Linear Algebra 1, Problem Sheet 1.
Epiphany 21/22.

1. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation defined by

$$T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x+y+z \\ -2y-2z \\ y+z \end{pmatrix}.$$

Find the matrix which represents T with respect to:

(a) the standard basis in both copies of $\mathbb{R}^3$;

(b) the ordered basis consisting of

$$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}$$

in both copies of $\mathbb{R}^3$.

2. * Find the characteristic polynomial of the matrix

$$M = \begin{pmatrix} A & C \\ \mathbf{0} & B \end{pmatrix},$$

where A denotes a $n \times n$ matrix, B a $m \times m$ matrix, C a $n \times m$ matrix, and $\mathbf{0}$ denotes the zero matrix of the appropriate dimensions, in this case $m \times n$.

3. Find the characteristic polynomials, the eigenvalues and the eigenspaces of the following matrices

$$\text{(i)} \begin{pmatrix} 7 & -4 \\ -8 & -7 \end{pmatrix}, \quad \text{(ii)} \begin{pmatrix} 5 & 2 & 3 \\ -13 & -6 & -11 \\ 4 & 2 & 4 \end{pmatrix}, \quad \text{(iii)} \begin{pmatrix} 3 & 1 & 1 \\ -15 & -5 & -5 \\ 6 & 2 & 2 \end{pmatrix}.$$

4. Show that each of the following matrices is similar to a diagonal matrix.

$$\text{(i)} \begin{pmatrix} 6 & 2 & 3 \\ -13 & -5 & -11 \\ 4 & 2 & 5 \end{pmatrix}; \quad \text{(ii)} \begin{pmatrix} 1+i & 4+4i \\ 0 & -1-i \end{pmatrix}; \quad \text{(iii)} \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

In each case write down an appropriate matrix which can be used to convert the given matrix to diagonal form.

5. Prove that, if λ is an eigenvalue of the linear transformation $T : V \to V$, then λ^k is an eigenvalue of T^k for each integer $k > 0$. Prove also that if $p(t)$ is a polynomial, then $p(\lambda)$ is an eigenvalue of $p(T)$.

6. Show that the linear transformation $S : \mathbb{R}[x]_n \to \mathbb{R}[x]_n$ given by $S\big(p(x)\big) = \frac{1}{x}\int_0^x p(y)\,dy$ is diagonalizable and find the eigenvalues and eigenvectors.

7. If A is a 2×2 matrix with characteristic polynomial $p_A(x) = c_0 + c_1 x + x^2$, show that $p_A(x) = x^2 - \mathrm{tr}(A)x + \det(A)$.

8. If A is a 3×3 matrix with characteristic polynomial $p_A(x) = c_0 + c_1 x + c_2 x^2 - x^3$, show that $p_A(x) = -x^3 + \mathrm{tr}(A)x^2 + \frac{\left(\mathrm{tr}(A^2) - (\mathrm{tr}A)^2\right)}{2}x + \det(A)$.

9. Find the characteristic equation for

$$B = \begin{pmatrix} 0 & 2 & 6 \\ 2 & -8 & -26 \\ -2 & 2 & 8 \end{pmatrix}$$

and hence show that $B^9 = 2^8 B$. [Hint: use Cayley-Hamilton.]

10. Which of the matrices

$$\begin{pmatrix} 11 & 5 & 8 \\ -30 & -16 & -30 \\ 9 & 5 & 10 \end{pmatrix}, \qquad \begin{pmatrix} -2 & -1 & -1 \\ 9 & 4 & 3 \\ -3 & -1 & 0 \end{pmatrix}, \qquad \begin{pmatrix} -4 & -1 & 1 \\ 17 & 4 & -5 \\ -5 & -1 & 2 \end{pmatrix}$$

are similar to diagonal matrices?

11. Prove that

$$\begin{pmatrix} 0 & -4 \\ -4 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} -1 & -5 \\ -3 & 1 \end{pmatrix}$$

are similar to each other.

12. Find two matrices $A, B \in M_n(\mathbb{R})$ with same characteristic polynomial, same eigenvalues, same determinant, same trace but not similar to one another.

13. Which of the matrices

$$\begin{pmatrix} -9 & -3 & -1 \\ 35 & 11 & 1 \\ -11 & -3 & 1 \end{pmatrix}, \qquad \begin{pmatrix} -1 & 2 & 0 \\ -1 & 1 & 0 \\ 4 & -2 & 1 \end{pmatrix}$$

are similar to diagonal matrices when we take (a) $\mathbb{R}$, (b) $\mathbb{C}$ as the underlying field of scalars?

