Linear Algebra 1, Solutions to exercises 1 to 25.
Epiphany 21/22.

1. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation defined by

$$T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x+y+z \\ -2y-2z \\ y+z \end{pmatrix}.$$

Find the matrix which represents T with respect to:

(a) the standard basis in both copies of $\mathbb{R}^3$;

(b) the ordered basis consisting of

$$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}$$

in both copies of $\mathbb{R}^3$.

Solution: (a) The matrix of T with respect to the standard basis of $\mathbb{R}^3$ is

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & -2 & -2 \\ 0 & 1 & 1 \end{pmatrix}.$$

(b) Let B be the matrix with respect to the new basis. Then $B = P^{-1}AP$ where

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & -2 & -2 \\ 0 & 1 & 1 \end{pmatrix}, \quad P = \begin{pmatrix} 1 & 1 & 3 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}.$$

First we find P^{-1}:

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 3 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 & 1 & 0 \\ 1 & 1 & 2 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 1 & 3 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & -1 \end{array}\right)$$
$$\to \left(\begin{array}{ccc|ccc} 1 & 1 & 3 & 1 & 0 & 0 \\ 0 & 1 & -2 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & -1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & -2 & 0 & 3 \\ 0 & 1 & 0 & 1 & 1 & -2 \\ 0 & 0 & 1 & 1 & 0 & -1 \end{array}\right)$$
$$\to \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -3 & -1 & 5 \\ 0 & 1 & 0 & 1 & 1 & -2 \\ 0 & 0 & 1 & 1 & 0 & -1 \end{array}\right).$$

Thus

$$P^{-1} = \begin{pmatrix} -3 & -1 & 5 \\ 1 & 1 & -2 \\ 1 & 0 & -1 \end{pmatrix}.$$

But then, since $B = P^{-1}AP$,

$$B = \begin{pmatrix} -3 & -1 & 5 \\ 1 & 1 & -2 \\ 1 & 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 \\ 0 & -2 & -2 \\ 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 & 3 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 5 & 9 & 3 \\ -5 & -8 & -6 \\ 1 & 1 & 3 \end{pmatrix}.$$

2. * Find the characteristic polynomial of the matrix

$$M = \begin{pmatrix} A & C \\ \mathbf{0} & B \end{pmatrix},$$

where A denotes a $n \times n$ matrix, B a $m \times m$ matrix, C a $n \times m$ matrix, and $\mathbf{0}$ denotes the zero matrix of the appropriate dimensions, in this case $m \times n$.

Solution: The characteristic polynomial for M is

$$p_M(x) = \det(M - xI) = \det \begin{pmatrix} A - xI_n & C \\ \mathbf{0} & B - xI_m \end{pmatrix},$$

where I_n, I_m, denotes respectively the $n \times n$, $m \times m$, identity matrix. By observing that

$$\begin{pmatrix} A - xI_n & C \\ \mathbf{0} & B - xI_m \end{pmatrix} = \begin{pmatrix} I_n & \mathbf{0} \\ \mathbf{0} & B - xI_m \end{pmatrix} \begin{pmatrix} A - xI_n & C \\ \mathbf{0} & I_m \end{pmatrix}$$

and using the fact that the determinant of the product of two matrices is the product of their determinants we can write

$$p_M(x) = \det(A - xI_n)\det(B - xI_m) = p_A(x) \cdot p_B(x).$$

3. Find the characteristic polynomials, the eigenvalues and the eigenspaces of the following matrices

$$\text{(i)} \begin{pmatrix} 7 & -4 \\ -8 & -7 \end{pmatrix}, \qquad \text{(ii)} \begin{pmatrix} 5 & 2 & 3 \\ -13 & -6 & -11 \\ 4 & 2 & 4 \end{pmatrix}, \qquad \text{(iii)} \begin{pmatrix} 3 & 1 & 1 \\ -15 & -5 & -5 \\ 6 & 2 & 2 \end{pmatrix}.$$

Solution: (i) Characteristic polynomial : $(x^2 - 81) = (x - 9)(x + 9)$; eigenvalues $9, -9$; corresponding eigenspaces:

$$\lambda = -9: \text{ span} \left\{ \begin{pmatrix} 1 \\ 4 \end{pmatrix} \right\}; \qquad \lambda = 9: \text{ span} \left\{ \begin{pmatrix} -2 \\ 1 \end{pmatrix} \right\}.$$

