Linear Algebra 1, Solutions to exercises 53 to 78.
Epiphany 21/22.

53. If the vector space $C[-1,1]$ of continuous real valued functions on the interval $[-1,1]$ is equipped with the inner product defined by $(f,g) = \int_{-1}^{1} f(t)g(t)\,dt$, find the linear polynomial $g(t)$ nearest to $f(t) = e^t$.

Solution: Note that the functions $1, t \in C[-1,1]$ are orthogonal and they form a basis for the subspace S of linear polynomials in $C[-1,1]$. Thus the linear polynomial closest to e^t is given by

$$\frac{(1,e^t)}{\|1\|^2}\,1 + \frac{(t,e^t)}{\|t\|^2}\,t.$$

But

$$\begin{aligned}(1,e^t) &= \int_{-1}^{1} e^t dt = e - e^{-1} = 2\sinh 1 \\ (t,e^t) &= \int_{-1}^{1} te^t dt = [te^t]_{-1}^{1} - \int_{-1}^{1} e^t dt = e + e^{-1} - (e - e^{-1}) = 2e^{-1},\end{aligned}$$

and $\|1\|^2 = 2$, $\|t\|^2 = \frac{2}{3}$. Thus the linear polynomial which is closest to e^t is $g(t) = \sinh 1 + 3e^{-1}\,t$.

54. Find an orthogonal matrix P such that P^tAP is diagonal, when

(i) $A = \begin{pmatrix} 11 & 8 \\ 8 & -1 \end{pmatrix}$, (ii) $A = \begin{pmatrix} 1 & 0 & -4 \\ 0 & 5 & 4 \\ -4 & 4 & 3 \end{pmatrix}$, (iii) $A = \begin{pmatrix} 5 & 7 & 7 \\ 7 & 5 & -7 \\ 7 & -7 & 5 \end{pmatrix}$.

Solution: (i) The characteristic polynomial of the matrix A is

$$p_A(t) = \det(A - tI) = t^2 - 10t - 75 = (t-15)(t+5),$$

so the eigenvalues are -5 and 15. We now calculate the corresponding eigenvectors:

$\lambda = -5$: $(A + 5I)\mathbf{u}_1 = \mathbf{0}$ has normalized solution $\mathbf{u}_2 = 5^{-1/2}(1,-2)^t$.

$\lambda = 15$: $(A - 15I)\mathbf{u}_2 = \mathbf{0}$ has normalized solution $\mathbf{u}_2 = 5^{-1/2}(2,1)^t$.

Thus $P^{-1}AP = \mathrm{diag}(-5,15)$ if

$$P = \frac{1}{\sqrt{5}}\begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}.$$

(ii) The characteristic polynomial of the matrix A is

$$\begin{aligned}\det(\lambda I - A) &= \det\begin{pmatrix} \lambda-1 & 0 & 4 \\ 0 & \lambda-5 & -4 \\ 4 & -4 & \lambda-3 \end{pmatrix} \\ &= (\lambda-1)\{(\lambda-5)(\lambda-3) - 16\} - 16(\lambda-5) \\ &= (\lambda-3)(\lambda+3)(\lambda-9),\end{aligned}$$

so the eigenvalues are $3, -3, 9$. We now calculate the corresponding eigenvectors:

$\lambda = 3$:

$$\begin{array}{rrrcl} 2x_1 & & +4x_3 & = & 0, \\ & -2x_2 & -4x_3 & = & 0, \\ 4x_1 & -4x_2 & & = 0. & \end{array}$$

Thus

$$\mathbf{u}_1 = \frac{1}{3}\begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix}$$

is a unit eigenvector.

$\lambda = -3$:

$$\begin{array}{rrrcl} -4x_1 & & +4x_3 & = & 0, \\ & -8x_2 & -4x_3 & = & 0, \\ 4x_1 & -4x_2 & -6x_3 & = & 0. \end{array}$$

Thus

$$\mathbf{u}_2 = \frac{1}{3}\begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}$$

is a unit eigenvector.

$\lambda = 9$:

$$\begin{array}{rrrcl} 8x_1 & & +4x_3 & = & 0, \\ & 4x_2 & -4x_3 & = & 0, \\ 4x_1 & -4x_2 & +6x_3 & = & 0. \end{array}$$

Thus

$$\mathbf{u}_3 = \frac{1}{3}\begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix}$$

is a unit eigenvector.

Thus if

$$P = \frac{1}{3}\begin{pmatrix} 2 & 2 & -1 \\ 2 & -1 & 2 \\ -1 & 2 & 2 \end{pmatrix}$$

then

$$P^{-1}AP = \begin{pmatrix} 3 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 9 \end{pmatrix}.$$

(iii) The characteristic polynomial of the matrix A is

$$\begin{aligned} \det(\lambda I - A) &= \det\begin{pmatrix} \lambda - 5 & -7 & -7 \\ -7 & \lambda - 5 & 7 \\ -7 & 7 & \lambda - 5 \end{pmatrix} \\ &= \det\begin{pmatrix} \lambda - 12 & 0 & \lambda - 12 \\ 0 & \lambda - 12 & -(\lambda - 12) \\ -7 & 7 & \lambda - 5 \end{pmatrix} \\ &= (\lambda - 12)^2(\lambda + 9). \end{aligned}$$

Thus the eigenvalues are $\lambda = 12, 12, -9$.

