Linear Algebra 1, Solutions to exercises 26 to 52.
Epiphany 21/22.

26. Decide which of the following bilinear functions defines an inner product:

    (i) $x_1y_1 - x_1y_3 - x_3y_1 + 2x_3y_3 + 4x_2y_2 + x_4y_4 + x_2y_4 + x_4y_2$ on $\mathbb{R}^4$;

    (ii) $2x_1y_1 + x_2y_2 + 2x_3y_2 + x_2y_3$ on $\mathbb{R}^3$;

    (iii) $2x_1y_1 + x_2y_2 - 2x_1y_3 - 2x_3y_1 - x_2y_3 - x_3y_2 + x_3y_3$ on $\mathbb{R}^3$;

    (iv) $4x_1y_1 + 2x_2y_2 + 6x_2y_3 + 6x_3y_2 + 18x_3y_3$ on $\mathbb{R}^3$;

    (v) $x_1y_1 + x_2y_2 - x_1y_3 - x_3y_1 + 3x_2y_3 + 3x_3y_2 + 11x_3y_3$ on $\mathbb{R}^3$.

    Solution: (i) Yes. Clearly symmetric and bilinear and

    $$(\mathbf{x}, \mathbf{x}) = (x_1 - x_3)^2 + x_3^2 + 3x_2^2 + (x_2 + x_4)^2 \geq 0$$

    with equality iff $x_1 = x_2 = x_3 = x_4 = 0$.

    (ii) No, not symmetric.

    (iii) No, not positive. ($(\mathbf{x}, \mathbf{x}) < 0$ for $\mathbf{x} = (3, 1, 3)$, for example).

    (iv) No, not strictly positive. $(\mathbf{x}, \mathbf{x}) = 4x_1^2 + 2(x_2 + 3x_3)^2$ so there are non-zero vectors with zero norm, for example $\mathbf{x} = (0, 3, -1)$.

    (v) Yes. Clearly symmetric and bilinear and

    $$(\mathbf{x}, \mathbf{x}) = (x_1 - x_3)^2 + (x_2 + 3x_3)^2 + x_3^2 \geq 0$$

    with equality iff $x_1 = x_2 = x_3 = 0$.

    You can also check positivity by checking whether the corresponding symmetric matrix is positive definite.

27. Show that the bilinear form on $\mathbb{R}^3$ defined by

    $$(\mathbf{x}, \mathbf{y}) = 6x_1y_1 - x_1y_2 - x_2y_1 + x_2y_2 - x_1y_3 - x_3y_1 + x_3y_3$$

    is an inner product on $\mathbb{R}^3$, and find the lengths of the vectors

    $$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 3 \\ -6 \\ 3 + 3\sqrt{3} \end{pmatrix}$$

    and the angle between them with respect to this inner product.

    Solution: Clearly $(\mathbf{x}, \mathbf{y})$ is symmetric and bilinear, so we have only to check positivity. But

    $$(\mathbf{x}, \mathbf{x}) = 6x_1^2 - 2x1x2 + x_2^2 + x_3^2 - 2x_1x_3 = 4x_1^2 + (x_1 - x_2)^2 + (x_1 - x_3)^2 \geq 0$$

    with equality iff $x_1 = x_2 = x_3 = 0$.

    Next note that

    $$\left\| \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\|^2 = 4, \quad \left\| \begin{pmatrix} 3 \\ -6 \\ 3 + 3\sqrt{3} \end{pmatrix} \right\|^2 = 144, \quad \left(\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ -6 \\ 3 + 3\sqrt{3} \end{pmatrix} \right) = 12.$$

Thus

$$\left\| \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\| = 2, \quad \left\| \begin{pmatrix} 3 \\ -6 \\ 3+3\sqrt{3} \end{pmatrix} \right\| = 12$$

and the angle θ between these vectors is given by

$$\cos\theta = \frac{1}{2}$$

so that $\theta = \pi/3$.

28. Find the angle between the vectors in $\mathbb{R}^4$ equipped with the standard inner product:

$$\text{(i)} \quad \begin{pmatrix} 1 \\ 2 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ -1 \\ 1 \end{pmatrix}; \qquad \text{(ii)} \quad \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}, \begin{pmatrix} 8 \\ -4 \\ -4 \\ 3 \end{pmatrix}; \qquad \text{(iii)} \quad \begin{pmatrix} 6 \\ 2 \\ -2 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ -1 \\ 3 \end{pmatrix}.$$

Solution: (i) Arcos $(2/7)$; (ii) $\pi/2$; (iii) $\pi/3$.

29. * Consider the vector space M_n of the $n \times n$ matrices with real coefficients, and the application $(\ ,\ ) : M_n \times M_n \mapsto \mathbb{R}$ given by

$$(A, B) = \mathrm{Tr}(A^t B)\,,$$

where A^t denotes the transpose of A and Tr denotes the trace. Show that $(\ ,\ )$ defines an inner product on M_n.

Solution: From linearity of the trace and the transposition of a matrix combined with the distributivity of matrix product it follows immediately that (A, B) is bilinear. Furthermore we have

$$(A, B) = \mathrm{Tr}(A^t B) = \mathrm{Tr}((B^t A)^t) = \mathrm{Tr}(B^t A) = (B, A)$$

since $\mathrm{Tr} A^t = \mathrm{Tr} A$, so $(\ ,\ )$ is also symmetric. We are left to prove that it is also positive definite. To this end we consider

$$(A, A) = \mathrm{Tr}(A^t A) = \sum_{i=1}^{n} \sum_{j=1}^{n} A^t_{ji} A_{ij} = \sum_{i=1}^{n} \sum_{j=1}^{n} (A_{ij})^2 \geq 0$$

since it is a sum of squares. Furthermore $(A, A) = 0$ if and only if $(A_{ij})^2$ is equal to zero for all $i, j = 1, ..., n$, i.e. if and only if $A = \mathbf{0}$ is the zero matrix thus proving that $(\ ,\ )$ defines an inner product on the vector space of $n \times n$ dimensional matrices with real coefficients.

30. Suppose that $\mathbb{C}^3$ is equipped with the standard inner product. Show that the vectors

$$\frac{1}{2} \begin{pmatrix} i \\ i \\ 1+i \end{pmatrix}, \quad \frac{1}{6} \begin{pmatrix} 3+3i \\ 1+i \\ -4 \end{pmatrix}$$

are mutually orthogonal unit vectors, and find an orthonormal basis for $\mathbb{C}^3$ which contains them.

