_v^\square)$$

to be the local deformation ring for $\bar{\rho}$. Our first observation, is that since each $R_v^\triangle$ is a $\Lambda_{\tilde{v}}$-module, we notice that R^{loc} inherits the structure of a $\widehat{\bigotimes}_{v\in S_l}\Lambda_{\tilde{v}} \cong \Lambda$-module. The isomorphism $\widehat{\bigotimes}_{v\in S_l}\Lambda_{\tilde{v}} \cong \Lambda$ is inherited from the group isomorphisms

$$T_n(\mathfrak{l}) \cong \prod_{v\in S_l} T_n\mathcal{O}_{F^+,v}(l) \cong \prod_{v\in S_l} T_n\mathcal{O}_{F,\tilde{v}}(l) \cong \prod_{v\in S_l} \bar{I}_{\tilde{v}}(l)^n$$

where the final isomorphism is given by local class field theory.

Notice, that by assumption on $\bar{\rho}$ and Lemma 6.2, that $R_v^\triangle[1/l]$ is smooth. We remark that $R_v^{\square,\tilde{r}}$ is the completion of a local ring on the moduli space of rank n framed $\tilde{r}$-discrete series representations, $X_{\tilde{r}}$. Since the map $X_{\tilde{r}} \to S_{\tilde{r}}$ given by 'forgetting the framing' is smooth, and the stack $S_{\tilde{r}}[1/l]$ is smooth over L by Proposition 6.1, we see that $\mathcal{O}_{X_{\tilde{r}},\bar{\rho}}[1/l]$ is regular, and hence, by an application of Lemma 2.10, we see that $R_v^{\square,\tilde{r}}[1/l]$ is regular.

Since the Level U is minimal, for ρ we have further, that $R_v^\square$ is regular for each $v \in R$. Hence, by Corollary 2.11, $R^{\mathrm{loc}}[1/l]$ is regular.

Let $\mathcal{S}$ be the following tuple

$$\mathcal{S} = (F : F^+, T, \tilde{T}, \epsilon^{1-n}\delta_{F/F^+}^{\mu_m}, \{R_v^{\triangle,ar} : v \in S_l\}, \{R_v^{\square,st} : v \in S_D\}, \{R_v^{\square} : v \in R\})$$

and say that $\rho : G_{F^+} \to \mathcal{G}(A)$ is a lifting of $\bar{\rho}$ to $A \in \mathcal{C}_\Lambda$ of type $\mathcal{S}$ if:

1. $\rho|_{G_F}$ lifts $\bar{r}_m$;
2. ρ is unramified outside T;
3. For $v \in S_D$, ρ_v is $\tilde{r}$-discrete series and gives rise to the morphism $R_v^{\square} \to A$ which factors through $R_v^{\square,\tilde{r}}$;
4. For $v \in S_l$, the restriction ρ_v and the Λ-structure on A give a morphism $R_v^{\square} \otimes \Lambda \to A$ which factors through $R_v^{\triangle}$;
5. $\nu \circ \rho = \epsilon^{1-n}\delta_{F/F^+}^{\mu_m}$.

By Proposition 2.2.9 of [CHT08], we can construct the universal deformation ring, $R_{\mathcal{S}}^{\mathrm{univ}}$, and the universal lifting ring $R_{\mathcal{S}}^{\square}$.

Let $h_0 = [F^+ : \mathbb{Q}]\frac{n(n-1)}{2} + [F^+ : \mathbb{Q}]\frac{n(1-(-1)^{\mu_{\mathfrak{m}}-1})}{2}$, and let h be an integer larger than both h_0 and $\dim[H^1_{\mathcal{L}^\perp}(G_{F^+,T}, \mathrm{ad}\bar{\rho}(1))]$. (Here, $H^1_{\mathcal{L}^\perp}(G_{F^+,T}, \mathrm{ad}\bar{\rho}(1))$ is a particular subspace of the cohomology group $H^1(G_{F^+,T}, \mathrm{ad}\bar{\rho}(1))$ of the Galois group $G_{F^+,T}$ of the maximal extension of F^+ unramified outside of T, defined in Proposition 4.4 of [Tho12].)