(ii) Characteristic polynomial : $-x(x-1)(x-2)$; eigenvalues $0, 1, 2$; corresponding eigenspaces:

$$\lambda = 0: \text{ span} \left\{ \begin{pmatrix} 1 \\ -4 \\ 1 \end{pmatrix} \right\}; \qquad \lambda = 1: \text{ span} \left\{ \begin{pmatrix} 1 \\ -5 \\ 2 \end{pmatrix} \right\}. \qquad \lambda = 2: \text{ span} \left\{ \begin{pmatrix} 1 \\ -3 \\ 1 \end{pmatrix} \right\}.$$

(iii) Characteristic polynomial : $-x^3$; only eigenvalue 0; corresponding eigenspace:

$$\lambda = 0: \text{ span} \left\{ \begin{pmatrix} -1 \\ 0 \\ 3 \end{pmatrix}, \begin{pmatrix} -1 \\ 3 \\ 0 \end{pmatrix} \right\}.$$

(There are many possible choices of basis for V_0 in this case).

4. Show that each of the following matrices is similar to a diagonal matrix.

$$\text{(i)} \begin{pmatrix} 6 & 2 & 3 \\ -13 & -5 & -11 \\ 4 & 2 & 5 \end{pmatrix}; \qquad \text{(ii)} \begin{pmatrix} 1+i & 4+4i \\ 0 & -1-i \end{pmatrix}; \qquad \text{(iii)} \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

In each case write down an appropriate matrix which can be used to convert the given matrix to diagonal form.

Solution: In each case we have $M^{-1}AM = D$:

(i)

$$M = \begin{pmatrix} 1 & 1 & 1 \\ -4 & -5 & -3 \\ 1 & 2 & 1 \end{pmatrix}, \quad D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix};$$

(ii)

$$M = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}, \quad D = \begin{pmatrix} 1+i & 0 \\ 0 & -1-i \end{pmatrix};$$

(iii)

$$M = \begin{pmatrix} -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}, \quad D = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Note that you might have found different matrices because the choice of eigenvectors is not unique.

5. Prove that, if λ is an eigenvalue of the linear transformation $T : V \to V$, then λ^k is an eigenvalue of T^k for each integer $k > 0$. Prove also that if $p(t)$ is a polynomial, then $p(\lambda)$ is an eigenvalue of $p(T)$.

Solution: If $T(\mathbf{v}) = \lambda \mathbf{v}$ then, by induction, $T^k(\mathbf{v}) = \lambda^k \mathbf{v}$ and hence $p(T)(\mathbf{v}) = p(\lambda)\mathbf{v}$.

6. Show that the linear transformation $S : \mathbb{R}[x]_n \to \mathbb{R}[x]_n$ given by $S\big(p(x)\big) = \frac{1}{x}\int_0^x p(y)\, dy$ is diagonalizable and find the eigenvalues and eigenvectors.

Solution: Suppose that $p(x) = a_0 + a_1 x + \dots + a_n x^n$ is an eigenvector of S. Then $Sp(x) = \lambda p(x)$ for some $\lambda \in R$, that is

$$\frac{a_0}{1} + \frac{a_1 x}{2} + \dots + \frac{a_n}{n+1} x^n = \lambda(a_0 + a_1 x + \dots + a_n x^n)$$

or, equivalently,

$$\frac{a_k}{k+1} = \lambda a_k, \qquad k = 0, \dots, n$$

So we can see that the possible eigenvalues are $\lambda_k = \frac{1}{k+1}$ with eigenfunction $p_k(x) = x^k$, for $k = 0, ..., n$.

7. If A is a 2×2 matrix with characteristic polynomial $p_A(x) = c_0 + c_1 x + x^2$, show that $p_A(x) = x^2 - \mathrm{tr}(A)x + \det(A)$.

Solution:

$$p_A(x) = \det(A - xI) = \det \begin{pmatrix} a_{11} - x & a_{12} \\ a_{21} & a_{22} - x \end{pmatrix}$$

and by expanding the determinant

$$p_A(x) = x^2 - \mathrm{tr}(A)x + \det(A).$$

8. If A is a 3×3 matrix with characteristic polynomial $p_A(x) = c_0 + c_1 x + c_2 x^2 - x^3$, show that $p_A(x) = -x^3 + \mathrm{tr}(A)x^2 + \frac{\left(\mathrm{tr}(A^2) - (\mathrm{tr}A)^2\right)}{2} x + \det(A)$.

Solution:

$$p_A(x) = \det(A - xI) = \det \begin{pmatrix} a_{11} - x & a_{12} & a_{13} \\ a_{21} & a_{22} - x & a_{23} \\ a_{31} & a_{32} & a_{33} - x \end{pmatrix}$$

and by expanding the determinant

$$p_A(x) = -x^3 + \mathrm{tr}(A)x^2 + \frac{(\mathrm{tr}(A^2) - (\mathrm{tr}A)^2)}{2}x + \det(A).$$

9. Find the characteristic equation for

$$B = \begin{pmatrix} 0 & 2 & 6 \\ 2 & -8 & -26 \\ -2 & 2 & 8 \end{pmatrix}$$

and hence show that $B^9 = 2^8 B$. [Hint: use Cayley-Hamilton.]