Corresponding eigenvectors:

$\lambda = 12$:

$$7x_1 - 7x_2 - 7x_3 = 0.$$

Clearly

$$\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

is an eigenvector, so if

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

is another eigenvector, mutually orthogonal to the first, then

$$\begin{array}{rrrcl} 7x & -7y & -7z & = & 0, \\ & y & -z & = & 0 \end{array}$$

from which we obtain

$$\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$$

as a solution. Thus we have mutually orthogonal unit eigenvectors:

$$\mathbf{u}_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \quad \mathbf{u}_2 = \frac{1}{\sqrt{6}} \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}.$$

$\lambda = -9$:

$$\begin{array}{rrrcl} -14x_1 & -7x_2 & -7x_3 & = & 0, \\ -7x_1 & -14x_2 & +7x_3 & = & 0, \\ -7x_1 & +7x_2 & -14x_3 & = & 0. \end{array}$$

Unit eigenvector:

$$\mathbf{u}_3 = \frac{1}{\sqrt{3}} \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}.$$

Thus if

$$P = \begin{pmatrix} 0 & \frac{2}{\sqrt{6}} & -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{pmatrix}$$

then

$$P^{-1}AP = \begin{pmatrix} 12 & 0 & 0 \\ 0 & 12 & 0 \\ 0 & 0 & -9 \end{pmatrix}.$$

55. (i) Let A be a real symmetric matrix. Show that there exists a real symmetric matrix B such that $B^2 = A$ if and only if the eigenvalues of A are all non-negative.

(ii) Find a real symmetric matrix C such that

$$C^5 = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

Solution: (i) If B is a real symmetric matrix then there exists an orthogonal matrix P such that

$$P^{-1}BP = D \qquad \text{where } D = \operatorname{diag}(\lambda_1, \ldots, \lambda_n).$$

Thus, if $B^2 = A$ then

$$P^{-1}AP = P^{-1}B^2P = D^2 = \operatorname{diag}(\lambda_1^2, \ldots, \lambda_n^2).$$

But then $\lambda_1^2, \ldots, \lambda_n^2$ are the eigenvalues of A and they are all non-negative.
Conversely, suppose that A is a real symmetric matrix all of whose eigenvalues $\mu_1, \ldots, \mu_n$ are non-negative. Thus we may write $\mu_1 = \lambda_1^2, \ldots, \mu_n = \lambda_n^2$ for some real numbers $\lambda_1, \ldots, \lambda_n$. Since A is a real symmetric matrix there exists an orthogonal matrix P such that

$$P^{-1}AP = \operatorname{diag}(\mu_1, \ldots, \mu_n) = D^2 \quad \text{where } D = \operatorname{diag}(\lambda_1, \ldots, \lambda_n).$$

But then writing $B = PDP^{-1}$ we have $B^2 = PD^2P^{-1} = A$.
(ii) The characteristic polynomial of

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

is given by

$$\det \begin{pmatrix} \lambda & -1 & -1 \\ -1 & \lambda & -1 \\ -1 & -1 & \lambda \end{pmatrix} = (\lambda - 2)(\lambda + 1)^2.$$

Thus the eigenvalues are $\lambda = 2, -1, -1$.
Corresponding eigenvectors:

$\lambda = 2$:

$$\begin{array}{rrrcl} 2x_1 & -x_2 & -x_3 & = & 0, \\ -x_1 & +2x_2 & -x_3 & = & 0, \\ -x_1 & -x_2 & +2x_3 & = & 0. \end{array}$$

Unit eigenvector:

$$\mathbf{u}_1 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

$\lambda = -1$:

$$x_1 + x_2 + x_3 = 0.$$

Mutually orthogonal unit eigenvectors:

$$\mathbf{u}_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \qquad \mathbf{u}_3 = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}.$$

Thus if

$$P = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & -\frac{2}{\sqrt{6}} \end{pmatrix}$$

then

$$P^{-1}AP = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Now define

$$C = P \begin{pmatrix} \sqrt[5]{2} & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} P^{-1},$$

so that $C^5 = A$. Then

$$\begin{aligned} C &= \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & -\frac{2}{\sqrt{6}} \end{pmatrix} \begin{pmatrix} \sqrt[5]{2} & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} \end{pmatrix} \\ &= \frac{1}{3} \begin{pmatrix} \sqrt[5]{2}-2 & \sqrt[5]{2}+1 & \sqrt[5]{2}+1 \\ \sqrt[5]{2}+1 & \sqrt[5]{2}-2 & \sqrt[5]{2}+1 \\ \sqrt[5]{2}+1 & \sqrt[5]{2}+1 & \sqrt[5]{2}-2 \end{pmatrix}. \end{aligned}$$

Some additional starred exercises

56. * Let V be an n-dimensional vector space over the reals and W a subspace of V with dimension $m \leq n$. Consider the set of linear transformations

$$U = \{T : V \mapsto V \text{ s.t. } T \text{ is linear and } \forall \mathbf{w} \in W \, \exists \, \alpha \in \mathbb{R} : \, T(\mathbf{w}) = \alpha \mathbf{w}\}.$$

Show that U is a vector subspace of $M_n(\mathbb{R})$ and compute its dimension.

Solution: To show that U is a subspace of $M_n(\mathbb{R})$ we need to show that if $T, S \in U$ then $T+S \in U$ and that if $T \in U$ then $\lambda T \in U$ for all $\lambda \in \mathbb{R}$.

If $T, S \in U$ then we have that for all $\mathbf{w} \in W$ there exist $\alpha, \beta \in \mathbb{R}$ such that $T(\mathbf{w}) = \alpha \mathbf{w}$ and $S(\mathbf{w}) = \beta \mathbf{w}$. This implies that $(T+S)(\mathbf{w}) = T(\mathbf{w}) + S(\mathbf{w}) = (\alpha + \beta)\mathbf{w}$ so $T + S \in U$.

Similarly if $T \in U$ and $\lambda \in \mathbb{R}$ we have that for all $\mathbf{w} \in W$ there exists $\alpha \in \mathbb{R}$ such that $T(\mathbf{w}) = \alpha \mathbf{w}$ so that $(\lambda T)(\mathbf{w}) = \lambda T(\mathbf{w}) = \lambda \alpha \mathbf{w}$ so $\lambda T \in U$. Hence U is a vector subspace of the space of linear transformations from V to V, i.e. the vector space of $n \times n$ real matrices.