After Thorne [Tho12], we will call a triple, $(Q, \tilde{Q}, \{\bar{\psi}_v\}_{v \in Q}$ a Taylor-Wiles triple if:

1. Q is a set of primes of F^+ which split in F;
2. for each $v \in Q$, $l|\mathrm{Nm}_{F^+}(v) - 1$
3. $|Q| = h$;
4. $\tilde{Q}$ is the set $\{\tilde{v}|v \in Q\}$;
5. for each $v \in Q$, $\bar{\rho}|_{G_v}$ splits as a direct sum into $\bar{s}_v \oplus \bar{\psi}_v$, with $\bar{\psi}$ the generalised eigenspace with eigenvalue $\bar{\alpha} \in \mathbb{F}$ of dimension d_v.

For any Taylor-Wiles set, Q, we can define a deformation problem $\mathcal{S}(Q)$, which is the same as $\mathcal{S}$, but in addition, we now allow $\rho_{\tilde{v}}$ for $v \in Q$ to ramify in the following way: $\rho_{\tilde{v}}$ splits as a direct sum $s \oplus \psi$, which lift $\bar{s}$ and $\bar{\psi}$ respectively, such that s is unramified, and $\psi|_{I_v} : I_v \to \mathrm{GL}_{d_v}$ factors through the scalar action on the underlying representation space. Using Proposition 2.2.9 in [CHT08] again, we can now take the universal deformation ring $R_{\mathcal{S}(Q)}^{\mathrm{univ}}$. Because stipulating that the local deformations at Taylor-Wiles primes are unramified is a closed condition, this presents us with a surjection $R_{\mathcal{S}(Q)}^{\mathrm{univ}} \twoheadrightarrow R_{\mathcal{S}}^{\mathrm{univ}}$. Further, we also have a natural map $R^{\mathrm{loc}} \to R_{\mathcal{S}(Q)}^{\mathrm{univ}}$ given by restrictions to the local subgroups at the level of functors.

Proposition 6.5. *For each $N \in \mathbb{N}$, we can find a Taylor-Wiles triple $(Q_N, \tilde{Q}_N, \{\bar{\psi}_v\}_{v\in Q})$ such that for all $v \in Q_N$, $l^N || Nm_F(v) - 1$, and the global deformation ring $R^{\text{univ}}_{\mathcal{S}(Q)}$ can be topologically generated over R^{loc} by $h - h_0$ generators.*

Proof. This follows from Lemma 4.4 of [Tho12] applied in the case of Theorem 8.6. ☐

In light of this proposition, set $R_\infty = R^{\text{loc}}[[X_1, ..., X_h]]$, $R_N = R^{\text{univ}}_{\mathcal{S}(Q_N)}$ and $R_0 = R^{\text{univ}}_{\mathcal{S}}$ so that we have surjections $R_\infty \twoheadrightarrow R_N$.

We now define some important subgroups of $G_D(\mathbb{A}^\infty_{F^+})$

Definition 6.6. *For $v \in Q_N$, suppose that $\bar{r}|_v = \bar{s} \oplus \bar{\psi}$, as before, with $\bar{\psi}$ a d_v dimensional semisimple unramified representation with all Frobenius eigenvalues equal. We take the group $U_i(\tilde{v})$ to be the subgroup of U_v of elements that take the form*

$$\begin{pmatrix} \varpi_{\tilde{v}}* & * \\ 0 & aI_{d_v} \end{pmatrix} (mod \tilde{v})$$

with $a = 1$ when $i = 1$, and arbitrary when $i = 0$. Set $U_i(Q) = U^v \times \prod_{v\in Q} U_i(\tilde{v})$