Solution: The characteristic polynomial of B is

$$p_B(x) = \det \begin{pmatrix} -x & 2 & 6 \\ 2 & -8 - x & -26 \\ -2 & 2 & 8 - x \end{pmatrix} = -x^3 + 4x.$$

Therefore $B^3 - 4B = 0$. Thus multiplying by B^6 we get $B^9 = 4B^7 = 4B\, B^3\, B^3 = 4B\, 4B\, 4B = 4^4 B$.

10. Which of the matrices

$$\begin{pmatrix} 11 & 5 & 8 \\ -30 & -16 & -30 \\ 9 & 5 & 10 \end{pmatrix}, \quad \begin{pmatrix} -2 & -1 & -1 \\ 9 & 4 & 3 \\ -3 & -1 & 0 \end{pmatrix}, \quad \begin{pmatrix} -4 & -1 & 1 \\ 17 & 4 & -5 \\ -5 & -1 & 2 \end{pmatrix}$$

are similar to diagonal matrices?

Solution: (i)

$$A = \begin{pmatrix} 11 & 5 & 8 \\ -30 & -16 & -30 \\ 9 & 5 & 10 \end{pmatrix},$$

Then $p_A(x) = -(x+1)(x-2)(x-4)$ so that the eigenvalues are real and distinct hence the matrix is diagonalizable, i.e. it is similar to a diagonal matrix.
There is no need to compute the eigenvectors in this case but we will do it anyway as an additional exercise. The eigenvalues and corresponding eigenvectors are as follows:

$\lambda = -1$: corresponding equations ($A\mathbf{x} = -\mathbf{x}$)

$$\begin{aligned} 11x_1 + 5x_2 + 8x_3 &= -x_1, \\ -30x_1 - 16x_2 + -30x_3 &= -x_2, \\ 9x_1 + x_2 + 10x_3 &= -x_3. \end{aligned}$$

Thus the (-1)-eigenspace is 1-dimensional and equal to

$$\mathrm{span}\left\{ \begin{pmatrix} 1 \\ -4 \\ 1 \end{pmatrix} \right\}.$$

$\lambda = 2$: corresponding equations ($A\mathbf{x} = 2\mathbf{x}$)

$$\begin{aligned} 11x_1 + 5x_2 + 8x_3 &= 2x_1, \\ -30x_1 - 16x_2 + -30x_3 &= 2x_2, \\ 9x_1 + x_2 + 10x_3 &= 2x_3. \end{aligned}$$

Thus the (2)-eigenspace is 1-dimensional and equal to

$$\operatorname{span}\left\{\begin{pmatrix} 1 \\ -5 \\ 2 \end{pmatrix}\right\}.$$

$\lambda = 4$: corresponding equations ($A\mathbf{x} = 4\mathbf{x}$)

$$\begin{aligned} 11x_1 + 5x_2 + 8x_3 &= 4x_1, \\ -30x_1 - 16x_2 + -30x_3 &= 4x_2, \\ 9x_1 + x_2 + 10x_3 &= 4x_3. \end{aligned}$$

Thus the (4)-eigenspace is 1-dimensional and equal to

$$\operatorname{span}\left\{\begin{pmatrix} 1 \\ -3 \\ 1 \end{pmatrix}\right\}.$$

Thus the basis

$$\left\{\begin{pmatrix} 1 \\ -4 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -5 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ -3 \\ 1 \end{pmatrix}\right\}.$$

diagonalizes A.

(ii)

$$A = \begin{pmatrix} -2 & -1 & -1 \\ 9 & 4 & 3 \\ -3 & -1 & 0 \end{pmatrix}$$

Then $p_A(x) = -x(x-1)^2$ so that the eigenvalues are real. The eigenvalues and corresponding eigenvectors are as follows:

$\lambda = 0$: corresponding equations ($A\mathbf{x} = \mathbf{0}$)

$$\begin{aligned} -2x_1 - x_2 - x_3 &= 0, \\ 9x_1 + 4x_2 + 3x_3 &= 0, \\ -3x_1 - x_2 &= 0. \end{aligned}$$

Thus, row reducing, we deduce that the (0)-eigenspace is the 1-dimensional subspace

$$\operatorname{span}\left\{\begin{pmatrix} 1 \\ -3 \\ 1 \end{pmatrix}\right\}.$$

