Linear Algebra 1, Problem Sheet 2.
Epiphany 21/22.

34. * Consider $V = \mathbb{R}^n$ with the following application

$$||\mathbf{v}||_\infty = \max_{1\leq i\leq n} |v_i|\,,$$

where $\mathbf{v} = (v_1, ..., v_n)$. Prove that $||\cdot||_\infty$ defines a norm on V. [This is called the ∞-norm, also called sup-norm, on V and it is not induced by an inner product.]

35. * Consider $V = \mathbb{R}^n$ with the following application

$$||\mathbf{v}||_1 = \sum_{i=1}^{n} |v_i|\,,$$

where $\mathbf{v} = (v_1, ..., v_n)$. Prove that $||\cdot||_1$ defines a norm on V. [This is called the ℓ_1-norm on V and it is not induced by an inner product.]

36. * Consider the vector space $V = C[a, b]$ of continuous functions on the interval $[a, b]$ with $-\infty < a < b < \infty$, and consider the application

$$||f||_1 = \int_a^b dx\,|f(x)|\,,$$

where $f \in V$. Prove that $||\cdot||_1$ defines a norm on V. [This is called the L_1-norm on V and it is not induced by an inner product.]

37. Apply Gram-Schmidt orthonormalisation to the basis $\left\{ \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}1\\2\\0\end{pmatrix}, \begin{pmatrix}1\\2\\3\end{pmatrix} \right\}$ of $\mathbb{R}^3$ equipped with the standard inner product. (But first guess the answer.)

38. Apply Gram-Schmidt orthonormalisation to the basis $\left\{ \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix} \right\}$ of $\mathbb{R}^3$ equipped with the inner product defined by $(\mathbf{x}, \mathbf{y}) = 2x_1y_1 + 2x_2y_2 + x_3y_3 - x_2y_3 - x_3y_2$.

39. If $\mathbb{R}^4$ is given the standard inner product, find an orthonormal basis for the subspace determined by the equation $x_1+x_2+x_3+x_4 = 0$, and extend this basis to an orthonormal basis for all of $\mathbb{R}^4$.

40. If $\mathbb{R}^4$ is given the standard inner product, find an orthonormal basis for the subspace determined by the equation $x_1+x_2-x_3-x_4 = 0$, and extend this basis to an orthonormal basis for all of $\mathbb{R}^4$.

41. * Let $(\ ,\ ) : V \times V \mapsto \mathbb{R}$ be an inner product on the n-dimensional vector space V and let U, W denote two vector subspaces of V. Prove the following

   (i) $W = W^{\perp\perp}$

   (ii) $U^\perp \cap W^\perp = (U + W)^\perp$

   (iii) $(U \cap W)^\perp = U^\perp + W^\perp$

42. Let $V = \mathbb{R}[t]_2$ be equipped with the inner product

$$(p, q) = \int_0^1 p(t)q(t)\, dt.$$

Use the Gram-Schmidt process to convert $\{1, t, t^2\}$ into an orthonormal basis $\{g_1, g_2, g_3\}$ for V.

43. Let $V = \mathbb{R}[t]_2$ be equipped with the inner product

$$(f, g) = \int_{-1}^1 f(t)g(t)\, dt,$$

and let $U = \{f \in V \mid f(-1) = f(1) = 0\}$. Find a basis for the orthogonal complement of U in V.

44. Consider $\mathbb{C}^4$ with the standard inner product. Find an orthonormal basis for the orthogonal complement of the subspace spanned by

$$\begin{pmatrix} 2 \\ 1 - \mathrm{i} \\ 0 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 0 \\ \mathrm{i} \\ 3 \end{pmatrix}.$$

45. Use the Gram-Schmidt process to show that every invertible $n \times n$ matrix A can be written in the form $A = BC$, where B is an orthogonal matrix and C is upper triangular. Find B, C when

$$A = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 2 & 3 \\ -1 & 1 & 3 \end{pmatrix}.$$

[Hint: Think about the columns of A as vectors.]

46. Let S consist of the following vectors in $\mathbb{R}^4$ with its standard inner product:

$$\mathbf{u}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \quad \mathbf{u}_2 = \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}, \quad \mathbf{u}_3 = \begin{pmatrix} -1 \\ -1 \\ 1 \\ 1 \end{pmatrix}, \quad \mathbf{u}_4 = \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix}.$$

   (a) Show that these vectors are all mutually orthogonal to each others, and that they form a basis of $\mathbb{R}^4$;

   (b) Write $\mathbf{w} = \big(6, 5, 3, 1\big)$ as a linear combination of $\mathbf{u}_1$, $\mathbf{u}_2$, $\mathbf{u}_3$, $\mathbf{u}_4$.