14. Given

$$A = \begin{pmatrix} -6 & 8 \\ -4 & 6 \end{pmatrix},$$

find P such that $P^{-1}AP$ is diagonal. Hence compute A^6.

15. Let A be an $n \times n$ complex matrix. Suppose that A has only one eigenvalue. Prove that if A is not of the form aI, then A is not similar to a diagonal matrix.

16. Given the matrix

$$A = \begin{pmatrix} -9 & -8 & -15 \\ 37 & 32 & 59 \\ -10 & -8 & -14 \end{pmatrix}.$$

find a matrix B such that $B^3 = A$. Is this B unique?

17. Solve the system of first-order differential equations

$$\dot{x}_1 = -3x_1 - 2x_2 - 6x_3, \quad \dot{x}_2 = -8x_1 - 3x_2 - 12x_3, \quad \dot{x}_3 = 5x_1 + 2x_2 + 8x_3,$$

subject to the initial conditions

$$x_1(0) = 1, \quad x_2(0) = \frac{3}{2}, \quad x_3(0) = -1.$$

18. Find the eigenvalues and eigenvectors of the linear transformation $T : \mathbb{R}[x]_4 \to \mathbb{R}[x]_4$ defined by $T(p(x)) = x^2\big(p(x+1) - 2p(x) + p(x-1)\big)$.

19. Define $T : M_n(\mathbb{R}) \to M_n(\mathbb{R})$ by $T(A) = A^t$. Prove that T has only two distinct eigenvalues, and that its eigenvectors span $M_n(\mathbb{R})$. Here $M_n(\mathbb{R})$ denotes the set of $n \times n$ real matrices, and A^t denotes the transpose of A.

20. Let S and T be linear transformations from an n-dimensional vector space to itself, and assume that $ST = TS$. If $\mathbf{v}$ is an eigenvector of S, and if $T(\mathbf{v})$ is not the zero vector, show that $T(\mathbf{v})$ is also an eigenvector of S. Hence show that if S has n distinct eigenvalues, then S and T have the same eigenvectors.

21. Compute the characteristic polynomial of

$$A = \begin{pmatrix} 16 & 14 & 10 \\ -9 & -9 & -5 \\ -2 & 13 & -9 \end{pmatrix},$$

and deduce that $A^4 = 16\,\mathbf{I}$. [Hint: Cayley-Hamilton.]

22. Find the general (real) solution of the system of first-order differential equations

$$\dot{x}_1 = 5x_1 + 5x_2, \quad \dot{x}_2 = -5x_1 - x_2.$$

23. * Let C be an element of $M_n(\mathbb{R})$ of the form:

$$C = \begin{pmatrix} c_0 & c_1 & c_2 & \cdots & c_{n-1} \\ c_{n-1} & c_0 & c_1 & \cdots & c_{n-2} \\ \vdots & & \ddots & & \vdots \\ c_1 & c_2 & \cdots & c_{n-1} & c_0 \end{pmatrix}$$

where each row is a just cyclic shift of the row above it. In a succint way we can write the entry $C_{ij} = c_{(i-j) \bmod n}$ where mod denotes the remainder of the integer division modulo n. This is called a *circulant* matrix and it's a particular type of a particular set of matrices called *Toeplitz* matrices. Show that if ω is an n-th root of unity, i.e. $\omega^n = 1$, then the vector $\mathbf{v} = (\omega^{\mathbf{0}}, \omega^{\mathbf{1}}, ..., \omega^{\mathbf{n-1}})^{\mathbf{t}}$ is an eigenvector of C and compute its corresponding eigenvalue.

24. * The *minimal polynomial* of a square matrix A is the polynomial $q(t) = t^m + a_1 t^{m-1} + \ldots + a_m$ of least degree such that $A^m + a_1 A^{m-1} + \ldots + a_m I = 0$. Show that if $A \in M_n(\mathbb{R})$, then the subset $\mathbb{R}[A]$ of $M_n(\mathbb{R})$ consisting of polynomials in A is a vector subspace whose dimension is the degree of the minimal polynomial of A. Find the dimension of $\mathbb{R}[A]$ when

$$A = \begin{pmatrix} 3 & 1 & -2 \\ -2 & 1 & 2 \\ 1 & 1 & 0 \end{pmatrix}.$$

25. (a) Let D be a diagonal 3×3 matrix with distinct entries. Prove that *every* diagonal 3×3 matrix can be expressed as a linear combination of I, D, D^2.

    (b) Prove that for $P \in GL_n(\mathbb{R})$ (that is P an invertible $n \times n$ real matrix) the map $A \mapsto P^{-1}AP$ defines a nonsingular linear transformation from $M_3(\mathbb{R})$ to itself.