Solution: First note that if

$$\mathbf{u}_1 = \frac{1}{2}\begin{pmatrix} i \\ i \\ 1+i \end{pmatrix}, \quad \mathbf{u}_2 = \frac{1}{6}\begin{pmatrix} 3+3i \\ 1+i \\ -4 \end{pmatrix}$$

then

$$\left\| \frac{1}{2}\begin{pmatrix} i \\ i \\ 1+i \end{pmatrix} \right\|^2 = \frac{1}{4}\{1+1+2\} = 1, \quad \left\| \frac{1}{6}\begin{pmatrix} 3+3i \\ 1+i \\ -4 \end{pmatrix} \right\|^2 = \frac{1}{36}\{18+2+16\} = 1.$$

so both vectors are unit vectors. Moreover,

$$\left\langle \frac{1}{2}\begin{pmatrix} i \\ i \\ 1+i \end{pmatrix}, \frac{1}{6}\begin{pmatrix} 3+3i \\ 1+i \\ -4 \end{pmatrix} \right\rangle = \frac{1}{12}\{i(3-3i) + i(1-i) + (1+i)(-4)\} = 0.$$

Thus the given vectors are also mutually orthogonal. To find a third vector

$$\mathbf{v} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

which is orthogonal to these, we have to solve the equations

$$0 = \langle \mathbf{v}, \mathbf{u}_1 \rangle = \frac{1}{2}\left(a(-i) + b(-i) + c(1-i)\right) = 0,$$

$$0 = \langle \mathbf{v}, \mathbf{u}_2 \rangle = \frac{1}{6}\left(a(3-3i) + b(1-i) + c(-4)\right) = 0.$$

The solutions are given by $b = -5/3a$, $c = (1-i)/3a$, so that in particular

$$\mathbf{v} = \begin{pmatrix} 3 \\ -5 \\ 1-i \end{pmatrix}$$

is a solution. (In fact every other solution is a multiple of this). But $\|\mathbf{v}\|^2 = 36$ so, setting

$$\mathbf{u}_3 = \frac{1}{6}\begin{pmatrix} 3 \\ -5 \\ 1-i \end{pmatrix}$$

we have that $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is an orthonormal basis.

31. Decide which of the following defines a Hermitian inner product on $\mathbb{C}^2$:

(i) $3z_1\bar{w}_1 + 4z_2\bar{w}_2$;

(ii) $z_1\bar{w}_2 + z_2\bar{w}_1$;

(iii) $z_1\bar{w}_1 + (1+i)z_2\bar{w}_2$;

(iv) $z_1\bar{w}_1 + z_2\bar{w}_2 + z_1 w_2$.

Solution: Write

$$\mathbf{z} = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}, \quad \mathbf{w} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}.$$

(i) Yes. If $\langle \mathbf{z}, \mathbf{w} \rangle = 3z_1\bar{w}_1 + 4z_2\bar{w}_2$ then clearly $\langle \mathbf{w}, \mathbf{z} \rangle = \overline{\langle \mathbf{z}, \mathbf{w} \rangle}$. Also $\langle \mathbf{z}, \mathbf{w} \rangle$ is $\mathbb{C}$-linear in $\mathbf{z}$. Finally $\langle \mathbf{z}, \mathbf{z} \rangle = 3|z_1|^2 + 4|z_2|^2 \geq 0$ with equality iff $z_1 = z_2 = 0$.

(ii) No. Does satisfy $\langle \mathbf{w}, \mathbf{z} \rangle = \overline{\langle \mathbf{z}, \mathbf{w} \rangle}$ but it is not positive definite.

In fact $\langle \mathbf{z}, \mathbf{z} \rangle = z_1 \bar{z}_2 + z_2 \bar{z}_1 = 2\mathrm{Re}\,(z_1 \bar{z}_2)$ so for example $\langle \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix} \rangle = -2 < 0$.

(iii) No. Does not satisfy $\langle \mathbf{w}, \mathbf{z} \rangle = \overline{\langle \mathbf{z}, \mathbf{w} \rangle}$.

(iv) No. Does not satisfy $\langle \mathbf{w}, \mathbf{z} \rangle = \overline{\langle \mathbf{z}, \mathbf{w} \rangle}$.

32. Show that $z_1 \bar{w}_1 + 2z_2 \bar{w}_2 + \frac{1+i}{\sqrt{2}} z_1 \bar{w}_2 + \frac{1-i}{\sqrt{2}} z_2 \bar{w}_1$ defines an inner product on $\mathbb{C}^2$. Using this inner product, find the norm of the vector

$$\mathbf{u} = \begin{pmatrix} -1 \\ \sqrt{2}i \end{pmatrix},$$

and determine all unit vectors which are orthogonal to it.

Solution: Write

$$z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}, \quad w = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}.$$

Define

$$\langle z, w \rangle = z_1 \bar{w}_1 + 2z_2 \bar{w}_2 + \frac{1+i}{\sqrt{2}} z_1 \bar{w}_2 + \frac{1-i}{\sqrt{2}} z_2 \bar{w}_1;$$

then

$$\overline{\langle z, w \rangle} = w_1 \bar{z}_1 + 2w_2 \bar{z}_2 + \frac{1-i}{\sqrt{2}} w_2 \bar{z}_1 + \frac{1+i}{\sqrt{2}} w_1 \bar{z}_2 = \langle w, z \rangle,$$

and $\langle z, w \rangle$ is linear in z. Also

$$\langle z, z \rangle \;=\; |z_1|^2 + 2|z_2|^2 + \frac{1}{\sqrt{2}}(z_1 \bar{z}_2 + z_2 \bar{z}_1) + \frac{i}{\sqrt{2}}(z_1 \bar{z}_2 - z_2 \bar{z}_1) \;=\; |z_1 + \frac{1-i}{\sqrt{2}} z_2|^2 + |z_2|^2.$$

Thus $\langle z, z \rangle \geq 0$ with equality if and only if $z_1 + \frac{1-i}{\sqrt{2}} z_2 = 0, z_2 = 0$, i.e. if and only if $z_1 = 0, z_2 = 0$, i.e. if and only if $z = 0$.

Using this inner product we have

$$\left\| \begin{pmatrix} -1 \\ \sqrt{2}i \end{pmatrix} \right\|^2 = |-1 + \frac{1-i}{\sqrt{2}} \sqrt{2}i|^2 + |\sqrt{2}i|^2 = 3.$$

Thus

$$\left\| \begin{pmatrix} -1 \\ \sqrt{2}i \end{pmatrix} \right\| = \sqrt{3}.$$

The vector z is orthogonal to

$$\begin{pmatrix} -1 \\ \sqrt{2}i \end{pmatrix}$$

if and only if

$$z_1(-1) + 2z_2(-i\sqrt{2}) + \frac{1+i}{\sqrt{2}} z_1(-i\sqrt{2}) + \frac{1-i}{\sqrt{2}} z_2(-1) = 0,$$

i.e. if and only if $z_1 = \frac{(-3+i)}{\sqrt{2}} z_2 = \lambda(-3+i)$, with $z_2 = \sqrt{2}\lambda$. Thus, such z are of the form

$$z = \lambda \begin{pmatrix} -3+i \\ \sqrt{2} \end{pmatrix},$$

for some $\lambda \in \mathbb{C}$. But

$$\left\| \begin{pmatrix} -3+i \\ \sqrt{2} \end{pmatrix} \right\|^2 = 6,$$

so that z is in addition a unit vector if and only if

$$z = \lambda \begin{pmatrix} -3+i \\ \sqrt{2} \end{pmatrix}, \qquad |\lambda|^2 = \frac{1}{6},$$

note that we can write $\lambda = e^{i\phi}/\sqrt{6}$ with $\phi \in [0, 2\pi]$.