To compute its dimension we first fix a basis of the subspace W given by $\{\mathbf{w}_1, ..., \mathbf{w}_m\}$ and complete to a basis of V with the remaining $n-m$ linearly independent vectors $\{\mathbf{v}_1, ..., \mathbf{v}_{n-m}\}$. If $T \in U$ we must have that $T(\mathbf{w_i}) = \alpha_i \mathbf{w_i}$ with $i = 1, ..., m$ for some real numbers α_i.

Now, choose $\mathbf{w} = \mathbf{w}_1 + \ldots + \mathbf{w}_m$. Then for any fixed T there exists some α such that

$$T(\mathbf{w}) = \alpha \mathbf{w} = \alpha \mathbf{w}_1 + \ldots + \alpha \mathbf{w}_m.$$

On the other hand, by linearity,

$$T(\mathbf{w}) = T(\mathbf{w}_1) + \ldots + T(\mathbf{w}_m) = \alpha_1 \mathbf{w}_1 + \ldots + \alpha_m \mathbf{w}_m.$$

Comparing both expression implies, since the $\mathbf{w}_i$ are linearly independent, that all α need to be equal.

Since T is linear once we have specified the value of α we know the action of T on any vector $\mathbf{w} \in W$. Then need to consider the action of T on the remaining $n-m$ vectors: $T(\mathbf{v_i}) = \mathbf{u}_i$ where $i = 1, ..., n-m$ and $\mathbf{u}_i \in V$, so for each of the $(n-m)$ remaining vectors $\mathbf{v_i}$ we need to specify the n coordinates of the vector $\mathbf{u}_i$, hence we need to specify $(n-m)n$ remaining entries for T. In total to specify T we must give $1+(n-m)n$ numbers, so that dim$U = 1 + (n-m)n$.

Using the basis $\{\mathbf{w}_1, ..., \mathbf{w}_m, \mathbf{v}_1, ..., \mathbf{v}_{n-m}\}$ the matrix form of T is

$$T = \begin{pmatrix} \alpha & 0 & \ldots & 0 & u_{1,m+1} & \ldots & u_{1,n} \\ 0 & \alpha & 0 & \vdots & u_{2,m+1} & \ldots & u_{2,n} \\ \vdots & 0 & \ddots & 0 & \vdots & & \vdots \\ 0 & \ldots & 0 & \alpha & & & \\ 0 & \ldots & 0 & 0 & \vdots & & \vdots \\ \vdots & \ddots & & \vdots & u_{n-1,m+1} & \ldots & u_{n-1,n} \\ 0 & \ldots & & 0 & u_{n,m+1} & \ldots & u_{n,n} \end{pmatrix}$$

57. * Let V be a real vector space with dimension n and $T : V \mapsto V$ a linear transformation.
   i) If $T^2 = 0$ show that $\dim \operatorname{Ker} T \geq \dim V/2$.
   ii) Show that $T^2 = 0$ and $\dim \operatorname{Ker} T = n/2$ and $\dim V = n$ is even if and only if $\operatorname{Ker} T = \operatorname{Im} T$.

   Solution: i) If $\mathbf{v} \in \mathsf{Im}\, T$ it means that there exists $\mathbf{w} \in V$ such that $T(\mathbf{w}) = \mathbf{v}$ but since $T^2 = 0$ we must have that $T(\mathbf{v}) = T(T(\mathbf{w})) = T^2(\mathbf{w}) = 0$ so $\mathbf{v} \in \mathsf{Ker}\, T$, hence $\mathsf{Im}\, T \subseteq \mathsf{Ker}\, T$ which in particular means $\dim \mathsf{Im}\, T \leq \dim \mathsf{Ker}\, T$. However we know from the rank-nullity theorem

   $$\dim V = \dim \mathsf{Im}\, T + \dim \mathsf{Ker}\, T \leq 2\dim \mathsf{Ker}\, T$$

   so that we must have $\dim \mathsf{Ker} T \geq \dim V/2$ as requested.

   ii) Suppose that $T^2 = 0$ and $\dim \mathsf{Ker} T = n/2$ and $\dim V = n$. From what we have proved at point i) we must have from the rank nullity theorem that $\dim \mathsf{Im}\, T = \dim V - \dim \mathsf{Ker}\, T = n/2$ so that $\dim \mathsf{Ker} T = \dim \mathsf{Im}\, T = n/2$. However we know from point i) that $\mathsf{Im}\, T \subseteq \mathsf{Ker}\, T$ but if the kernel and the image must have equal dimension we can only have $\mathsf{Im}\, T = \mathsf{Ker}\, T$.

   Viceversa if $\mathsf{Im}\, T = \mathsf{Ker}\, T$ we have obviously that $\dim \mathsf{Ker}\, T = \dim \mathsf{Im}\, T$ and from the rank-nullity theorem $n = \dim V = 2 \dim \mathsf{Ker}\, T$ so that n must be even and $\dim \mathsf{Ker}\, T = n/2$. Furthermore we have that for every vector $\mathbf{w} \in V$ that either $T(\mathbf{w}) = 0$, i.e. $\mathbf{w} \in \mathsf{Ker}\, T$, so that obviously $T^2(\mathbf{w}) = T(\mathbf{0}) = 0$ or alternatively $T(\mathbf{w}) = \mathbf{v} \neq \mathbf{0}$ which means that $\mathbf{v} \in \mathsf{Im}\, T$, however since by hypothesis $\mathsf{Im}\, T = \mathsf{Ker}\, T$ we have also $\mathbf{v} \in \mathsf{Ker}\, T$ so that $0 = T(\mathbf{v}) = T(T(\mathbf{w})) = T^2(\mathbf{w})$ hence $T^2 = 0$.