$\lambda = 1$: corresponding equations ($A\mathbf{x} = \mathbf{x}$)

$$\begin{aligned} -2x_1 - x_2 - x_3 &= x_1, \\ 9x_1 + 4x_2 + 3x_3 &= x_2, \\ -3x_1 - x_2 &= x_3. \end{aligned}$$

Thus the $(+1)$-eigenspace is 2-dimensional and equal to

$$\operatorname{span}\left\{\begin{pmatrix} -1 \\ 0 \\ 3 \end{pmatrix}, \begin{pmatrix} -1 \\ 3 \\ 0 \end{pmatrix}\right\}.$$

Thus the subspace spanned by the eigenvectors

$$\text{span}\left\{\begin{pmatrix}1\\-3\\1\end{pmatrix}, \begin{pmatrix}-1\\0\\3\end{pmatrix}, \begin{pmatrix}-1\\3\\0\end{pmatrix}\right\}$$

is 3-dimensional and A is hence diagonalizable.

(iii)

$$A = \begin{pmatrix}-4 & -1 & 1\\ 17 & 4 & -5\\ -5 & -1 & 2\end{pmatrix}.$$

Then once again $p_A(x) = -x(x-1)^2$ so that the eigenvalues are real. The eigenvalues and corresponding eigenvectors are as follows:

$\lambda = 0$: corresponding equations ($A\mathbf{x} = \mathbf{0}$)

$$\begin{aligned}-4x_1 - x_2 + x_3 &= 0,\\ 17x_1 + 4x_2 - 5x_3 &= 0,\\ -5x_1 - x_2 + 2x_3 &= 0.\end{aligned}$$

Thus, row reducing, we deduce that the (0)-eigenspace is the 1-dimensional subspace

$$\text{span}\left\{\begin{pmatrix}1\\-3\\1\end{pmatrix}\right\}.$$

$\lambda = 1$: corresponding equations ($A\mathbf{x} = \mathbf{x}$)

$$\begin{aligned}-4x_1 - x_2 + x_3 &= x_1,\\ 17x_1 + 4x_2 - 5x_3 &= x_2,\\ -5x_1 - x_2 + 2x_3 &= x_3.\end{aligned}$$

Thus the $(+1)$-eigenspace is 1-dimensional and equal to

$$\text{span}\left\{\begin{pmatrix}1\\-4\\1\end{pmatrix}\right\}.$$

Thus the subspace spanned by the eigenvectors is only 2-dimensional and so A is not diagonalizable.

11. Prove that

$$\begin{pmatrix}0 & -4\\ -4 & 0\end{pmatrix} \quad \text{and} \quad \begin{pmatrix}-1 & -5\\ -3 & 1\end{pmatrix}$$

are similar to each other.

Solution: Both have eigenvalues -4 and 4, so are similar to

$$\begin{pmatrix}-4 & 0\\ 0 & 4\end{pmatrix}.$$

12. Find two matrices $A, B \in M_n(\mathbb{R})$ with same characteristic polynomial, same eigenvalues, same determinant, same trace but not similar to one another.

Solution: Let $A = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$ and $B = \begin{pmatrix} \lambda & x \\ 0 & \lambda \end{pmatrix}$ with $x \in \mathbb{R}$ and $x \neq 0$. It is simple to check that A and B have same characteristic polynomial, hence same eigenvalues, hence same trace and same determinant. However A is similar to a diagonal matrix while B cannot be diagonalized, hence it is not similar to A. [For more details see the final chapter of the notes on the Jordan normal form]

13. Which of the matrices

$$\begin{pmatrix} -9 & -3 & -1 \\ 35 & 11 & 1 \\ -11 & -3 & 1 \end{pmatrix}, \qquad \begin{pmatrix} -1 & 2 & 0 \\ -1 & 1 & 0 \\ 4 & -2 & 1 \end{pmatrix}$$

are similar to diagonal matrices when we take (a) $\mathbb{R}$, (b) $\mathbb{C}$ as the underlying field of scalars?

Solution: For

$$A = \begin{pmatrix} -9 & -3 & -1 \\ 35 & 11 & 1 \\ -11 & -3 & 1 \end{pmatrix}$$

the characteristic polynomial is $-(x+1)(x-2)^2$ and the eigenvalues $\lambda = -1, 2$ are real. But the eigenspace corresponding to $\lambda = 2$ is 1-dimensional, so A is not diagonalizable.
For

$$A = \begin{pmatrix} -1 & 2 & 0 \\ -1 & 1 & 0 \\ 4 & -2 & 1 \end{pmatrix}$$

the characteristic polynomial is $-(x-1)(x^2+1)$. Thus A has three distinct eigenvalues of which exactly one is real, $\lambda = 1, i, -i$, and so A is diagonalizable over the complex numbers but not over the reals.

