Linear Algebra 1, Problem Sheet 3.
Epiphany 21/22.

65. If A is a real $n \times n$ matrix, show that A is skew-symmetric (anti-symmetric) if and only if $\mathbf{x}^t A\mathbf{x} = 0$ for all $\mathbf{x} \in \mathbb{R}^n$.

66. Let M_n be the vector space of $n \times n$ matrices with real coefficients. Show that $M_n = \mathrm{Skew}_n \oplus \mathrm{Sym}_n$ where $\mathrm{Skew}_n = \{A \in M_n \mid A^t = -A\}$ and $\mathrm{Sym}_n = \{A \in M_n \mid A^t = A\}$. What are the dimensions of M_n, Skew_n and Sym_n as vector spaces over the field of real numbers?

67. * Consider the vector space M_n of $n \times n$ matrices with real coefficients with the inner product $(A, B) = \mathrm{Tr}(A^t B)$. Find the orthogonal complement to the vector subspace Sym_n with respect to this inner product.

68. * Let $\sigma \in S_n$ denote a permutation of n elements, the matrix R_σ, with respect to the standard basis $\{v_1, ..., v_n\}$ of $\mathbb{R}^n$, is associated with the linear transformation that permutes the basis vector with σ, i.e. $R_\sigma v_i = v_{\sigma(i)}$. Prove that R_σ is orthogonal. [HINT: How can you write $(R_\sigma)^{-1}$ in terms of another permutation?]

69. If A is a complex $n \times n$ matrix, show that A is Hermitian if and only if $\mathbf{x}^* A\mathbf{x}$ is real for all $\mathbf{x} \in \mathbb{C}^n$.

70. Find a unitary matrix P such that $P^* AP$ is diagonal when

$$A = \begin{pmatrix} 2 & 1+i \\ 1-i & 3 \end{pmatrix}.$$

71. Show that the determinant of a unitary matrix is of unit modulus.

72. A unitary matrix of determinant $+1$ is *special unitary.* Show that every unitary matrix A can be written in the form $A = kB$ where $k \in \mathbb{C}$ is of unit modulus and B is special unitary.

73. Show that every special unitary 2×2 matrix is of the form

$$\begin{pmatrix} a & -\bar{c} \\ c & \bar{a} \end{pmatrix},$$

with $a, c \in \mathbb{C}$ and $a\bar{a} + c\bar{c} = 1$.

74. Show that, if n is odd, every real orthogonal $n \times n$ matrix A has $\det(A)$ as an eigenvalue. (Note that, for any real orthogonal matrix A, $\det(A) = \pm 1$).

75. Identify the polynomial $f(x) = c_0 + c_1 x + \ldots + c_{n-1}x^{n-1} + x^n$ for which the integral $\int_{-1}^{1} f(x)^2\, dx$ has the smallest value. (Hint: Consider $f(x)$ as a linear combination of Legendre polynomials $P_0(x), \ldots, P_n(x)$, taking P_k to be normalized by $P_k(x) = x^k + \ldots$.

76. (a) Verify by direct computation that the Laguerre operator

$$\mathcal{L}_l = x\frac{d^2}{dx^2} + (1-x)\frac{d}{dx}$$

on the space $\mathbb{R}[x]$ of polynomials in x is symmetric with respect to the inner product given by the formula $(f, g) = \int_0^{+\infty} f(x)g(x)e^{-x}dx$.

(b) Find the matrix and the characteristic polynomial of the Laguerre operator $\mathcal{L}_l$ on the space $\mathbb{R}[x]_2$ (use the basis $\{1,\ x,\ x^2\}$).

(c) What are all the eigenvalues of the Laguerre operator on $\mathbb{R}[x]$?

(d) Find all the eigenfunctions of the Laguerre operator on the space $\mathbb{R}[x]_2$.

(e) Find the Laguerre polynomial of degree 5. (For simplicity use the convention in which Laguerre polynomials have leading coefficient 1, even if this is not compatible with them having unit norm.)

77. (a) Verify by direct computation that the Hermite operator

$$\mathcal{L}_H = \frac{d^2}{dx^2} - 2x\frac{d}{dx}$$

on the space $\mathbb{R}[x]$ is symmetric with respect to the inner product given by the formula $(f,g) = \int_{-\infty}^{+\infty} f(x)g(x)e^{-x^2}dx$.