47. Let U be the vector subspace of $\mathbb{R}^4$ defined by

$$x_1 + x_2 + x_3 + x_4 = 0, \qquad x_1 - x_2 + x_3 - x_4 = 0.$$

Find orthonormal bases for U and its orthogonal complement, when $\mathbb{R}^4$ is equipped with the standard inner product.

48. In $\mathbb{R}^4$ equipped with the standard inner product, find the projection of $\mathbf{a} = (1, 2, 0, -1)$ on the plane V spanned by $\mathbf{v}_1 = (1, 0, 0, 1)$ and $\mathbf{v}_2 = (1, 1, 2, 0)$. (First construct an orthonormal basis for V.)

49. Let U be a vector subspace of $\mathbb{R}^n$, equipped with the standard inner product, and suppose that $\mathbf{v}$ is an element of $\mathbb{R}^n$ not in U. Then we know that there is a unique point $\mathbf{u}_0$ in U such that, for all $\mathbf{u} \in U$, we have $\|\mathbf{v} - \mathbf{u}_0\| \leq \|\mathbf{v} - \mathbf{u}\|$; and $\mathbf{v} - \mathbf{u}_0$ is orthogonal to U. Find $\mathbf{u}_0$ if U is the plane $x_1 - 2x_2 + 2x_3 = 0$ in $\mathbb{R}^3$ and $\mathbf{v} = \big(1, 0, 0\big)$.

50. Let V be the space $C[-1, 1]$ equipped with the inner product $(f, g) = \int_{-1}^{1} f(t)g(t)\, dt$. Let S be the subspace of V spanned by $\{1, t, t^2\}$. Construct an orthonormal basis $\{g_1, g_2, g_3\}$ for S, and find the function $h \in S$ closest to t^3.

51. Find the point in the 3-plane $2x_1 - x_2 + 2x_3 + 2x_4 = 0$ in $\mathbb{R}^4$, with standard Euclidean inner product, which is nearest to the point $\mathbf{a} = (1, 2, 1, 2)$.

52. Find the point in the 2-plane in $\mathbb{R}^4$ defined by $x_1 + x_2 + x_3 + x_4 = 0$, $x_1 - x_2 + x_3 - x_4 = 0$, which is nearest to the point $\mathbf{v} = (1, 2, 1, 2)$ with standard Euclidean inner product.

53. If the vector space $C[-1, 1]$ of continuous real valued functions on the interval $[-1, 1]$ is equipped with the inner product defined by $(f, g) = \int_{-1}^{1} f(t)g(t)\, dt$, find the linear polynomial $g(t)$ nearest to $f(t) = e^t$.

54. Find an orthogonal matrix P such that $P^t AP$ is diagonal, when

$$\text{(i)} \quad A = \begin{pmatrix} 11 & 8 \\ 8 & -1 \end{pmatrix}, \qquad \text{(ii)} \quad A = \begin{pmatrix} 1 & 0 & -4 \\ 0 & 5 & 4 \\ -4 & 4 & 3 \end{pmatrix}, \qquad \text{(iii)} \quad A = \begin{pmatrix} 5 & 7 & 7 \\ 7 & 5 & -7 \\ 7 & -7 & 5 \end{pmatrix}.$$

55. (i) Let A be a real symmetric matrix. Show that there exists a real symmetric matrix B such that $B^2 = A$ if and only if the eigenvalues of A are all non-negative.

(ii) Find a real symmetric matrix C such that

$$C^5 = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

Some additional starred exercises

56. * Let V be an n-dimensional vector space over the reals and W a subspace of V with dimension $m \leq n$. Consider the set of linear transformations

$$U = \{T : V \mapsto V \,\text{s.t.}\, T \,\text{is linear and}\, \forall \mathbf{w} \in W \,\exists\, \alpha \in \mathbb{R} : \, T(\mathbf{w}) = \alpha \mathbf{w}\}.$$

Show that U is a vector subspace of $M_n(\mathbb{R})$ and compute its dimension.