14. Given

$$A = \begin{pmatrix} -6 & 8 \\ -4 & 6 \end{pmatrix},$$

find P such that $P^{-1}AP$ is diagonal. Hence compute A^6.

Solution: The characteristic polynomial of

$$A = \begin{pmatrix} -6 & 8 \\ -4 & 6 \end{pmatrix}$$

is $x^2 - 4$ with eigenvectors

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

Let P be the matrix whose columns are these vectors. Thus

$$P^{-1}AP = D := \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}, \qquad P = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}.$$

Thus $A = PDP^{-1}$ so

$$\begin{aligned} A^6 = PD^6P^{-1} &= P \begin{pmatrix} 2^6 & 0 \\ 0 & 2^6 \end{pmatrix} P^{-1} \\ &= 2^6 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \end{aligned}$$

We could have arrived at the same conclusion by using Hamilton-Cayley: from the characteristic polynomial we know that $A^2 = 4\,\mathbf{I}$ so, taking the cube of this equation, we get $A^6 = 64\,\mathbf{I}$. Generically by induction $A^{2n} = 4^n\,\mathbf{I}$.

15. Let A be an $n \times n$ complex matrix. Suppose that A has only one eigenvalue. Prove that if A is not of the form aI, then A is not similar to a diagonal matrix.

Solution: If A is similar to $\operatorname{diag}(\lambda_1, \ldots, \lambda_n)$, then the characteristic polynomial is $(x-\lambda_1)\ldots(x-\lambda_n)$. And if A has only one eigenvalue, then $\lambda_1 = \ldots = \lambda_n = \lambda$, say, and so A is similar to λI. But the only matrix similar to λI is λI.

16. Given the matrix

$$A = \begin{pmatrix} -9 & -8 & -15 \\ 37 & 32 & 59 \\ -10 & -8 & -14 \end{pmatrix}.$$

find a matrix B such that $B^3 = A$. Is this B unique?

Solution: We have $A = M^{-1}D^3M$ where

$$M = \begin{pmatrix} 1 & 1 & 2 \\ 2 & 0 & -2 \\ 3 & 1 & 1 \end{pmatrix}, \qquad D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad M^{-1} = \begin{pmatrix} -1 & -1/2 & 1 \\ 4 & 5/2 & -3 \\ -1 & -1 & 1 \end{pmatrix}.$$

Thus if $B = M^{-1}DM$ then $B^3 = A$. This gives

$$B = \begin{pmatrix} -3 & -2 & -3 \\ 13 & 8 & 11 \\ -4 & -2 & -2 \end{pmatrix},$$

This B is not unique, since either of the positive entries in D could acquire a phase ω such that $\omega^3 = 1$ without changing D^3.

17. Solve the system of first-order differential equations

$$\dot{x}_1 = -3x_1 - 2x_2 - 6x_3, \quad \dot{x}_2 = -8x_1 - 3x_2 - 12x_3, \quad \dot{x}_3 = 5x_1 + 2x_2 + 8x_3,$$

subject to the initial conditions

$$x_1(0) = 1, \quad x_2(0) = \frac{3}{2}, \quad x_3(0) = -1.$$

Solution: Write the equations as $\dot{\mathbf{x}} = A\mathbf{x}$ where

$$A = \begin{pmatrix} -3 & -2 & -6 \\ -8 & -3 & -12 \\ 5 & 2 & 8 \end{pmatrix}.$$

Then A has eigenvalues $2, 1, -1$ with corresponding eigenvectors

$$\begin{pmatrix} -2 \\ -4 \\ 3 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}.$$

Thus, if

$$P = \begin{pmatrix} -2 & 1 & 1 \\ -4 & 1 & 2 \\ 3 & -1 & -1 \end{pmatrix},$$

then

$$P^{-1}AP = D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

So, putting $\mathbf{x} = P\mathbf{y}$, the equations reduce to $\dot{\mathbf{y}} = D\mathbf{y}$ which have solutions

$$y_1 = c_1 e^{2t}, \quad y_2 = c_2 e^t, \quad y_3 = c_3 e^{-t}.$$

Thus

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -2 & 1 & 1 \\ -4 & 1 & 2 \\ 3 & -1 & -1 \end{pmatrix} \begin{pmatrix} c_1 e^{2t} \\ c_2 e^t \\ c_3 e^{-t} \end{pmatrix} = \begin{pmatrix} -2c_1 e^{2t} + c_2 e^t + c_3 e^{-t} \\ -4c_1 e^{2t} + c_2 e^t + 2c_3 e^{-t} \\ 3c_1 e^{2t} - c_2 e^t - c_3 e^{-t} \end{pmatrix}.$$