58. * [Nilpotency] A square matrix N is said to be nilpotent if $N^k = 0$ for some positive integer $k \in \mathbb{N}$. The smallest such k such that $N^{k-1} \neq 0$ but $N^k = 0$ is called the degree of nilpotency of N. Show that if N is nilpotent with degree k, then the matrix $A = I + N$ is invertible and its inverse is given by

   $$A^{-1} = I - N + N^2 - N^3 + ... + (-1)^{k-1} N^{k-1} \,.$$

   Solution: We just need to compute

   $$\begin{aligned}
   &(I + N)(I - N + N^2 - N^3 + ... + (-1)^{k-1}N^{k-1}) \\
   &= (I - N + N^2 - N^3 + ... + (-1)^{k-1}N^{k-1}) + N(I - N + N^2 - N^3 + ... + (-1)^{k-1}N^{k-1}) \\
   &= I - N + N^2 - N^3 + ... + (-1)^{k-1}N^{k-1} + N - N^2 + ... + (-1)^{k-2}N^{k-1} + (-1)^{k-1}N^k \\
   &= I + (-1)^{k-1}N^k = I \,,
   \end{aligned}$$

   where in the final step we use the fact that N is nilpotent, i.e. $N^k = 0$.

59. * Show that if N is a nilpotent matrix and λ is an eigenvalue of N with eigenvector $\mathbf{v} \neq \mathbf{0}$ then necessarily $\lambda = 0$. In particular deduce that the characteristic polynomial of every $n \times n$ nilpotent matrix N is $p_N(t) = (-t)^n$. [i.e. a nilpotent matrix has only vanishing eigenvalues]

   Solution: Since N is nilpotent we must have that

   $$N^k \mathbf{v} = \mathbf{0} \,,$$

   however we also have

   $$N^k \mathbf{v} = \mathbf{N^{k-1}}(\mathbf{N}\mathbf{v}) = \lambda \mathbf{N^{k-1}} \mathbf{v} = ... = \lambda^{\mathbf{k}} \mathbf{v} \,,$$

so necessarily $\lambda = 0$.

For the second part we know that $p_N(t)$ is a degree n polynomial whose roots are the eigenvalues of N and we have just proved that the only eigenvalue of N is 0 so we can write the characteristic polynomial of N as $p_N(t) = \prod_{i=1}^n (\lambda_i - t) = (-t)^n$ since all the eigenvalues λ_i are vanishing.

60. * Show that if N is nilpotent than $\det(I + N) = 1$. Viceversa if N is a matrix such that $\det(I + xN) = 1$ for every x then show that N is nilpotent. [Hint: use the previous exercise].

Solution: We know that $p_N(t) = (-t)^n$ where N is a $n \times n$ nilpotent matrix and $p_N(t) = \det(N - tI)$ so we simply have that $p_N(-1) = \det(N + I) = 1^n = 1$.

For the second part if N is an $n \times n$ matrix we must have that

$$1 = \det(I + xN) = x^n \det(N - (-x^{-1})I) = x^n p_N(-x^{-1}),$$

so that $p_N(-x^{-1}) = x^{-n}$ or equivalently $p_N(t) = (-t)^n$ and from Cayley-Hamilton we have $N^n = 0$, so N is nilpotent. Note that n is not necessarily the nilpotency index of N.

61. * [Quadratic forms] Let $V = \mathbb{R}^2$ with a bilinear form $Q(\mathbf{v}, \mathbf{w})$ which we assume symmetric, i.e. $Q(\mathbf{v}, \mathbf{w}) = Q(\mathbf{w}, \mathbf{v})$, but not necessarily positive definite. The function $\phi_Q : V \mapsto \mathbb{R}$ defined by $\phi_Q(\mathbf{v}) = Q(\mathbf{v}, \mathbf{v})$ is called the (associated) *quadratic form*, note: it is called quadratic because $\phi_Q(\lambda \mathbf{v}) = \lambda^2 \phi_Q(\mathbf{v})$. Show that in terms of the coordinates $\mathbf{v} = (x, y)^t$, the set of points satisfying $\phi_Q(\mathbf{v}) = Q(\mathbf{v}, \mathbf{v}) = 1$ is either describing an ellipse, an hyperbola, two parallel lines or the empty set.

Solution: Using the matrix version we have that

$$Q(\mathbf{v}, \mathbf{v}) = \mathbf{v}^t A \mathbf{v} = (x, y) \begin{pmatrix} a & b \\ b & c \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = ax^2 + 2bxy + cy^2 ,$$

where we used the fact the the bilinear form Q is symmetric, translating into the fact that its matrix representation is a symmetric matrix.

We know that since A is a symmetric matrix we can diagonalize it with an orthogonal matrix R, such that $R^{-1} = R^t$, so $R^t A R = \mathrm{diag}(a_1, a_2)$ with a_1, a_2 the eigenvalues of A. Let us define

$$\begin{pmatrix} x \\ y \end{pmatrix} = R \begin{pmatrix} X \\ Y \end{pmatrix} ,$$

so we have

$$\begin{aligned} Q(\mathbf{v}, \mathbf{v}) &= (x, y) \begin{pmatrix} a & b \\ b & c \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = (X, Y) R^t \begin{pmatrix} a & b \\ b & c \end{pmatrix} R \begin{pmatrix} X \\ Y \end{pmatrix} \\ &= (X, Y) \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} = a_1 X^2 + a_2 Y^2 \end{aligned}$$

To understand the set of solutions to the equation $a_1 X^2 + a_2 Y^2 = 1$ we need to distinguish few cases.