(b) Find the matrix and the characteristic polynomial of the Hermite operator $\mathcal{L}_H$ on the space $\mathbb{R}[x]_3$ (use the basis $\{1, x, x^2, x^3\}$). What is the set of all eigenvalues of $\mathcal{L}_H$ as an operator on the space $\mathbb{R}[x]_3$?

(c) What are all the eigenvalues of the Hermite operator on $\mathbb{R}[x]$?

(d) Find all eigenfunctions of the Hermite operator on the space $\mathbb{R}[x]_3$.

(e) Find the Hermite polynomial of degree 5. (For simplicity use the convention in which Hermite polynomials have leading coefficient 1, even if this is not compatible with them having unit norm.)

78. (a) Verify by direct computation that the Legendre operator

$$\mathcal{L}_L = (1-x^2)\frac{d^2}{dx^2} - 2x\frac{d}{dx}$$

on the space $C[-1,1]$ is symmetric with respect to the inner product given by the formula $(f,g) = \int_{-1}^{1} f(x)g(x)dx$.

(b) Find the characteristic polynomial of the Legendre operator $\mathcal{L}_L$ on the space $\mathbb{R}[x]_4$.

(c) What is the set of all eigenvalues of $\mathcal{L}_L$ as an operator on $\mathbb{R}[x]$?

(d) Find the Legendre polynomial of degree 5. (For simplicity use the convention in which Legendre polynomials have leading coefficient 1, even if this is not compatible with them having unit norm.)

79. (a) Verify by direct computation that the Chebyshev-I operator

$$\mathcal{L}_I = (1-x^2)\frac{d^2}{dx^2} - x\frac{d}{dx}$$

on the space $C[-1,1]$ is symmetric with respect to the inner product given by the formula $(f,g) = \int_{-1}^{1} f(x)g(x)(1-x^2)^{-1/2}dx$.

(b) Find the matrix and the characteristic polynomial of the Chebyshev-I operator $\mathcal{L}_I$ on the space $\mathbb{R}[x]_3$ (use the basis $\{1, x, x^2, x^3\}$).

(c) Hence find the Chebyshev-I polynomials of degree 2 and 3.

(d) What is the set of all eigenvalues of $\mathcal{L}_I$ as an operator on the space of all polynomials $\mathbb{R}[x]$?

(e) Find the Chebyshev-I polynomial of degree 5. (For simplicity use the convention in which Chebyshev-I polynomials have leading coefficient 1, even if this is not compatible with them having unit norm.)

80. (a) Verify by direct computation that the Chebyshev-II operator

$$\mathcal{L}_{II} = (1-x^2)\frac{d^2}{dx^2} - 3x\frac{d}{dx}$$

on the space $C[-1,1]$ is symmetric with respect to the inner product given by the formula $(f,g) = \int_{-1}^{1} f(x)g(x)(1-x^2)^{1/2}dx$.

(b) Find the matrix and the characteristic polynomial of the Chebyshev-II operator $\mathcal{L}_{II}$ on the space $\mathbb{R}[x]_4$ (use the basis $\{1, x, x^2, x^3, x^4\}$).

(c) What is the set of all eigenvalues of $\mathcal{L}_{II}$ as an operator on the space of all polynomials $\mathbb{R}[x]$?

(d) Find the Chebyshev-II polynomial of degree 5. (For simplicity use the convention in which Chebyshev-II polynomials have leading coefficient 1, even if this is not compatible with them having unit norm.)

81. Let $F[2\pi]$ be the vector space of all real 2π-periodic infinitely differentiable functions in one variable t with the inner product $(f,g) = \int_{-\pi}^{\pi} f(t)g(t)dt$.

(a) Prove that the operator $L = d^2/dt^2$ on $F[2\pi]$ is symmetric.

(b) Find all eigenvalues and eigenfunctions of the above operator L on $F[2\pi]$.

82. Let $F[2\pi]$ be the vector space from the above problem 81 and consider the operator $L_1 = d/dt$ on $F[2\pi]$.

(a) Prove that L_1 is skew-symmetric, i.e. for any $f, g \in F[2\pi]$, we have $(L_1 f, g) = -(f, L_1 g)$.