But then the initial conditions give

$$\begin{pmatrix} -2c_1 + c_2 + c_3 \\ -4c_1 + c_2 + 2c_3 \\ 3c_1 - c_2 - c_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 3/2 \\ -1 \end{pmatrix},$$

so that $c_1 = 0, c_2 = 1/2, c_3 = 1/2$. Thus

$$x_1 = \frac{1}{2}(e^t + e^{-t}) = \cosh t, \quad x_2 = \frac{1}{2}(e^t + 2e^{-t}) = \cosh t + \frac{e^{-t}}{2}, \quad x_3 = -\frac{1}{2}(e^t + e^{-t}) = -\cosh t.$$

18. Find the eigenvalues and eigenvectors of the linear transformation $T : \mathbb{R}[x]_4 \to \mathbb{R}[x]_4$ defined by $T(p(x)) = x^2\big(p(x+1) - 2p(x) + p(x-1)\big)$.

Solution: We take the standard basis $\{1, x, x^2, x^3, x^4\}$ of $\mathbb{R}[x]_4$ and find the matrix of T with respect to this basis. Since

$$T(1) = 0, \quad T(x) = 0, \quad T(x^2) = 2x^2, \quad T(x^3) = 6x^3, \quad T(x^4) = 12x^4 + 2x^2,$$

the required matrix is

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 2 \\ 0 & 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 0 & 12 \end{pmatrix}.$$

The eigenvalues are $0, 0, 2, 6, 12$. The polynomials in $\mathbb{R}[x]_4$ corresponding to each eigenvectors in order:

$$1, \ x, \ x^2, x^3, x^4 + \frac{x^2}{5}.$$

19. Define $T : M_n(\mathbb{R}) \to M_n(\mathbb{R})$ by $T(A) = A^t$. Prove that T has only two distinct eigenvalues, and that its eigenvectors span $M_n(\mathbb{R})$. Here $M_n(\mathbb{R})$ denotes the set of $n \times n$ real matrices, and A^t denotes the transpose of A.

Solution: Clearly $T^2 = I$. If $T\mathbf{v} = \lambda\mathbf{v}$, then applying T again gives $T^2\mathbf{v} = \lambda^2\mathbf{v}$; so $\lambda^2 = 1$ and hence $\lambda = \pm 1$. Now the $\lambda = 1$ eigenspace V_1 consists of all the symmetric matrices, and the $\lambda = -1$ eigenspace V_{-1} consists of all the antisymmetric matrices. Since each $A \in M_n(\mathbb{R})$ can be written in the form

$$A = \frac{1}{2}(A + A^t) + \frac{1}{2}(A - A^t),$$

it is clear that the eigenvectors span $M_n(\mathbb{R})$. Indeed $M_n(\mathbb{R}) = V_1 \oplus V_{-1}$. (Note the dimension of V_1 is $n(n+1)/2$, and the dimension of V_{-1} is $n(n-1)/2$; these two numbers add up to n^2 which is the dimension of M_n.)

20. Let S and T be linear transformations from an n-dimensional vector space to itself, and assume that $ST = TS$. If $\mathbf{v}$ is an eigenvector of S, and if $T(\mathbf{v})$ is not the zero vector, show that $T(\mathbf{v})$ is also an eigenvector of S. Hence show that if S has n distinct eigenvalues, then S and T have the same eigenvectors.

Solution: Suppose that $S\mathbf{v} = \lambda\mathbf{v}$ for some $\mathbf{v} \neq \mathbf{0}$. Then $ST\mathbf{v} = TS\mathbf{v} = \lambda T\mathbf{v}$. Thus if $T\mathbf{v} \neq \mathbf{0}$, then $T\mathbf{v}$ is also an eigenvector of S corresponding to the eigenvalue λ.

Now suppose that S has distinct eigenvalues $\lambda_1, \ldots, \lambda_n$ with corresponding eigenvectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$. Thus $S\mathbf{v}_i = \lambda_i\mathbf{v}_i$ for $i = 1, \ldots, n$, and the eigenspace V_i corresponding to λ_i is given by $V_i = \mathrm{span}\{\mathbf{v}_i\}$. (Note that V_i contains $\mathrm{span}\{\mathbf{v}_i\}$ but cannot be any bigger for dimensional reasons.) If $T\mathbf{v}_i \neq \mathbf{0}$ then $T\mathbf{v}_i \in V_i = \mathrm{span}\{\mathbf{v}_i\}$. Thus for all $i = 1, \ldots, n$, $T\mathbf{v}_i = \mu_i\mathbf{v}_i$ for some μ_i. But then $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is a basis for V and each element is an eigenvector of T. Since T has at most n linearly independent eigenvectors, it follows that S and T have the same eigenvectors.