- $a_1 > 0$, $a_2 > 0$ describes an ellipse and when $a_1 = a_2$ a circle which is a special case;
- $a_1 > 0$, $a_2 < 0$ describes an hyperbola;
- $a_1 < 0$, $a_2 > 0$ describes an hyperbola;
- $a_1 < 0$, $a_2 < 0$ has no solution, so we get the empty set;
- $a_1 > 0$, $a_2 = 0$ describes two vertical parallel lines $X = \pm\frac{1}{\sqrt{a_1}}$;

- $a_1 = 0$, $a_1 > 0$ describes two horizzontal parallel lines $Y = \pm\frac{1}{\sqrt{a_2}}$;
- $a_1 < 0$, $a_2 = 0$ has no solution, so we get the empty set;
- $a_1 = 0$, $a_2 < 0$ has no solution, so we get the empty set.

62. * [Dual space] Let V be an n-dimensional real vector space and consider the space $V^* = \{\phi : V \mapsto \mathbb{R}\,, \text{s.t. } \phi \text{ is linear}\}$. Show that V^* is a real vector space called the dual space of V. Show that if $\{\mathbf{v_1}, ..., \mathbf{v_n}\}$ is a basis for V then the set of $\phi^{(i)} \in V^*$, $i = 1, ..., n$, defined by $\phi^{(i)}(\mathbf{v_j}) = \delta^i_j$ span a basis for V^* called the dual basis, where δ^i_j is the Kronecker delta, so that V^* has exactly the same dimension as V.

Solution: The fact that V^* is a real vector space is almost immediate from its definition since for every $\phi, \psi \in V^*$ and $\alpha, \beta \in \mathbb{R}$ we have that obviously $\alpha\phi + \beta\psi$ is also a linear map from V to $\mathbb{R}$ simply by

$$\begin{aligned}(\alpha\phi + \beta\psi)(a\mathbf{v} + b\mathbf{w}) &= \alpha\phi(a\mathbf{v} + b\mathbf{w}) + \beta\psi(a\mathbf{v} + b\mathbf{w})\\ &= a\alpha\phi(\mathbf{v}) + b\alpha\phi(\mathbf{w}) = +a\beta\psi(\mathbf{v}) + b\beta\psi(\mathbf{w})\\ &= a(\alpha\phi + \beta\psi)(\mathbf{v}) + b(\alpha\phi + \beta\psi)(\mathbf{w})\end{aligned}$$

so that $a(\alpha\phi + \beta\psi) \in V^*$ and hence V^* is a vector space.

Let us fix a basis $\{\mathbf{v_1}, ..., \mathbf{v_n}\}$ for V, then to uniquely specify an element $\phi \in V^*$ we simply need to fix the real numbers $\alpha_i = \phi(\mathbf{v}_i)$ with $i = 1, ..., n$ and from here using linearity of ϕ we can obtain $\phi(\mathbf{v})$ for a general vector $\mathbf{v} = \sum_{i=1}^n a_i\mathbf{v}_i$

$$\phi(\mathbf{v}) = \phi\left(\sum_{i=1}^n a_i\mathbf{v}_i\right) = \sum_{i=1}^n a_i\phi(\mathbf{v}_i) = \sum_{i=1}^n a_i\alpha_i\,.$$

If we defined $\phi^{(i)} \in V^*$, $i = 1, ..., n$, by $\phi^{(i)}(\mathbf{v_j}) = \delta^i_j$, e.g. $\phi^{(1)}(\mathbf{v}_1) = 1$ and $\phi^{(1)}(\mathbf{v}_{i\neq 1}) = 0$ we can span all the elements ϕ of V^* using

$$\phi = \sum_{i=1}^n \phi(\mathbf{v}_i)\phi^{(i)} = \sum_{i=1}^n \alpha_i\phi^{(i)}\,,$$

since by linearity as we wrote above

$$\begin{aligned}\phi(\mathbf{v}) = \phi\left(\sum_{j=1}^n a_j\mathbf{v}_ij\right) &= \sum_{i=1}^n \phi(\mathbf{v}_i)\phi^{(i)}\left(\sum_{j=1}^n a_j\mathbf{v}_j\right))\\ &= \sum_{i=1}^n\sum_{j=1}^n \alpha_i a_j\phi^{(i)}(\mathbf{v}_j) = \sum_{i=1}^n\sum_{j=1}^n \alpha_i a_j\delta_{ij} = \sum_{i=1}^n a_i\alpha_i\,,\end{aligned}$$

exactly as written above. So the $\phi^{(i)}$ span the whole V^*.

They are also linearly independent if in fact we had numbers α_i not all vanishing such that $\phi = \sum_{i=1}^n \alpha_i\phi^{(i)} = 0$ this would mean that ϕ applied to any vector of V would give 0 but in particular $0 = \phi(\mathbf{v}_1) = \sum_{i=1}^n \alpha_i\phi^{(i)}(\mathbf{v}_1) = \sum_{i=1}^n \alpha_i\delta_{i1} = \alpha_1$ so α_1 must vanish. Repeating the same argument for $\mathbf{v}_2, ...$ we find that all α_i must vanish hence the $\phi^{(i)}$ are linearly independent and form a basisi for V^*. Obviously they are n in number so that the dual space has the same dimension as the original space V.

63. * Consider a real n-dimensional inner product space $\{V, (\cdot,\cdot)\}$. Show that for every vector $\mathbf{v} \in V$ we can construct the application $\phi_\mathbf{v} : V \mapsto \mathbb{R}$ defined by $\phi_\mathbf{v}(\mathbf{w}) = (\mathbf{w}, \mathbf{v})$. Prove that $\phi_\mathbf{v} \in V^*$. [This is telling you that V and V^* are isomorphic, however this is not a *natural* isomorphism in the sense that it dependes on your choice of inner product.]