Set Δ_N be the maximal l-power quotient of $U_0(Q_N)/U_1(Q_N) \cong \prod_{v\in Q_N} k(\tilde{v})^\times$. We may view Δ_N as the maximal l-quotient of $\prod_{v\in Q_N} k(\tilde{v})^\times \cong (\mathbb{Z}/l^N)^q$. We claim there is an action of Δ_N on the ring $R^{\text{univ}}_{\mathcal{S}(Q)}$. The map, $\det \circ r^{\text{univ}}_N : I_{F,\tilde{v}} \to (R^{\text{univ}}_{\mathcal{S}(Q)})^\times$, given by the determinant of the universal deformation $r^{\text{univ}}_N := r^{\text{univ}}_{\mathcal{S}(Q_N),\bar{\rho}}$, factors through the kernel of $(R^{\text{univ}}_{\mathcal{S}(Q)})^\times \to \mathbb{F}^\times$, which is an abelian l-power group. By local class field theory, there is an isomorphism $I^{\text{ab}}_{F,\tilde{v}} \to \mathcal{O}^\times_{F,\tilde{v}}$, and the l-power quotient of this group is the l-power quotient of $k(\tilde{v})^\times$. We hence see that there is a map $\Delta_N \to (R^{\text{univ}}_{\mathcal{S}(Q_N)})^\times$ and thus a ring map $\Lambda[\Delta_N] \to R^{\text{univ}}_{\mathcal{S}(Q)}$, so that $R^{\text{univ}}_{\mathcal{S}(Q_N)}$ inherits the structure of a finitely generated $\Lambda[\Delta_N]$-algebra. Notice that if a_N is the augmentation ideal of $\Lambda[\Delta_N]$, then $R^{\text{univ}}_{\mathcal{S}(Q_N)}/a_N$ is the ring of the universal deformation ring which parametrises Galois deformations of type $\mathcal{S}$. (These deformations are required to be unramified at places above Q_N.) Note, that by choice of Q_N, that $\Delta_N \cong (\mathbb{Z}/l^n\mathbb{Z})^h$.

As in Chapter 5, we can construct the Hecke operators $\mathbb{T}_{N,1} := \mathbb{T}^{T\cup Q_N,\text{ord}}(U_1(Q_N)(l^\infty), \mathcal{O})$ and through a map $\mathbb{T}^{T\cup Q_N,\text{ord}}(U_1(Q_N)(l^\infty), \mathcal{O}) \to \mathbb{T}^{T,\text{ord}}(U(l^\infty), \mathcal{O})$ we can lift our choice of maximal ideal $\mathfrak{m}$ to a maximal ideal $\mathfrak{m}_N \subset \mathbb{T}_{N,1}$. As in Proposition 6.4, we can construct a representation $r_{\mathfrak{m}_N} : G_{F^+} \to \mathcal{G}_n(\mathbb{T}_{N,1})$ which by the proof of Theorem 6.8 of [Tho12] gives us an $\mathcal{S}(Q_N)$-lifting of $\bar{\rho}$. Hence, we get a surjection $R^{\text{univ}}_{\mathcal{S}(Q)} \twoheadrightarrow \mathbb{T}_{N,1}$ for each N.

6.4 Patching

We now define a module H_N over $\mathbb{T}^{T\cup Q_N,\text{ord}}(U_1(Q_N)(l^\infty), \mathcal{O})_m$ for each set Q_N.

Define the space of automorphic forms $S^{\mathrm{ord}}(U_i(Q_N)(l^\infty), L/\mathcal{O})_m$ as before, and set $H_0 = S^{\mathrm{ord}}(U(l^\infty), L/\mathcal{O})_m^\vee$. In Proposition 5.9 of [Tho12], Thorne describes a projection Pr_v on $S^{\mathrm{ord}}(U_i(Q_N)(l^\infty), L/\mathcal{O})_m$, and in Theorem 6.8, modules

$$H_{i,N} := \prod_{v \in Q_N} \Pr_v[S^{\mathrm{ord}}(U_i(Q_N)(l^\infty), L/\mathcal{O})_m]^\vee$$

with the following properties:

Proposition 6.7. *[Tho12]*

1. *H_{1,Q_N} is a free $\Lambda[\Delta_{Q_N}]$-module, and restriction to $S^{ord}(U_0(Q_N)(l^\infty), L/\mathcal{O})_m$ gives an isomorphism $H_{1,Q_N}/a_N \cong H_{0,Q_N}$.*
2. *The map*
$$(\prod_{v \in Q_N} \Pr_{\tilde{v}})^\vee : H_{0,Q_N} \to H_0$$
is an isomorphism.

Theorem 6.8 (Patching)**.** *Let $R \twoheadrightarrow \mathbb{T}$ be a surjective Λ-algebra homomorphism, with $\mathbb{T}$ a finite Λ-algebra. Suppose we have the following data:*

1. *Integers $t, h \geqslant 1$;*
2. *a finite $\mathbb{T}$-module H;*
3. *$S_N = \Lambda[(\mathbb{Z}/l^n\mathbb{Z})^h] \cong \Lambda[\Delta_{Q_N}]$ with augmentation ideal $\mathfrak{a}_N$, with inverse limit $S'_\infty := \varprojlim \Lambda[\Delta_{Q_N}] \cong \Lambda[[Y_1, ..., Y_h]]$;*
4. *a ring $S_\infty = S'_\infty \widehat{\otimes}_{\mathcal{O}} \mathcal{T}$, where $\mathcal{T} = \mathcal{O}[[X_1, ..., X_{|T|n^2}]]$*
5. *For each $N \geqslant 1$ have*
   - *(a) $R_N \twoheadrightarrow \mathbb{T}_N$ are S_N-algebra homomorphisms, such that reduction modulo $\mathfrak{a}_N$ reduces the map to $R \twoheadrightarrow \mathbb{T}$.*
   - *(b) a finite $\mathbb{T}_N$-module H_N, which is finite and free over S_N, whose rank is independent of N;*
6. *An S_∞-algebra R_∞ such that $R_\infty \twoheadrightarrow R_N$ with kernel $\ker(S_\infty \to S_N)R_\infty$.*

Then there is an $R_\infty \otimes S_\infty$-module H_∞, such that

1. *$H_\infty / aH_\infty \cong H$,*
2. *H_∞ is a finite free S_∞-module.*
3. *The action of S_∞ on H_∞ factors through that of R_∞.*

Proof. The details of the Taylor-Wiles-Kisin patching method used here is essentially no different to chapter 4.3 of [Ger18]. One can also find details in chapter 8 of [Tho12], under the heading 'another patching argument'. □

Theorem 6.9. *The module $H_0[1/l]$ is a finite locally free $R^{univ}_{\mathcal{S}}[1/l]$-module.*

Proof. We calculate that $\dim(S_\infty) = \dim(\Lambda)+h+|T|n^2 = n[F^+ : \mathbb{Q}]n+h+|T|n^2$, and that

$$\begin{aligned}\dim(R_\infty) &= 1 + \sum_{v \in S_l}([F_{\tilde{v}} : \mathbb{Q}_l]\frac{n(n+1)}{2} + n^2) + n^2|S_D \cup R| + h - h_0 \\ &= [F^+ : \mathbb{Q}]\frac{n(n+1)}{2} + |T|n^2 + h - h_0 \\ &= [F^+ : \mathbb{Q}]n + |T|n^2 + h - [F^+ : \mathbb{Q}]\frac{n(1-(-1)^{\mu_{\mathfrak{m}}-n})}{2}\end{aligned}$$

Consider the module $H^{\square}_\infty$. Since $H^{\square}_\infty$ is a finite free S_∞ module, and that the action of S_∞ factors through R_∞ we see that

$$\dim(S_\infty) = \operatorname{depth}_{S_\infty}(H^{\square}_\infty) \leqslant \operatorname{depth}_{R_\infty}(H^{\square}_\infty) \leqslant \dim(R_\infty)$$

and thus, the only possible way for this inequality to hold is if equality holds throughout, and $\mu_m \equiv n \bmod 2$, and $H^{\square}_\infty$ is a maximal Cohen-Macaulay R_∞ module.