33. If the vector space $C[-\pi, \pi]$ of continuous complex valued functions on the interval $[-\pi, \pi]$ is equipped with the inner product defined by

$$(f, g) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, \overline{g(x)}\, dx,$$

where $\overline{g(x)}$ denotes the complex conjugate of $g(x)$, show that

$$e^{i\,n\,x}$$

with $n \in \mathbb{N}$, i.e. $n = 0, 1, 2...$, are mutually orthogonal unit vectors in $C[-\pi, \pi]$.

Solution: We first observe that if k is an integer different from 0, then

$$\int_{-\pi}^{\pi} e^{ikx}\, dx = 0.$$

Also, for n and m integers different from each others

$$\int_{-\pi}^{\pi} e^{i(-n)x} e^{imx}\, dx = \frac{\sin((m-n)\pi)}{(m-n)\pi}$$

which vanishes for all $m, n \in \mathbb{N}$, with $m \neq n$. So $e^{i\,n\,x}$ is orthogonal to $e^{i\,m\,x}$ for $m \neq n$. To check their norms we simply compute

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i(-n)x} e^{inx}\, dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} dx = 1\,,$$

so they are all mutually orthogonal unit vectors in $C[-\pi, \pi]$

34. * Consider $V = \mathbb{R}^n$ with the following application

$$||\mathbf{v}||_\infty = \max_{1 \le i \le n} |v_i|\,,$$

where $\mathbf{v} = (v_1, ..., v_n)$. Prove that $||\cdot||_\infty$ defines a norm on V. [This is called the ℓ_∞-norm, also called sup-norm, on V and it is not induced by an inner product.]

Solution: To prove that $||\cdot||_\infty$ defines a norm we must show absolute homogeneity, triangle inequality and separation of points. First we note that

$$||a \cdot \mathbf{v}||_\infty = \max_{1 \le i \le n} |a v_i| = \max_{1 \le i \le n} |a||v_i| = |a| \max_{1 \le i \le n} |v_i|$$

for every $a \in \mathbb{R}$.

Secondly

$$||\mathbf{v} + \mathbf{w}||_\infty = \max_{1 \le i \le n} |v_i + w_i| \le \max_{1 \le i \le n} |v_i| + |w_i| \le \max_{1 \le i \le n} |v_i| + \max_{1 \le j \le n} |w_j| \le ||\mathbf{v}||_\infty + ||\mathbf{w}||_\infty.$$

Where we used the fact that $|a+b| \le |a| + |b|$ together with $\max_i (a_i + b_i) \le \max_i a_i + \max_j b_j$ since $a_i \le \max_j a_j$ for every $1 \le j \le n$. This proves the triangle inequality.

Finally if $||\mathbf{v}||_\infty = 0$ it means that $\max_{1 \le i \le n} |v_i| = 0$ and since $|v_j| \le \max_{1 \le i \le n} |v_i|$ for every $1 \le j \le n$ we deduce that $v_j = 0$ for every $1 \le j \le n$, i.e. $\mathbf{v} = \mathbf{0}$. The viceversa is obvious.

35. * Consider $V = \mathbb{R}^n$ with the following application

$$||\mathbf{v}||_1 = \sum_{i=1}^{n} |v_i| \,,$$

where $\mathbf{v} = (v_1, ..., v_n)$. Prove that $|| \cdot ||_1$ defines a norm on V. [This is called the ℓ_1-norm on V and it is not induced by an inner product.]

Solution: To prove that $||\cdot||_1$ defines a norm we must show absolute homogeneity, triangle inequality and separation of points. First we note that for every $\lambda \in \mathbb{R}$ we have

$$||\lambda\mathbf{v}||_1 = \sum_{i=1}^{n} |\lambda\, v_i| = |\lambda| \sum_{i=1}^{n} |v_i| = |\lambda| \cdot ||\mathbf{v}||_1 \,,$$

hence homogeneity holds. Secondly we have

$$||\mathbf{v} + \mathbf{w}||_1 = \sum_{i=1}^{n} |v_i + w_i| \leq \sum_{i=1}^{n} |v_i| + |w_i| \leq ||\mathbf{v}||_1 + ||\mathbf{w}||_1 \,,$$

which proves triangle inequality. Finally if $\mathbf{v} = \mathbf{0}$ obviously we have $||\mathbf{v}||_1 = 0$, viceversa if $\mathbf{v} \neq \mathbf{0}$ there is at least one component of $\mathbf{v}$ which does not vanish, say $v_j \neq 0$, and we have

$$||\mathbf{v}||_1 = \sum_{i=1}^{n} |v_i| \geq |v_j| > 0$$

hence separation of points holds and $|| \cdot ||_1$ defines a norm.

36. * Consider the vector space $V = C[a, b]$ of continuous functions on the interval $[a, b]$ with $-\infty < a < b < \infty$, and consider the application

$$||f||_1 = \int_a^b dx\, |f(x)| \,,$$

where $f \in V$. Prove that $|| \cdot ||_1$ defines a norm on V. [This is called the L_1-norm on V and it is not induced by an inner product.]

Solution: To prove that $||\cdot||_1$ defines a norm we must show absolute homogeneity, triangle inequality and separation of points. First we note that for every $\lambda \in \mathbb{R}$ we have

$$||\lambda f||_1 = \int_a^b dx\, |\lambda\, f(x)| = |\lambda| \int_a^b dx\, |f(x)| = |\lambda| \cdot ||f||_1 \,,$$

hence homogeneity holds. Secondly we have

$$||f + g||_1 = \int_a^b dx\, |f(x) + g(x)| \leq \int_a^b dx\, (|f(x)| + |g(x)|) \leq ||f||_1 + ||g||_1 \,,$$

where we used the fact that $|f(x) + g(x)| \leq |f(x)| + |g(x)|$ for every $x \in [a, b]$, thus proving triangle inequality. Finally if $f(x) = 0$ obviously we have$||f||_1 = 0$. Viceversa if $f(x) \neq 0$ there is a point $x_0 \in [a, b]$ such that $|f(x_0)| = c \neq 0$ and from continuity we know that there exists an interval $(x_0 - \epsilon, x_0 + \epsilon) \subset [a, b]$ with $\epsilon > 0$ such that $|f(x)| > c/2$ for every $x \in (x_0 - \epsilon, x_0 + \epsilon)$ this means that

$$||f||_1 \int_a^b dx\, |f(x)| \geq \int_{x_0-\epsilon}^{x_0+\epsilon} dx |f(x)| > \int_{x_0-\epsilon}^{x_0+\epsilon} dx \frac{c}{2} > c\,\epsilon > 0 \,,$$

hence separation of points holds and $|| \cdot ||_1$ defines a norm.