Solution: Let us consider $\phi_{\mathbf{v}} = (\cdot, \mathbf{v})$ we want to show that it belongs to V^* so we need to prove that it is a real valued linear transformation but both this facts follow almost trivially from the properties of the inner product. For every $\mathbf{w}_1, \mathbf{w}_2 \in V$ and for every $a, b \in \mathbb{R}$ we have that

$$\phi_{\mathbf{v}}(a\mathbf{w}_1 + b\mathbf{w}_2) = (a\mathbf{w}_1 + b\mathbf{w}_2, \mathbf{v}) = a(\mathbf{w}_1, \mathbf{v}) + b(\mathbf{w}_2, \mathbf{v}) = a\phi_{\mathbf{v}}(\mathbf{w}_1) + b\phi_{\mathbf{v}}(\mathbf{w}_2)\,,$$

so that $\phi_{\mathbf{v}}$ is linear and real valued since $(\mathbf{w}, \mathbf{v}) \in \mathbb{R}$ for every $\mathbf{w}, \mathbf{v} \in V$.

Note that if we choose $\{\mathbf{v}_1, ..., \mathbf{v}_n\}$ to be an orthonormal basis for $(\cdot, \cdot)$ we have that $\{\phi_{\mathbf{v}_1}, ..., \phi_{\mathbf{v}_n}\}$ form precisely the basis $\phi^{(i)}$ described in the previous exercise, i.e. $\phi_{\mathbf{v}_i}(\mathbf{v}_j) = (\mathbf{v}_j, \mathbf{v}_i) = \delta_{ij}$.

64. * Consider a real n-dimensional vector space V, its dual V^* and the double-dual

$$V^{**} = \{\Phi : V^* \mapsto \mathbb{R}\,, \text{ s.t. } \Phi \text{ is linear}\}\,.$$

Show that for every vector $\mathbf{v} \in V$, the application $\Phi_{\mathbf{v}} : V^* \mapsto \mathbb{R}$ defined by $\Phi_{\mathbf{v}}(\phi) = \phi(\mathbf{v})$, for every $\phi \in V^*$, is an element of V^{**}, i.e. $\Phi_{\mathbf{v}} \in V^{**}$. [This is telling you that there is a natural isomorphism between V and V^{**} given by evaluation on a specific vector.]

Solution: Note that an element $\Phi \in V^{**}$ is a linear transformation that "eats" an element $\phi \in V^*$ and returns a real number, but we know that an element $\phi \in V^*$ is a linear transformation that "eats" an element $\mathbf{v} \in V$ and returns a real number, so basically the object $\Phi_{\mathbf{v}}$ take any element $\phi \in V^*$ and returns its evaluation on the vector $\mathbf{v}$ given by $\phi(\mathbf{v})$.

It is very simple to see that this all process preserves linearity, i.e. for every $\mathbf{v} \in V$ and for every $\phi, \psi \in V^*$, for every $\alpha, \beta \in \mathbb{R}$, we have

$$\Phi_{\mathbf{v}}(\alpha\phi + \beta\psi) = (\alpha\phi + \beta\psi)(\mathbf{v}) = \alpha\phi(\mathbf{v}) + \beta\psi(\mathbf{v}) = \alpha\Phi_{\mathbf{v}}(\phi) + \beta\Phi_{\mathbf{v}}(\psi)\,.$$

So that $\Phi_{\mathbf{v}} \in V^*$. It is also straightforward to show that if $\{\mathbf{v}_1, ..., \mathbf{v}_n\}$ is a basis for V then $\{\Phi_{\mathbf{v}_1}, ..., \Phi_{\mathbf{v}_n}\}$ is a basis for V^{**} so these two spaces are naturally isomorphic, i.e. they are identical and we simply call the same objects with different names.

65. If A is a real $n \times n$ matrix, show that A is skew-symmetric (anti-symmetric) if and only if $\mathbf{x}^t A\mathbf{x} = 0$ for all $\mathbf{x} \in \mathbb{R}^n$.

Solution: Assume that A is skew-symmetric and so $A^t = -A$. Since you can think of $\mathbf{x}^t A\mathbf{x}$ as just a 1×1 matrix (i.e. a number) then it must be equal to its transpose:

$$\mathbf{x}^t A\mathbf{x} = \left(\mathbf{x}^t A\mathbf{x}\right)^t = \mathbf{x}^t A^t\mathbf{x} = -\mathbf{x}^t A\mathbf{x},$$

hence we must have $\mathbf{x}^t A\mathbf{x} = 0$ for all $\mathbf{x} \in \mathbb{R}^n$. Conversely, if $\mathbf{x}^t A\mathbf{x} = 0$ for each $\mathbf{x} \in \mathbb{R}^n$ then

$$\begin{aligned} 0 &= (\mathbf{x}^t + \mathbf{y}^t)A(\mathbf{x} + \mathbf{y}) \\ &= \mathbf{x}^t A\mathbf{x} + \mathbf{x}^t A\mathbf{y} + \mathbf{y}^t A\mathbf{x} + \mathbf{y}^t A\mathbf{y} \\ &= \mathbf{x}^t A\mathbf{y} + \mathbf{y}^t A\mathbf{x}. \end{aligned}$$

Letting $\mathbf{x}$ and $\mathbf{y}$ run through the standard basis for $\mathbb{R}^n$ we see that, writing $A = (a_{ij})$ then $a_{ij} + a_{ji} = 0$ as required.

66. Let M_n be the vector space of $n \times n$ matrices with real coefficients. Show that $M_n = \mathrm{Skew}_n \oplus \mathrm{Sym}_n$ where $\mathrm{Skew}_n = \{A \in M_n \,|\, A^t = -A\}$ and $\mathrm{Sym}_n = \{A \in M_n \,|\, A^t = A\}$. What are the dimensions of M_n, Skew_n and Sym_n as vector spaces over the field of real numbers?

Solution: Let $A \in M_n$ then $A = (A + A^t)/2 + (A - A^t)/2$ and clearly $(A + A^t)/2 \in \mathsf{Sym}_n$ while $(A - A^t)/2 \in \mathsf{Skew}_n$. If $A \in \mathsf{Sym}_n$ and also $A \in \mathsf{Skew}_n$ then $A = -A$ so $A = 0$ so $M_n = \mathsf{Skew}_n \oplus \mathsf{Sym}_n$.