Now, consider the generic fibre. Let $m \subseteq R_\infty[1/l]$ be a maximal ideal. Since localisation commutes with tensor products, we see that

$$\left(\bigotimes_{\mathcal{O}, v \in T} R_v\right)[1/l] \cong \bigotimes_{L, v \in T}(R_v[1/l]).$$

By Lemma 2.10, we see that

$$R_\infty[1/l]^\wedge_m = \left(\widehat{\bigotimes}_{\mathcal{O}, v\in T} R_v\right)[1/l]^\wedge_m \cong \left(\bigotimes_{\mathcal{O}, v \in T} R_v\right)[1/l]^\wedge_m$$

and so we see that $R_\infty[1/l]^\wedge_m$ is a power series ring tensor product of formally smooth rings. Since it is formally smooth, any finitely generated $R_\infty[1/l]_m$-module has finite projective dimension, and by the Auslander Buchsbaum formula, is projective. This shows that $H^{\square}_\infty[1/l]_m$ is a free $R_\infty[1/l]_m$-module, this shows that $H^{\square}_\infty[1/l]$ is a locally finite free $R_\infty[1/l]$-module. It follows that $H_0[1/l]$ is a locally finite free $R^{univ}_{\mathcal{S}}[1/l]$-module. □

Corollary 6.10. $R^{univ}_{\mathcal{S}}[1/l] = \mathbb{T}[1/l]$.

Proof. Let I be the kernel of the surjection $R^{univ}_{\mathcal{S}}[1/l] \to \mathbb{T}[1/l]$. Choose any maximal ideal m of $R^{univ}_{\mathcal{S}}[1/l]$. Since localisation is an exact functor, we get a short exact sequence

$$0 \to I_m \to R^{univ}_{\mathcal{S}}[1/l]_m \to \mathbb{T}[1/l]_m \to 0.$$

Note that the action of $R^{univ}_{\mathcal{S}}[1/l]_m$ on $H_0[1/l]_m$ factors through $\mathbb{T}[1/l]_m$, so that I_m annihilates all of $H_0[1/l]_m$. Since this is a free module, this shows that I_m is trivial. Since this is true for every m, this shows that $\operatorname{Supp}(I) = \varnothing$ and hence $I = 0$. Hence the surjection above is an isomorphism $R^{univ}_{\mathcal{S}}[1/l] \cong \mathbb{T}[1/l]$. □

Remark. We finally want to remark on an application of Theorem 6.9. Whenever M is a locally free coherent sheaf on a connected space X, the rank function

$$\begin{aligned} X &\to \mathbb{N} \cup \{0\} \\ x &\mapsto \mathrm{Rank}_x(M) \end{aligned}$$

is locally constant. Therefore, the rank of a geometrically connected component can be calculated by calculating the rank at any special point $x \in X$. In our special case, the rank of the module $H_0[1/l]$ can be interpreted as the number of distinct automorphic forms with a given set of Hecke eigenvalues. Which can again, be interpreted as the multiplicity of the Galois representation determined by said Hecke eigenvalues inside the space of automorphic forms. We have shown that for these automorphic forms, the multiplicity is determined only by the connected component that the representation $\rho_{\mathfrak{m}}$ lies on. By Lemma 4.2 of [Ger18], we see that the minimal primes of $R_\infty[1/l]$ biject with the minimal primes of Λ, and thus we have a bijection with those of $R_{\mathcal{S}}^{univ}[1/l]$. Thus, if one could show that for each component of SpecΛ, there is an automorphic form of some classical weight had multiplicity 1, then all the Hida families of forms would also have multiplicity 1. Thus these results have an application to 'multiplicity problems'.

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