37. Apply Gram-Schmidt orthonormalisation to the basis $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \right\}$ of $\mathbb{R}^3$ equipped with the standard inner product. (But first guess the answer.)

Solution: Let $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \mathbf{v}_3 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$. First note that $\|\mathbf{v}_1\|^2 = 1$, so set $\mathbf{u}_1 = \mathbf{v}_1$.

Next set

$$\tilde{\mathbf{u}}_2 = \mathbf{v}_2 - (\mathbf{v}_2, \mathbf{u}_1)\mathbf{u}_1 = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}.$$

Then $\|\tilde{\mathbf{u}}_2\|^2 = 4$, so set $\mathbf{u}_2 = \frac{\tilde{\mathbf{u}}_2}{2} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$. Finally define

$$\tilde{\mathbf{u}}_3 = \mathbf{v}_3 - (\mathbf{v}_3, \mathbf{u}_1)\mathbf{u}_1 - (\mathbf{v}_3, \mathbf{u}_2)\mathbf{u}_2 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} - 2\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix}.$$

Then $\|\tilde{\mathbf{u}}_3\|^2 = 9$, so set $\mathbf{u}_3 = \frac{\tilde{\mathbf{u}}_3}{3} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$. Thus applying Gram-Schmidt to $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ gives the standard orthonormal basis $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$ of $\mathbb{R}^3$ (as it had to).

38. Apply Gram-Schmidt orthonormalisation to the basis $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$ of $\mathbb{R}^3$ equipped with the inner product defined by $(\mathbf{x}, \mathbf{y}) = 2x_1y_1 + 2x_2y_2 + x_3y_3 - x_2y_3 - x_3y_2$.

Solution: Let

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Since the inner product is given by

$$(\mathbf{x}, \mathbf{y}) = 2x_1y_1 + 2x_2y_2 + x_3y_3 - x_2y_3 - x_3y_2,$$

we first note that $\|\mathbf{v}_1\|^2 = 2$, so set

$$\mathbf{u}_1 = \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

Next note that $(\mathbf{v}_1, \mathbf{v}_2) = 0$ and $\|\mathbf{v}_2\|^2 = 2$, so set $\mathbf{u}_2 = \frac{1}{\sqrt{2}}\mathbf{v}_2$

$$\mathbf{u}_2 = \frac{\mathbf{v}_2}{\|\mathbf{v}_2\|} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

Finally define

$$\tilde{\mathbf{v}}_3 = \mathbf{v}_3 - (\mathbf{v}_3, \mathbf{u}_1)\mathbf{u}_1 - (\mathbf{v}_3, \mathbf{u}_2)\mathbf{u}_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} - 0 - (-\frac{1}{\sqrt{2}}) \begin{pmatrix} 0 \\ 1/\sqrt{2} \\ 0 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix},$$

and note that $\|\tilde{\mathbf{v}}_3\|^2 = 1/2$. Therefore set

$$\mathbf{u}_3 = \frac{\tilde{\mathbf{v}}_3}{\|\tilde{\mathbf{v}}_3\|} = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\2\end{pmatrix}.$$

Thus applying Gram-Schmidt to $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ gives the orthonormal basis

$$\left\{\frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\0\end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\0\end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\2\end{pmatrix}\right\}.$$

39. If $\mathbb{R}^4$ is given the standard inner product, find an orthonormal basis for the subspace determined by the equation $x_1+x_2+x_3+x_4 = 0$, and extend this basis to an orthonormal basis for all of $\mathbb{R}^4$.

Solution: The vectors

$$\begin{pmatrix}1\\-1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\-1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\\-1\end{pmatrix}$$

are a basis for the subspace

$$U = \left\{\begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix} \mid x_1+x_2+x_3+x_4 = 0\right\},$$

since they are linearly independent and dim $U = 4-1 = 3$. Let

$$v_1 = \begin{pmatrix}1\\-1\\0\\0\end{pmatrix}, \quad v_2 = \begin{pmatrix}0\\1\\-1\\0\end{pmatrix}, \quad v_3 = \begin{pmatrix}0\\0\\1\\-1\end{pmatrix}.$$

$$\begin{aligned}
\mathbf{u_1} &= \begin{pmatrix}\frac{\sqrt{2}}{2}\\-\frac{\sqrt{2}}{2}\\0\\0\end{pmatrix},\\
\tilde{\mathbf{v}}_\mathbf{2} &= \mathbf{v_2} - (\mathbf{v_2}, \mathbf{u_1})\mathbf{u_1} = \begin{pmatrix}0\\1\\-1\\0\end{pmatrix} - \begin{pmatrix}-1/2\\1/2\\0\\0\end{pmatrix} = \begin{pmatrix}1/2\\1/2\\-1\\0\end{pmatrix},\\
\mathbf{u_2} &= \begin{pmatrix}\frac{\sqrt{3}\sqrt{2}}{6}\\\frac{\sqrt{3}\sqrt{2}}{6}\\-\frac{\sqrt{3}\sqrt{2}}{3}\\0\end{pmatrix},\\
\tilde{\mathbf{v}}_\mathbf{3} &= \mathbf{v_3} - (\mathbf{v_3}, \mathbf{u_1})\mathbf{u_1} - (\mathbf{v_3}, \mathbf{u_2})\mathbf{u_2} = \begin{pmatrix}0\\0\\1\\-1\end{pmatrix} - 0 - \begin{pmatrix}-1/3\\-1/3\\2/3\\0\end{pmatrix} = \begin{pmatrix}1/3\\1/3\\1/3\\-1\end{pmatrix},\\
\mathbf{u_3} &= \begin{pmatrix}\frac{\sqrt{3}}{6}\\\frac{\sqrt{3}}{6}\\\frac{\sqrt{3}}{6}\\-\frac{\sqrt{3}}{2}\end{pmatrix}.
\end{aligned}$$

The vector

$$\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

is orthogonal to U, so

$$\frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

completes the orthonormal basis of $\mathbb{R}^4$.