Remark. Notice that in terms of matrices this may be stated as follows. If A and B are $n \times n$ matrices such that $AB = BA$, and A has distinct eigenvalues, then there exists an invertible matrix P such that $P^{-1}AP$ and $P^{-1}BP$ are both diagonal. The matrices A and B are said to be "simultaneously diagonalizable".

21. Compute the characteristic polynomial of

$$A = \begin{pmatrix} 16 & 14 & 10 \\ -9 & -9 & -5 \\ -2 & 13 & -9 \end{pmatrix},$$

and deduce that $A^4 = 16\,\mathbf{I}$. [Hint: Cayley-Hamilton.]

Solution: The characteristic polynomial is $-x^3 - 2x^2 - 4x - 8$. Thus by the Cayley-Hamilton theorem, $A^3 + 2A^2 + 4A + 8I = 0$. Multiplying by $A - 2\mathbf{I}$ then gives $A^4 - 16\mathbf{I} = 0$.

22. Find the general (real) solution of the system of first-order differential equations

$$\dot{x}_1 = 5x_1 + 5x_2, \quad \dot{x}_2 = -5x_1 - x_2.$$

Solution: Write the equations as $\dot{\mathbf{x}} = A\mathbf{x}$, where

$$A = \begin{pmatrix} 5 & 5 \\ -5 & -1 \end{pmatrix}.$$

Then A has eigenvalues $2 \pm 4i$, with corresponding eigenvectors

$$\begin{pmatrix} 1 \mp 2i \\ 1 \pm 2i \end{pmatrix}.$$

Thus, if

$$P = \begin{pmatrix} 1-2i & 1+2i \\ 1+2i & 1-2i \end{pmatrix},$$

then

$$P^{-1}AP = D = \begin{pmatrix} 2+4i & 0 \\ 0 & 2-4i \end{pmatrix}.$$

So, putting $\mathbf{x} = P\mathbf{y}$, the equations reduce to $\dot{\mathbf{y}} = D\mathbf{y}$, which have solutions

$$y_1 = ae^{2t}e^{4it}, \quad y_2 = \bar{a}e^{2t}e^{-4it}.$$

Here a is a complex constant. (We need y_1 and y_2 to be complex conjugates of each other, in order for x_1 and x_2 to be real.) Transforming back to $\mathbf{x}$ then gives

$$x_1 = 2e^{2t}[(c_1+2c_2)\cos(4t)+(2c_1-c_2)\sin(4t)], \quad x_2 = 2e^{2t}[(c_1-2c_2)\cos(4t)-(2c_1+c_2)\sin(4t)],$$

where c_1 and c_2 are real constants (with $a = c_1 + ic_2$).

23. * Let C be an element of $M_n(\mathbb{R})$ of the form:

$$C = \begin{pmatrix} c_0 & c_1 & c_2 & \dots & c_{n-1} \\ c_{n-1} & c_0 & c_1 & \dots & c_{n-2} \\ \vdots & & \ddots & & \vdots \\ c_1 & c_2 & \dots & c_{n-1} & c_0 \end{pmatrix}$$

where each row is just a cyclic shift of the row above it. In a succint way we can write the entry $C_{ij} = c_{(i-j) \bmod n}$ where mod denotes the remainder of the integer division modulo n. This is called a *circulant* matrix and it's a particular type of a particular set of matrices called *Toeplitz* matrices. Show that if ω is an n-th root of unity, i.e. $\omega^n = 1$, then the vector $\mathbf{v} = (\omega^{\mathbf{0}}, \omega^{\mathbf{1}}, ..., \omega^{\mathbf{n-1}})^{\mathbf{t}}$ is an eigenvector of C and compute its corresponding eigenvalue.

Solution: If we compute the k-th entry of the vector of $C\mathbf{v}$ we have

$$(C\mathbf{v})_k = \sum_{i=0}^{k-1} c_{n-k+i} v_i + \sum_{i=k}^{n-1} c_{i-k} v_i$$

where for simplicity we start counting vectors components from zero, i.e. $\mathbf{v} = (v_0, ..., v_{n-1})^t$. By changing dummy variables of summation we have

$$\begin{aligned}(C\mathbf{v})_k &= \sum_{i=n-k}^{n-1} c_i v_{i+k-n} + \sum_{i=0}^{n-k-1} c_i v_{k+i} \\ &= \sum_{i=0}^{n-k-1} c_i v_{k+i} + \sum_{i=n-k}^{n-1} c_i v_{i+k-n} \\ &= \sum_{i=0}^{n-k-1} c_i \omega^{k+i} + \sum_{i=n-k}^{n-1} c_i \omega^{i+k-n} \\ &= \omega^k \sum_{i=0}^{n-k-1} c_i \omega^i + \omega^{k-n} \sum_{i=n-k}^{n-1} c_i \omega^i = \left(\sum_{i=0}^{n-1} c_i \omega^i \right) \omega^k = \lambda v_k\end{aligned}$$