We have $\mathrm{dim} M_n = n^2$ while $\mathrm{dim}\mathsf{Skew}_n = \frac{n(n-1)}{2}$ and $\mathrm{dim}\mathsf{Sym}_n = \frac{n(n+1)}{2}$. Note that $\mathrm{dim} M_n = \mathrm{dim}\mathsf{Skew}_n + \mathrm{dim}\mathsf{Sym}_n$ as expected.

67. * Consider the vector space M_n of $n \times n$ matrices with real coefficients with the inner product $(A, B) = \mathrm{Tr}(A^t B)$. Find the orthogonal complement to the vector subspace Sym_n with respect to this inner product.

Solution: To obtain the orthogonal complement to the vector subspace, Sym_n, formed by the symmetric matrices we first notice that if B is an anti-symmetric matrix, i.e. $B^t = -B$, we have that for every $A \in \mathsf{Sym}_n$

$$(B, A) = \mathrm{Tr}(B^t A) = -\mathrm{Tr}(BA) = -\mathrm{Tr}(AB) = -\mathrm{Tr}(A^t B) = -(A, B) = -(B, A)$$

which implies that if B is an anti-symmetric matrix $(B, A) = 0$ for every $A \in \mathsf{Sym}_n$, i.e. $\mathsf{Skew}_n \subseteq (\mathsf{Sym}_n)^\perp$. To show that $(\mathsf{Sym}_n)^\perp = \mathsf{Skew}_n$ we can make use of the fact that

$$\dim(\mathsf{Sym}_n)^\perp = \dim M_n - \dim\mathsf{Sym}_n = n^2 - \frac{n(n+1)}{2} = \frac{n(n-1)}{2} = \dim\mathsf{Skew}_n\,.$$

68. * Let $\sigma \in S_n$ denote a permutation of n elements, the matrix R_σ, with respect to the standard basis $\{v_1, ..., v_n\}$ of $\mathbb{R}^n$, is associated with the linear transformation that permutes the basis vector with σ, i.e. $R_\sigma v_i = v_{\sigma(i)}$. Prove that R_σ is orthogonal. [HINT: How can you write $(R_\sigma)^{-1}$ in terms of another permutation?]

Solution: The inverse of R_σ is simply given by $R_{\sigma^{-1}}$ where σ^{-1} denotes the inverse permutation of σ, i.e. $\sigma^{-1}(\sigma(i)) = i$. So in particular if $R_\sigma v_i = v_{\sigma(i)}$ we have that $(R_\sigma)^{-1} v_{\sigma(i)} = R_{\sigma^{-1}} v_{\sigma(i)} = v_i$ so we have that $(R_\sigma)^{-1} = R_{\sigma^{-1}} = (R_\sigma)^t$, hence R_σ is an orthogonal matrix for every $\sigma \in S_n$.

69. If A is a complex $n \times n$ matrix, show that A is Hermitian if and only if $\mathbf{x}^* A\mathbf{x}$ is real for all $\mathbf{x} \in \mathbb{C}^n$.

Solution: Assume that A is Hermitian and so $A^* = A$. Since you can think of $\mathbf{x}^* A\mathbf{x}$ as just a 1×1 matrix (i.e. a complex number) if we take its star it is the same as taking just its complex conjugate

$$(\mathbf{x}^* A\mathbf{x})^* = \left(\overline{\mathbf{x}^* A\mathbf{x}}\right)^t = \overline{\mathbf{x}^* A\mathbf{x}},$$

however we also have

$$(\mathbf{x}^* A\mathbf{x})^* = \mathbf{x}^* A^* \mathbf{x} = \mathbf{x}^* A\mathbf{x},$$

and we deduce that $\mathbf{x}^* A\mathbf{x} = \overline{\mathbf{x}^* A\mathbf{x}}$ as required. Conversely, if $\mathbf{x}^* A\mathbf{x}$ is real for each $\mathbf{x} \in \mathbb{C}^n$ then

$$(\mathbf{x}^* + \mathbf{y}^*)A(\mathbf{x} + \mathbf{y}) = \mathbf{x}^* A\mathbf{x} + \mathbf{x}^* A\mathbf{y} + \mathbf{y}^* A\mathbf{x} + \mathbf{y}^* A\mathbf{y}$$

is real and hence $\mathbf{x}^* A\mathbf{y} + \mathbf{y}^* A\mathbf{x}$ is also real. Letting $\mathbf{x}$ and $\mathbf{y}$ run through the standard basis for $\mathbb{C}^n$ we see that, writing $A = (a_{jk})$ then $a_{jk} + a_{kj}$ is real. Letting $\mathbf{x}$ run through i times a standard basis vector for $\mathbb{C}^n$ and $\mathbf{y}$ run through the standard basis for $\mathbb{C}^n$ we see that $ia_{jk} - ia_{kj}$ is real. So we have that both $a_{jk} + a_{kj}$ and $ia_{jk} - ia_{kj}$ must be real numbers, say $a_{jk} + a_{kj} = 2x$ and $ia_{jk} - ia_{kj} = 2y$ with $x, y \in \mathbb{R}$. We can thus write $a_{jk} = x - iy$ and $a_{kj} = x + iy$ which implies $a_{kj} = \overline{a_{jk}}$. This gives the result.

70. Find a unitary matrix P such that $P^* AP$ is diagonal when

$$A = \begin{pmatrix} 2 & 1+i \\ 1-i & 3 \end{pmatrix}.$$

Solution:

$$p_A(t) = \det(A - tI) = (2-t)(3-t) - (1+i)(1-i) = t^2 - 5t + 4 = (t-1)(t-4).$$

The eigenvalues of A are the roots of this polynomial, namely $\lambda = 1, 4$.