40. If $\mathbb{R}^4$ is given the standard inner product, find an orthonormal basis for the subspace determined by the equation $x_1+x_2-x_3-x_4 = 0$, and extend this basis to an orthonormal basis for all of $\mathbb{R}^4$.

Solution: The vectors

$$\begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix}$$

form a basis for the subspace

$$U = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \mid x_1 + x_2 - x_3 - x_4 = 0 \right\},$$

since they are linearly independent and $\dim U = 4 - 1 = 3$. Let

$$v_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \quad v_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix}.$$

$$\mathbf{u_1} = \begin{pmatrix} \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} \\ 0 \\ 0 \end{pmatrix},$$

$$\tilde{\mathbf{v}}_2 = \mathbf{v_2} - (\mathbf{v_2}, \mathbf{u_1})\mathbf{u_1} = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} - \begin{pmatrix} -1/2 \\ 1/2 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1/2 \\ 1/2 \\ 1 \\ 0 \end{pmatrix},$$

$$\mathbf{u_2} = \begin{pmatrix} \frac{\sqrt{3}\sqrt{2}}{6} \\ \frac{\sqrt{3}\sqrt{2}}{6} \\ \frac{\sqrt{3}\sqrt{2}}{3} \\ 0 \end{pmatrix},$$

$$\tilde{\mathbf{v}}_3 = \mathbf{v_3} - (\mathbf{v_3}, \mathbf{u_1})\mathbf{u_1} - (\mathbf{v_3}, \mathbf{u_2})\mathbf{u_2} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix} - 0 - \begin{pmatrix} 1/3 \\ 1/3 \\ 2/3 \\ 0 \end{pmatrix} = \begin{pmatrix} -1/3 \\ -1/3 \\ 1/3 \\ -1 \end{pmatrix},$$

$$\mathbf{u_3} = \begin{pmatrix} -\frac{\sqrt{3}}{6} \\ -\frac{\sqrt{3}}{6} \\ \frac{\sqrt{3}}{6} \\ -\frac{\sqrt{3}}{2} \end{pmatrix}.$$

The vector $\begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}$ is orthogonal to U, so $\frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}$ completes the orthonormal basis of $\mathbb{R}^4$.

41. * Let $(\ ,\ ) : V \times V \mapsto \mathbb{R}$ be an inner product on the n-dimensional vector space V and let U, W denote two vector subspaces of V. Prove the following

    (i) $W = W^{\perp\perp}$

    (ii) $U^{\perp} \cap W^{\perp} = (U + W)^{\perp}$

    (iii) $(U \cap W)^{\perp} = U^{\perp} + W^{\perp}$

    Solution: (i) We simply need to remember that $W^{\perp} = \{v \in V \text{s.t.} (v, w) = 0 \forall w \in W\}$ so the orthogonal complement of the orthogonal complement of W will surely contain W, i.e. $W \subseteq W^{\perp\perp}$. To see that W is indeed equal to $W^{\perp\perp}$ we just need to remember that $\dim V = \dim W + \dim W^{\perp}$.

    (ii) If $w \in U^{\perp} \cap W^{\perp}$ it means that $(w, u) = 0$ for all $u \in U$ and also for all $u \in W$, which means for all $u \in U + W$, hence $w \in (U + W)^{\perp}$, so we have the inclusion $U^{\perp} \cap W^{\perp} \subseteq (U + W)^{\perp}$. To prove the equality we just observe that since trivially $U \subseteq U + W$ and $W \subseteq U + W$ we have $(U + W)^{\perp} \subseteq U^{\perp}$ and $(U + W)^{\perp} \subseteq W^{\perp}$ hence $(U + W)^{\perp} \subseteq U^{\perp} \cap W^{\perp}$ thus proving the equality.

    (iii) If we apply what we have learnt at (ii) to the subspaces $U^{\perp}$ and $W^{\perp}$ we have that $U^{\perp\perp} \cap W^{\perp\perp} = (U^{\perp} + W^{\perp})^{\perp}$ and using (i) we obtain $U \cap W = (U^{\perp} + W^{\perp})^{\perp}$ which reduces to (iii) by taking the orthogonal complement.

42. Let $V = \mathbb{R}[t]_2$ be equipped with the inner product

$$(p, q) = \int_0^1 p(t)q(t)\, dt.$$

Use the Gram-Schmidt process to convert $\{1, t, t^2\}$ into an orthonormal basis $\{g_1, g_2, g_3\}$ for V.

Solution: Let $f_1 = 1$, $f_2 = t$, $f_3 = t^2$. It is clear that $(f_1, f_1) = 1$, so $g_1 = 1$. Now $(f_2, g_1) = 1/2$, so $\tilde{f}_2 = f_2 - (f_2, g_1)g_1 = t - 1/2$. Now $(\tilde{f}_2, \tilde{f}_2) = 1/12$, so $g_2 = \sqrt{3}(2t-1)$. Also $(f_3, g_1) = 1/3$ and $(f_3, g_2) = \sqrt{3}/6$, so

$$\tilde{f}_3 = f_3 - (f_3, g_1)g_1 - (f_3, g_2)g_2 = t^2 - (t - 1/2) - 1/3.$$

Finally $(\tilde{f}_3, \tilde{f}_3) = 1/180$, so $g_3 = \sqrt{5}(6t^2 - 6t + 1)$.

43. Let $V = \mathbb{R}[t]_2$ be equipped with the inner product

$$(f, g) = \int_{-1}^{1} f(t)g(t)\, dt,$$

and let $U = \{f \in V \mid f(-1) = f(1) = 0\}$. Find a basis for the orthogonal complement of U in V.

Solution: Recall that $V = \mathbb{R}[t]_2$ is the vector space of polynomials with real coefficients, of degree at most 2, and that $\dim V = 3$ with $1, t, t^2$ forming a basis. Now note that since $f(-1) = f(1) = 0$ if and only if $(t+1)(t-1)$ divides $f(t)$, it follows that $U = \text{span}\{(t+1)(t-1)\}$. Suppose $g(t) = a_0 + a_1 t + a_2 t^2$. Then $g(t) \in U^\perp$ if and only if $(g(t), (t+1)(t-1)) = 0$, and doing the integral shows that this is equivalent to

$$\frac{-4}{15} a_2 + 0\, a_1 + \frac{-4}{3} a_0 = 0.$$

So, for example taking $a_2 = -5a_0$ and $a_1 \in \mathbb{R}$ in turn, we see that $U^\perp$ has basis $5t^2 - 1, t$.