    (c) Prove that if $A \in M_3(\mathbb{R})$ has distinct real eigenvalues, then the set $\mathbb{R}[A]$ of all polynomials in A is a 3-dimensional subspace of $M_3(\mathbb{R})$.

26. Decide which of the following bilinear functions defines an inner product:

    (i) $x_1y_1 - x_1y_3 - x_3y_1 + 2x_3y_3 + 4x_2y_2 + x_4y_4 + x_2y_4 + x_4y_2$ on $\mathbb{R}^4$;

    (ii) $2x_1y_1 + x_2y_2 + 2x_3y_2 + x_2y_3$ on $\mathbb{R}^3$;

    (iii) $2x_1y_1 + x_2y_2 - 2x_1y_3 - 2x_3y_1 - x_2y_3 - x_3y_2 + x_3y_3$ on $\mathbb{R}^3$;

    (iv) $4x_1y_1 + 2x_2y_2 + 6x_2y_3 + 6x_3y_2 + 18x_3y_3$ on $\mathbb{R}^3$;

    (v) $x_1y_1 + x_2y_2 - x_1y_3 - x_3y_1 + 3x_2y_3 + 3x_3y_2 + 11x_3y_3$ on $\mathbb{R}^3$.

27. Show that the bilinear form on $\mathbb{R}^3$ defined by

$$(\mathbf{x}, \mathbf{y}) = 6x_1y_1 - x_1y_2 - x_2y_1 + x_2y_2 - x_1y_3 - x_3y_1 + x_3y_3$$

is an inner product on $\mathbb{R}^3$, and find the lengths of the vectors

$$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 3 \\ -6 \\ 3 + 3\sqrt{3} \end{pmatrix}$$

and the angle between them with respect to this inner product.

28. Find the angle between the vectors in $\mathbb{R}^4$ equipped with the standard inner product:

$$\text{(i)} \quad \begin{pmatrix} 1 \\ 2 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ -1 \\ 1 \end{pmatrix}; \qquad \text{(ii)} \quad \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}, \begin{pmatrix} 8 \\ -4 \\ -4 \\ 3 \end{pmatrix}; \qquad \text{(iii)} \quad \begin{pmatrix} 6 \\ 2 \\ -2 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ -1 \\ 3 \end{pmatrix}.$$

29. * Consider the vector space M_n of the $n \times n$ matrices with real coefficients, and the application $(\ ,\ ) : M_n \times M_n \mapsto \mathbb{R}$ given by

$$(A, B) = \mathrm{Tr}(A^t B),$$

where A^t denotes the transpose of A and Tr denotes the trace. Show that $(\ ,\ )$ defines an inner product on M_n.

30. Suppose that $\mathbb{C}^3$ is equipped with the standard inner product. Show that the vectors

$$\frac{1}{2}\begin{pmatrix} i \\ i \\ 1+i \end{pmatrix}, \quad \frac{1}{6}\begin{pmatrix} 3+3i \\ 1+i \\ -4 \end{pmatrix}$$

are mutually orthogonal unit vectors, and find an orthonormal basis for $\mathbb{C}^3$ which contains them.

31. Decide which of the following defines a Hermitian inner product on $\mathbb{C}^2$:

   (i) $3z_1\bar{w}_1 + 4z_2\bar{w}_2$;

   (ii) $z_1\bar{w}_2 + z_2\bar{w}_1$;

   (iii) $z_1\bar{w}_1 + (1+i)z_2\bar{w}_2$;

   (iv) $z_1\bar{w}_1 + z_2\bar{w}_2 + z_1w_2$.

32. Show that $z_1\bar{w}_1 + 2z_2\bar{w}_2 + \frac{1+i}{\sqrt{2}}z_1\bar{w}_2 + \frac{1-i}{\sqrt{2}}z_2\bar{w}_1$ defines a hermitian inner product on $\mathbb{C}^2$. Using this inner product, find the norm of the vector

$$\mathbf{u} = \begin{pmatrix} -1 \\ \sqrt{2}i \end{pmatrix},$$

and determine all unit vectors which are orthogonal to it.

33. If the vector space $C[-\pi,\pi]$ of continuous complex valued functions on the interval $[-\pi,\pi]$ is equipped with the inner product defined by

$$(f,g) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,\overline{g(x)}\,dx,$$

where $\overline{g(x)}$ denotes the complex conjugate of $g(x)$, show that

$$e^{inx}$$

with $n \in \mathbb{N}$, i.e. $n = 0, 1, 2...$, are mutually orthogonal unit vectors in $C[-\pi,\pi]$.

Note: Starred, e.g. 1.*, exercises are more advanced/complicated.