(b) Deduce that the only eigenfunctions for L_1 in $F[2\pi]$ are constant functions (with the zero eigenvalue).

(c) Let $F_{\mathbb{C}}[2\pi]$ be the complexification of the above $F[2\pi]$. Prove that the only eigenvalues for L_1 on $F_{\mathbb{C}}[2\pi]$ are complex numbers $\{ni \mid n \in \mathbb{Z}\}$ and for each such number $\lambda_n = ni$, there is only one (up to a non-zero scalar factor) eigenfunction e^{int} in $F_{\mathbb{C}}[2\pi]$.

(d) Prove that the functions $1, e^{it}, e^{-it}, e^{2it}, e^{-2it}, \ldots, e^{nit}, e^{-int}, \ldots$ are orthogonal in $F_{\mathbb{C}}[2\pi]$. What are the norms of those functions?

83. Prove that each of the following sets forms a group under ordinary multiplication.

(a) $\{2^k\}$ where $k \in \mathbb{Z}$.

(b) $\{\frac{1+2m}{1+2n}\}$ where $m,\ n \in \mathbb{Z}$.

(c) $\{\cos\theta + i\sin\theta\}$ where θ runs over all rational numbers.

84. Think of the integers $\mathbb{Z}$ as points equally spaced along the real line. Define two kinds of transformations on $\mathbb{Z}$:

(1) Translations of the form T_a (where a is an integer) which have the effect of translating $\mathbb{Z}$ a places to the right (if $a \geq 0$; or $-a$ places to the left if $a < 0$) using the formula $n \mapsto n + a$.

(2) Reflections of the form R_c (where c is an integer) which have the effect of reflecting $\mathbb{Z}$ in the point $\frac{c}{2}$ using the formula $n \mapsto c - n$.

Work out the effect of composing the following pairs of transformations: (a) $T_b T_a$, (b) $R_d T_a$, (c) $T_b R_c$, (d) $R_d R_c$. [In each case, because these are *functions* the compositions have to be evaluated from right to left; e.g., $T_b T_a$ means first do T_a and then do T_b.]

Now let A be the set of all such T_a and R_c. Show that A is a group and that we can find examples of elements $g,\ h \in A$ such that $gh \neq hg$, $g^2 = h^2 = e$ and $\forall s > 0,\ (gh)^s \neq e$.

85. Let G be the set of all 2×2 matrices of the form $\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$ where $a \in \mathbb{R}$. Show that G is an abelian group under matrix multiplication. What is it isomorphic to?

86. (a) Let G be the set of all 2×2 matrices of the form $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$ where $a,\ b,\ d \in \mathbb{R}$, and $ad \neq 0$. Show that G is a group under matrix multiplication.

    (b) With G as in part (a), define $Z(G) = \{g \in G \,|\, \text{such that}, \forall h \in G,\ gh = hg\}$. Identify the elements of $Z(G)$ and show that it is also a group. [$Z(G)$ is called the *centre* of G.]

87. (a) The modular group is defined by $SL(2, \mathbb{Z}) = \{A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \text{ with } a, b, c, d \in \mathbb{Z} \text{ and } \det A = 1\}$. Show that $SL(2, \mathbb{Z})$ is indeed a group under matrix multiplication (you may assume associativity).

    (b) Show that $T = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ belongs to $SL(2, \mathbb{Z})$ and compute T^n with $n \in \mathbb{Z}$ (for negative integers T^{-n} means $(T^{-1})^n$). What is the connection with Exercise 85 ?

88. Let G be a group such that for every element $g \in G$, $g^2 = e$. Show that G is abelian (i.e. $gf = fg$ for any $f, g \in G$).

89. Show that the group $\mathbb{Z}_8^\times$ has order 4. Is it isomorphic either to $\mathbb{Z}_4$ or to the Klein group V?

90. Write down the group table of the multiplicative group $\mathbb{Z}_9^\times$. Is this group isomorphic to $\mathbb{Z}_n$ for any n?

91. Write down the group table for $\mathbb{Z}_2 \times \mathbb{Z}_2$, the direct product of two copies of the cyclic group of order two, and compute its order. Is this group isomorphic to any group discussed during lectures?

Note: Starred, e.g. 1.*, exercises are more advanced/complicated.