44. Consider $\mathbb{C}^4$ with the standard inner product. Find an orthonormal basis for the orthogonal complement of the subspace spanned by

$$\begin{pmatrix} 2 \\ 1-\mathrm{i} \\ 0 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 0 \\ \mathrm{i} \\ 3 \end{pmatrix}.$$

Solution: Let

$$U = \text{span}\left\{ \begin{pmatrix} 2 \\ 1-i \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ i \\ 3 \end{pmatrix} \right\}.$$

Then $U^\perp$ is the space of solutions of the system of linear equations

$$\begin{aligned} 2z_1 + (1+i)z_2 \qquad\quad + z_4 &= 0, \\ z_1 \qquad\qquad - iz_3 + 3z_4 &= 0. \end{aligned}$$

Using elementary row operations to bring these equations to row reduced echelon form, we have

$$\begin{pmatrix} 2 & 1+i & 0 & 1 \\ 1 & 0 & -i & 3 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -i & 3 \\ 2 & 1+i & 0 & 1 \end{pmatrix} \to$$
$$\begin{pmatrix} 1 & 0 & -i & 3 \\ 0 & 1+i & 2i & -5 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -i & 3 \\ 0 & 1 & 1+i & -\frac{5}{2}(1-i) \end{pmatrix}.$$

Thus z_3, z_4 are free variables and the solutions are given by

$$\begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix} = \lambda \begin{pmatrix} i \\ -(1+i) \\ 1 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} -3 \\ \frac{5}{2}(1-i) \\ 0 \\ 1 \end{pmatrix}, \qquad \lambda, \mu \in \mathbb{C}.$$

Set

$$\mathbf{v}_1 = \begin{pmatrix} i \\ -(1+i) \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} -3 \\ \frac{5}{2}(1-i) \\ 0 \\ 1 \end{pmatrix}.$$

Then $\|\mathbf{v}_1\|^2 = 1 + 2 + 1 = 4$, so set

$$\mathbf{u}_1 = \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|} = \frac{1}{2} \begin{pmatrix} i \\ -(1+i) \\ 1 \\ 0 \end{pmatrix}.$$

Next set

$$\begin{aligned} \tilde{\mathbf{u}}_2 &= \mathbf{v}_2 - (\mathbf{v}_2, \mathbf{u}_1)\mathbf{u}_1 \\ &= \begin{pmatrix} -3 \\ \frac{5}{2}(1-i) \\ 0 \\ 1 \end{pmatrix} - \frac{1}{4}((-3)(-i) + \frac{5}{2}(1-i)(-1)(1-i)) \begin{pmatrix} i \\ -(1+i) \\ 1 \\ 0 \end{pmatrix} \\ &= \frac{1}{2} \begin{pmatrix} -2 \\ 1-i \\ -4i \\ 2 \end{pmatrix}. \end{aligned}$$

Finally set

$$\mathbf{u}_2 = \frac{\mathbf{v}_2}{\|\mathbf{v}_2\|} = \frac{1}{\sqrt{26}} \begin{pmatrix} -2 \\ 1-i \\ -4i \\ 2 \end{pmatrix}.$$

Then $\{\mathbf{u}_1, \mathbf{u}_2\}$ is the required orthonormal basis.

45. Use the Gram-Schmidt process to show that every invertible $n \times n$ matrix A can be written in the form $A = BC$, where B is an orthogonal matrix and C is upper triangular. Find B, C when

$$A = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 2 & 3 \\ -1 & 1 & 3 \end{pmatrix}.$$

[Hint: Think about the columns of A as vectors.]

Solution: Suppose the columns of A are $\mathbf{v}_1, \ldots, \mathbf{v}_n$. Since A is invertible these vectors form a basis for $\mathbb{R}^n$. Recall that applying the Gram-Schmidt process we replace $\mathbf{v}_1, \ldots, \mathbf{v}_n$ by a set of orthonormal vectors $\mathbf{u}_1, \ldots, \mathbf{u}_n$ and that, for each $k = 1, \ldots, n$, $\text{span}\{\mathbf{u}_1, \ldots, \mathbf{u}_n\} = \text{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ so that, for suitable $a_{ij} \in \mathbb{R}$,

$$\mathbf{v}_k = c_{1k}\mathbf{u}_1 + \ldots + c_{kk}\mathbf{u}_k.$$

But then $A = BC$, where B is the matrix whose columns are $\mathbf{u}_1, \ldots, \mathbf{u}_n$ and C is the upper triangular matrix

$$\begin{pmatrix} c_{11} & c_{12} & \ldots & c_{1n} \\ 0 & c_{22} & \ldots & c_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & c_{nn} \end{pmatrix}.$$

Since $\mathbf{u}_1, \ldots, \mathbf{u}_n$ are mutually orthogonal unit vectors $B^t B = I$ and B is an orthogonal matrix. If

$$A = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 2 & 3 \\ -1 & 1 & 3 \end{pmatrix}$$

then $\mathbf{v}_1 = \big(1, 0, -1\big)$, $\mathbf{v}_2 = \big(0, 2, 1\big)$, $\mathbf{v}_3 = \big(-1, 3, 3\big)$ and applying Gram-Schmidt we find that

$$\mathbf{u}_1 = \frac{1}{\sqrt{2}}\,\big(1, 0, -1\big)\,, \quad \mathbf{u}_2 = \frac{1}{3\sqrt{2}}\,\big(1, 4, 1\big)\,, \quad \mathbf{u}_3 = \frac{1}{3}\,\big(2, -1, 2\big)\,.$$

Also

$$(\mathbf{v}_2, \mathbf{u}_1) = -\frac{1}{\sqrt{2}}, \quad (\mathbf{v}_3, \mathbf{u}_1) = -2\sqrt{2}, \quad (\mathbf{v}_3, \mathbf{u}_2) = \frac{14}{3\sqrt{2}},$$

$$\tilde{\mathbf{v}_2} = \frac{3}{\sqrt{2}}\mathbf{u}_2, \qquad \tilde{\mathbf{v}_2} = \frac{1}{3}\mathbf{u}_3.$$

Thus

$$\mathbf{v}_1 = \sqrt{2}\mathbf{u}_1, \quad \mathbf{v}_2 = -\frac{1}{\sqrt{2}}\mathbf{u}_1 + \frac{3}{\sqrt{2}}\mathbf{u}_2, \quad \mathbf{v}_3 = -2\sqrt{2}\mathbf{u}_1 + \frac{14}{3\sqrt{2}}\mathbf{u}_2 + \frac{1}{3}\mathbf{u}_3.$$

Hence

$$\begin{pmatrix} 1 & 0 & -1 \\ 0 & 2 & 3 \\ -1 & 1 & 3 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{3\sqrt{2}} & \frac{2}{3} \\ 0 & \frac{4}{3\sqrt{2}} & \frac{-1}{3} \\ -\frac{1}{\sqrt{2}} & \frac{1}{3\sqrt{2}} & \frac{2}{3} \end{pmatrix} \begin{pmatrix} \sqrt{2} & -\frac{1}{\sqrt{2}} & -2\sqrt{2} \\ 0 & \frac{3}{\sqrt{2}} & \frac{14}{3\sqrt{2}} \\ 0 & 0 & \frac{1}{3} \end{pmatrix}.$$