57. * Let V be a real vector space with dimension n and $T : V \mapsto V$ a linear transformation.

i) If $T^2 = 0$ show that $\dim \operatorname{Ker} T \geq \dim V/2$.

ii) Show that $T^2 = 0$ and $\dim \operatorname{Ker} T = n/2$ and $\dim V = n$ is even if and only if $\operatorname{Ker} T = \operatorname{Im} T$.

58. * [Nilpotency] A square matrix N is said to be nilpotent if $N^k = 0$ for some positive integer $k \in \mathbb{N}$. The smallest such k such that $N^{k-1} \neq 0$ but $N^k = 0$ is called the degree of nilpotency of N. Show that if N is nilpotent with degree k, then the matrix $A = I + N$ is invertible and its inverse is given by

$$A^{-1} = I - N + N^2 - N^3 + ... + (-1)^{k-1} N^{k-1}.$$

59. * Show that if N is a nilpotent matrix and λ is an eigenvalue of N with eigenvector $\mathbf{v} \neq \mathbf{0}$ then necessarily $\lambda = 0$. In particular deduce that the characteristic polynomial of every $n \times n$ nilpotent matrix N is $p_N(t) = (-t)^n$. [i.e. a nilpotent matrix has only vanishing eigenvalues]

60. * Show that if N is nilpotent than $\det(I + N) = 1$. Viceversa if N is a matrix such that $\det(I + xN) = 1$ for every x then show that N is nilpotent. [Hint: use the previous exercise].

61. * [Quadratic forms] Let $V = \mathbb{R}^2$ with a bilinear form $Q(\mathbf{v}, \mathbf{w})$ which we assume symmetric, i.e. $Q(\mathbf{v}, \mathbf{w}) = Q(\mathbf{w}, \mathbf{v})$, but not necessarily positive definite. The function $\phi_Q : V \mapsto \mathbb{R}$ defined by $\phi_Q(\mathbf{v}) = Q(\mathbf{v}, \mathbf{v})$ is called the (associated) *quadratic form*, note: it is called quadratic because $\phi_Q(\lambda \mathbf{v}) = \lambda^2 \phi_Q(\mathbf{v})$. Show that in terms of the coordinates $\mathbf{v} = (x, y)^t$, the set of points satisfying $\phi_Q(\mathbf{v}) = Q(\mathbf{v}, \mathbf{v}) = 1$ is either describing an ellipse, an hyperbola, two parallel lines or the empty set.

62. * [Dual space] Let V be an n-dimensional real vector space and consider the space $V^* = \{\phi : V \mapsto \mathbb{R}\,, \text{s.t. } \phi \text{ is linear}\}$. Show that V^* is a real vector space called the dual space of V. Show that if $\{\mathbf{v_1}, ..., \mathbf{v_n}\}$ is a basis for V then the set of $\phi^{(i)} \in V^*$, $i = 1, ..., n$, defined by $\phi^{(i)}(\mathbf{v_j}) = \delta^i_j$ span a basis for V^* called the dual basis, where δ^i_j is the Kronecker delta, so that V^* has exactly the same dimension as V.

63. * Consider a real n-dimensional inner product space $\{V, (\cdot, \cdot)\}$. Show that for every vector $\mathbf{v} \in V$ we can construct the application $\phi_{\mathbf{v}} : V \mapsto \mathbb{R}$ defined by $\phi_{\mathbf{v}}(\mathbf{w}) = (\mathbf{w}, \mathbf{v})$. Prove that $\phi_{\mathbf{v}} \in V^*$. [This is telling you that V and V^* are isomorphic, however this is not a *natural* isomorphism in the sense that it dependes on your choice of inner product.]

64. * Consider a real n-dimensional vector space V, its dual V^* and the double-dual

$$V^{**} = \{\Phi : V^* \mapsto \mathbb{R}\,, \text{ s.t. } \Phi \text{ is linear}\}\,.$$

Show that for every vector $\mathbf{v} \in V$, the application $\Phi_{\mathbf{v}} : V^* \mapsto \mathbb{R}$ defined by $\Phi_{\mathbf{v}}(\phi) = \phi(\mathbf{v})$, for every $\phi \in V^*$, is an element of V^{**}, i.e. $\Phi_{\mathbf{v}} \in V^{**}$. [This is telling you that there is a natural isomorphism between V and V^{**} given by evaluation on a specific vector.]

Note: Starred, e.g. 1.*, exercises are more advanced/complicated.