where the eigenvalue is $\lambda = \sum_{i=0}^{n-1} c_i \omega^i$ and in the last line we used crucially the fact that ω is an n-th root of unity, i.e. $\omega^n = \omega^{-n} = 1$. We can write all the roots of unity as $\omega_m = e^{2\pi i \frac{m}{n}}$ with $m \in \{0, ..., n-1\}$, hence we have an eigenvalue of the form

$$\lambda_m = \sum_{k=0}^{n-1} c_k \exp(2\pi i \frac{k\,m}{n})$$

which is called *Discrete Fourier Transform* (DFT) of the sequence $\{c_k\}$.

24. * The *minimal polynomial* of a square matrix A is the polynomial $q(t) = t^m + a_1 t^{m-1} + \ldots + a_m$ of least degree such that $A^m + a_1 A^{m-1} + \ldots + a_m I = 0$. Show that if $A \in M_n(\mathbb{R})$, then the subset $\mathbb{R}[A]$ of $M_n(\mathbb{R})$ consisting of polynomials in A is a vector subspace whose dimension is the degree of the minimal polynomial of A. Find the dimension of $\mathbb{R}[A]$ when

$$A = \begin{pmatrix} 3 & 1 & -2 \\ -2 & 1 & 2 \\ 1 & 1 & 0 \end{pmatrix}.$$

Solution: Suppose that the degree of the minimal polynomial of A is m. Then

$$I,\ A,\ \ldots,\ A^m \in \operatorname{span}\{I,\ A,\ \ldots,\ A^{m-1}\}$$

and hence, by induction,

$$A^k \in \operatorname{span}\{I,\ A,\ \ldots,\ A^{m-1}\}$$

for all integers $k \geq 0$. Thus $I, A, \ldots, A^{m-1}$ span $\mathbb{R}[A]$. But $I, A, \ldots, A^{m-1}$ are linearly independent for any linear dependence relation among them would give rise to a polynomial equation for A of degree strictly less than m. Thus $\{I,\ A,\ \ldots,\ A^{m-1}\}$ is a basis for $\mathbb{R}[A]$ and so $\dim \mathbb{R}[A] = m$. In the case of the given matrix A the minimal polynomial is $(x-1)^2(x-2)$ (cf. question 13), so that $\dim \mathbb{R}[A] = 3$.

25. (a) Let D be a diagonal 3×3 matrix with distinct entries. Prove that *every* diagonal 3×3 matrix can be expressed as a linear combination of I, D, D^2.

(b) Prove that for $P \in GL_n(\mathbb{R})$ (that is P an invertible $n \times n$ real matrix) the map $A \mapsto P^{-1}AP$ defines a nonsingular linear transformation from $M_3(\mathbb{R})$ to itself.

(c) Prove that if $A \in M_3(\mathbb{R})$ has distinct real eigenvalues, then the set $\mathbb{R}[A]$ of all polynomials in A is a 3-dimensional subspace of $M_3(\mathbb{R})$.

Solution: (a) If

$$D = \operatorname{diag}(\lambda_1,\ \lambda_2,\ \lambda_3)$$

where $\lambda_1, \lambda_2, \lambda_3$ are distinct, then $I,\ D,\ D^2$ are linearly independent in $M_3(\mathbb{R})$, for otherwise there would be a quadratic equation with three distinct solutions. Since the vector subspace U of $M_3(\mathbb{R})$ consisting of diagonal matrices has dimension 3, it follows that I, D, D^2 is a basis.

(b) It is elementary to check linearity. The transformation is nonsingular since it has inverse $A \mapsto PAP^{-1}$.

(c) First note that for any A, the set $\mathbb{R}[A]$ is a vector subspace of $M_3(\mathbb{R})$. If $A \in M_3(\mathbb{R})$ has distinct real eigenvalues $\lambda_1,\ \lambda_2,\ \lambda_3$, then

$$D = \operatorname{diag}(\lambda_1,\ \lambda_2,\ \lambda_3) = P^{-1}AP$$

for some invertible matrix P, and so by (b) the set $\mathbb{R}[A]$ is isomorphic to the vector subspace U of diagonal matrices. By (a) this has dimension 3. In fact note that $\{I, A, A^2\}$ is a basis for $\mathbb{R}[A]$.