46. Let S consist of the following vectors in $\mathbb{R}^4$ with its standard inner product:

$$\mathbf{u}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \quad \mathbf{u}_2 = \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}, \quad \mathbf{u}_3 = \begin{pmatrix} -1 \\ -1 \\ 1 \\ 1 \end{pmatrix}, \quad \mathbf{u}_4 = \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix}.$$

(a) Show that these vectors are all mutually orthogonal to each others, and that they form a basis of $\mathbb{R}^4$;

(b) Write $\mathbf{w} = \big(6, 5, 3, 1\big)$ as a linear combination of $\mathbf{u}_1$, $\mathbf{u}_2$, $\mathbf{u}_3$, $\mathbf{u}_4$.

Solution: (a) It is easy to check that $(\mathbf{u}_i, \mathbf{u}_j) = 0$ for all $i \neq j$. Therefore, S is orthogonal and linearly independent. Since $\mathbb{R}^4$ has dimension 4 and there are 4 vectors, this means it is a basis.

(b) The coordinates of W appear as the projection of $\mathbf{w}$ into the space spanned by the $\mathbf{u}_j$. Hence

$$\begin{aligned} \mathbf{w} &= \frac{(\mathbf{w}, \mathbf{u}_1)}{(\mathbf{u}_1, \mathbf{u}_1)}\,\mathbf{u}_1 + \frac{(\mathbf{w}, \mathbf{u}_2)}{(\mathbf{u}_2, \mathbf{u}_2)}\,\mathbf{u}_2 + \frac{(\mathbf{w}, \mathbf{u}_3)}{(\mathbf{u}_3, \mathbf{u}_3)}\,\mathbf{u}_3 + \frac{(\mathbf{w}, \mathbf{u}_4)}{(\mathbf{u}_4, \mathbf{u}_4)}\,\mathbf{u}_4 \\ &= \frac{15}{4}\,\mathbf{u}_1 + \frac{3}{4}\,\mathbf{u}_2 - \frac{7}{4}\,\mathbf{u}_3 - \frac{1}{4}\,\mathbf{u}_4. \end{aligned}$$

47. Let U be the vector subspace of $\mathbb{R}^4$ defined by

$$x_1 + x_2 + x_3 + x_4 = 0, \qquad x_1 - x_2 + x_3 - x_4 = 0.$$

Find orthonormal bases for U and its orthogonal complement, when $\mathbb{R}^4$ is equipped with the standard inner product.

Solution:

$$\begin{aligned} U &= \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} : x_1 + x_2 + x_3 + x_4 = 0,\ x_1 - x_2 + x_3 - x_4 = 0 \right\} \\ &= \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} : x_1 + x_3 = 0,\ x_2 + x_4 = 0 \right\}. \end{aligned}$$

Thus a basis for U is

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix} \right\}.$$

These vectors are clearly orthogonal and have length $\sqrt{2}$. Thus

$$\left\{ \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{-1}{\sqrt{2}} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ 0 \\ \frac{-1}{\sqrt{2}} \end{pmatrix} \right\}$$

is an orthonormal basis for U. The orthogonal complement $U^{\perp}$ of U is spanned by

$$\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

so that an orthonormal basis for $U^{\perp}$ is

$$\left\{ \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix} \right\}.$$

48. In $\mathbb{R}^4$ equipped with the standard inner product, find the projection of $\mathbf{a} = (1, 2, 0, -1)$ on the plane V spanned by $\mathbf{v}_1 = (1, 0, 0, 1)$ and $\mathbf{v}_2 = (1, 1, 2, 0)$. (First construct an orthonormal basis for V.)

Solution: We need an orthonormal basis for the plane V. So

$$\mathbf{u}_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix},$$

$$\widetilde{\mathbf{v}}_2 = \mathbf{v_2} - (\mathbf{v_2}, \mathbf{u_1})\mathbf{u_1} = \begin{pmatrix} 1 \\ 1 \\ 2 \\ 0 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 \\ 2 \\ 4 \\ -1 \end{pmatrix},$$

and

$$\mathbf{u}_2 = \frac{1}{\sqrt{22}}\begin{pmatrix} 1 \\ 2 \\ 4 \\ -1 \end{pmatrix}.$$

Then the V-component of $\mathbf{a}$ is

$$(\mathbf{a}, \mathbf{u}_1)\mathbf{u}_1 + (\mathbf{a}, \mathbf{u}_2)\mathbf{u}_2 = \frac{3}{11}\begin{pmatrix} 1 \\ 2 \\ 4 \\ -1 \end{pmatrix}.$$

49. Let U be a vector subspace of $\mathbb{R}^n$, equipped with the standard inner product, and suppose that $\mathbf{v}$ is an element of $\mathbb{R}^n$ not in U. Then we know that there is a unique point $\mathbf{u}_0$ in U such that, for all $\mathbf{u} \in U$, we have $\|\mathbf{v} - \mathbf{u}_0\| \leq \|\mathbf{v} - \mathbf{u}\|$; and $\mathbf{v} - \mathbf{u}_0$ is orthogonal to U. Find $\mathbf{u}_0$ if U is the plane $x_1 - 2x_2 + 2x_3 = 0$ in $\mathbb{R}^3$ and $\mathbf{v} = \big(1, 0, 0\big)$.

Solution: We need an orthonormal basis $\{\mathbf{u_1}, \mathbf{u_2}\}$ for the plane U. Start with any two vectors in U, say $\mathbf{w_1} = (0, 1, 1)$ and $\mathbf{w_2} = (2, 1, 0)$. Then $\mathbf{u_1} = (0, 1, 1)/\sqrt{2}$,

$$\widetilde{\mathbf{w}}_2 = \mathbf{w_2} - (\mathbf{w_2}, \mathbf{u_1})\mathbf{u_1} = (2, \frac{1}{2}, -\frac{1}{2})$$

and $\mathbf{u_2} = (4, 1, -1)/(3\sqrt{2})$. Now $\mathbf{u_0}$ is the projection of $\mathbf{v}$ onto U, namely

$$\mathbf{u_0} = (\mathbf{v}, \mathbf{u}_1)\mathbf{u}_1 + (\mathbf{v}, \mathbf{u}_2)\mathbf{u}_2 = \frac{2}{9}(4, 1, -1).$$

50. Let V be the space $C[-1, 1]$ equipped with the inner product $(f, g) = \int_{-1}^{1} f(t)g(t)\,dt$. Let S be the subspace of V spanned by $\{1, t, t^2\}$. Construct an orthonormal basis $\{g_1, g_2, g_3\}$ for S, and find the function $h \in S$ closest to t^3.

Solution: If $V = C[-1, 1]$, $\mathbf{v} = t^3$ and $S = \operatorname{span}\{1, t, t^2\}$, then we first apply Gram-Schmidt orthonormalization to $\{1, t, t^2\}$ to obtain an orthonormal basis for S. Let us write $f_1 = 1$, $f_2 = t$, $f_3 = t^2$. Then $\|f_1\|^2 = \int_{-1}^{1} dt = 2$, so set $g_1 = \frac{1}{\sqrt{2}}$. Now note that

$$(f_2, f_1) = \int_{-1}^{1} t\,dt = 0 \quad \text{and} \quad \|f_2\|^2 = \int_{-1}^{1} t^2\,dt = \frac{2}{3},$$

so set

$$g_2 = \frac{f_2}{\|f_2\|} = \sqrt{\frac{3}{2}}\,t.$$

Finally set

$$\tilde{f}_3 = f_3 - (f_3, g_1)g_1 - (f_3, g_2)g_2.$$

Since

$$(f_3, f_1) = \int_{-1}^{1} t^2\,dt = \frac{2}{3} \quad \text{and} \quad (f_3, f_2) = \int_{-1}^{1} t^3\,dt = 0,$$

we have

$$\tilde{f}_3 = t^2 - \frac{2}{3}.\frac{1}{2} = t^2 - \frac{1}{3}.$$

Then

$$\|\tilde{f}_3\|^2 = \int_{-1}^{1} (t^2 - \frac{1}{3})^2\, dt = \int_{-1}^{1} (t^4 - \frac{2}{3}t^2 + \frac{1}{9})\, dt = \frac{2}{5} - \frac{4}{9} + \frac{2}{9} = \frac{8}{45},$$

so set

$$g_3 = \frac{\tilde{f}_3}{\|\tilde{f}_3\|} = \frac{3\sqrt{5}}{2\sqrt{2}}\,(t^2 - \frac{1}{3}).$$

The function in S closest to t^3 is then

$$h = (t^3, g_1)g_1 + (t^3, g_2)g_2 + (t^3, g_3)g_3.$$

But

$$\begin{aligned} (t^3, g_1) &= \frac{1}{\sqrt{2}} \int_{-1}^{1} t^3\, dt = 0 \\ (t^3, g_2) &= \sqrt{\frac{3}{2}} \int_{-1}^{1} t^4\, dt = \frac{2}{5}\sqrt{\frac{2}{3}} \\ (t^3, g_3) &= \frac{3\sqrt{5}}{2\sqrt{2}} \int_{-1}^{1} (t^5 - \frac{1}{3}t^3)\, dt = 0. \end{aligned}$$

Thus the function $h \in S$ closest to t^3 is

$$h(t) = \frac{2}{5}\sqrt{\frac{3}{2}}\sqrt{\frac{3}{2}}\, t = \frac{3}{5}\, t.$$

51. Find the point in the 3-plane $2x_1 - x_2 + 2x_3 + 2x_4 = 0$ in $\mathbb{R}^4$, with standard Euclidean inner product, which is nearest to the point $\mathbf{a} = (1, 2, 1, 2)$.

Solution: The 3-plane U defined by $2x_1 - x_2 + 2x_3 + 2x_4 = 0$ has normal

$$\begin{pmatrix} 2 \\ -1 \\ 2 \\ 2 \end{pmatrix}$$

and thus unit normal

$$\mathbf{e} = \frac{1}{\sqrt{13}} \begin{pmatrix} 2 \\ -1 \\ 2 \\ 2 \end{pmatrix}.$$

Thus $U^\perp = \operatorname{span}\{\mathbf{e}\}$. Given $\mathbf{v} \in V$, we may write $\mathbf{v}$ in the form

$$\mathbf{v} = \mathbf{u} + \tilde{\mathbf{u}} \qquad \text{for unique } \mathbf{u} \in U,\ \tilde{\mathbf{u}} \in U^\perp,$$

and $\tilde{\mathbf{u}} = (\mathbf{v}, \mathbf{e})\mathbf{e}$. Thus the nearest point in U to $\mathbf{v}$ is $\mathbf{u} = \mathbf{v} - (\mathbf{v}, \mathbf{e})\mathbf{e}$. In particular, when

$$\mathbf{v} = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 2 \end{pmatrix}$$

we have

$$\begin{aligned} \mathbf{u} &= \begin{pmatrix} 1 \\ 2 \\ 1 \\ 2 \end{pmatrix} - \frac{1}{13}(1 \cdot 2 + 2 \cdot (-1) + 1 \cdot 2 + 2 \cdot 2) \begin{pmatrix} 2 \\ -1 \\ 2 \\ 2 \end{pmatrix} \\ &= \begin{pmatrix} 1 \\ 2 \\ 1 \\ 2 \end{pmatrix} - \frac{6}{13} \begin{pmatrix} 2 \\ -1 \\ 2 \\ 2 \end{pmatrix} = \frac{1}{13} \begin{pmatrix} 1 \\ 32 \\ 1 \\ 14 \end{pmatrix}. \end{aligned}$$

52. Find the point in the 2-plane in $\mathbb{R}^4$ defined by $x_1+x_2+x_3+x_4=0$, $x_1-x_2+x_3-x_4=0$, which is nearest to the point $\mathbf{v}=(1,2,1,2)$ with standard Euclidean inner product.

Solution: Let U denote the plane defined in the question. Clearly

$$\left\{\begin{pmatrix}1\\0\\-1\\0\end{pmatrix},\begin{pmatrix}0\\1\\0\\-1\end{pmatrix}\right\}$$

is a basis for U. We need an orthonormal basis for U. Clearly these two vectors are orthogonal and they each have length $\sqrt{2}$. Hence an orthonormal basis is

$$\mathbf{u}_1=\frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\-1\\0\end{pmatrix},\qquad \mathbf{u}_2=\frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\0\\-1\end{pmatrix}.$$

The point nearest of U nearest to $\mathbf{v}$ is the orthogonal projection $\mathbf{u}$ of $\mathbf{v}$ onto U. This is given by

$$\mathbf{u}=(\mathbf{v},\mathbf{u}_1)\,\mathbf{u}_1+(\mathbf{v},\mathbf{u}_2)\,\mathbf{u}_2.$$

It is clear that $(\mathbf{v},\mathbf{u}_1)=0$ and $(\mathbf{v},\mathbf{u}_2)=0$ and so

$$\mathbf{u}=\begin{pmatrix}0\\0\\0\\0\end{pmatrix}.$$