$\lambda = 1$:
$$\begin{pmatrix} 1 & 1+i \\ 1-i & 2 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$
Hence an eigenvector is
$$\mathbf{v}_1 = \begin{pmatrix} -1-i \\ 1 \end{pmatrix}.$$
But $\|\mathbf{v}_1\|^2 = 3$. So a unit eigenvector is
$$\mathbf{u}_1 = \frac{1}{\sqrt{3}} \begin{pmatrix} -1-i \\ 1 \end{pmatrix}.$$

$\lambda = 4$:
$$\begin{pmatrix} -2 & 1+i \\ 1-i & -1 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$
Hence an eigenvector is
$$\mathbf{v}_2 = \begin{pmatrix} 1 \\ 1-i \end{pmatrix}.$$
But $\|\mathbf{v}_2\|^2 = 3$, so a unit eigenvector is
$$\mathbf{u}_2 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1-i \end{pmatrix}.$$

Define P to be the matrix whose columns are $\mathbf{u}_1$ and $\mathbf{u}_2$:
$$P = \frac{1}{\sqrt{3}} \begin{pmatrix} -1-i & 1 \\ 1 & 1-i \end{pmatrix}.$$
Then P is unitary, and $P^*AP = \mathrm{diag}(1\ 4)$.

71. Show that the determinant of a unitary matrix is of unit modulus.

Solution: If $AA^* = I$ then, since
$$\det(AA^*) = \det A \det A^* = \det A\, \overline{\det A^t} = \det A\, \overline{\det A},$$
we have $\det A\, \overline{\det A} = 1$.

72. A unitary matrix of determinant $+1$ is *special unitary*. Show that every unitary matrix A can be written in the form $A = kB$ where $k \in \mathbb{C}$ is of unit modulus and B is special unitary.

Solution: Suppose A is a unitary $n \times n$ matrix. Then $\det A = e^{i\theta}$ for some θ. Then we may write $A = e^{i\theta/n}B$ where $B = e^{-i\theta/n}A$ and
$$\det B = \det(e^{-i\theta/n}A) = \det(e^{-i\theta/n}I)\det A = e^{-i\theta}e^{i\theta} = 1.$$
Thus $A = kB$ where $k \in \mathbb{C}$ is of unit modulus and B is special unitary. If $A = k'B'$, $k' \in \mathbb{C}$ and B' special unitary, is another decomposition then taking determinants we see that $k^n = k'^n$ so that $k' = \zeta k$, for some n-th root of unity ζ, and consequently $B' = \zeta^{-1}B$. Thus the decomposition is only unique up to multiplication by n-th roots of unity in the above sense.

73. Show that every special unitary 2×2 matrix is of the form
$$\begin{pmatrix} a & -\bar{c} \\ c & \bar{a} \end{pmatrix},$$
with a, $c \in \mathbb{C}$ and $a\bar{a} + c\bar{c} = 1$.

Solution: Let A be a special unitary 2×2-matrix. Then $AA^* = I$ and $\det A = 1$. Thus, writing

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

for some $a,\, b,\, c,\, d \in \mathbb{C}$, we have

$$\begin{pmatrix} \bar{a} & \bar{c} \\ \bar{b} & \bar{d} \end{pmatrix} = A^* = A^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} d & -b \\ -b & a \end{pmatrix}.$$

Thus $b = -\bar{c}, d = \bar{a}$, and

$$A = \begin{pmatrix} a & -\bar{c} \\ c & \bar{a} \end{pmatrix}$$

with $a\bar{a} + b\bar{b} = 1$.

74. Show that, if n is odd, every real orthogonal $n \times n$ matrix A has $\det(A)$ as an eigenvalue. (Note that, for any real orthogonal matrix A, $\det(A) = \pm 1$).

Solution: Let A be a real $n \times n$ orthogonal matrix and let λ be an eigenvalue of A with eigenvector $\mathbf{x} \in \mathbb{R}^n$. Since A is orthogonal, we have $\|\mathbf{x}\| = \|A\mathbf{x}\| = |\lambda|\, \|\mathbf{x}\|$. Thus each of A's eigenvalues has unit modulus. Consider the characteristic polynomial of A. This is a polynomial in x with real coefficients. Therefore its complex roots occur in conjugate pairs. Suppose that A has $2a$ complex eigenvalues occurring in conjugate pairs, b eigenvalues that are $+1$ and c eigenvalues that are -1. Then $2a+b+c = n$, which is assumed to be odd, and $\det(A) = (-1)^c$. Suppose that $\det(A) = +1$ then c is even and so b is odd. Thus A has an odd number of eigenvalues $+1$. Suppose that $\det(A) = -1$ then c is odd. Thus A has an odd number of eigenvalues -1.

75. Identify the polynomial $f(x) = c_0 + c_1 x + \ldots + c_{n-1}x^{n-1} + x^n$ for which the integral $\int_{-1}^{1} f(x)^2\, dx$ has the smallest value. (Hint: Consider $f(x)$ as a linear combination of Legendre polynomials $P_0(x), \ldots, P_n(x)$, taking P_k to be normalized by $P_k(x) = x^k + \ldots$.

Solution: First observe that since, for all k, $P_k(x)$ is a polynomial of degree k, we may write

$$f(x) = c_0 + c_1 x + \ldots + c_{n-1}x^{n-1} + x^n = a_0 P_0(x) + \ldots + a_n P_n(x),$$

with $a_n = 1$. Since $P_0(x), \ldots, P_n(x)$ are mutually orthogonal, we get

$$\int_{-1}^{1} f(x)^2\, dx = a_0^2 \|P_0(x)\|^2 + \ldots + a_n^2 \|P_n(x)\|^2 \geq a_n^2 \|P_n(x)\|^2$$

with equality if and only if $a_0 = \ldots = a_{n-1} = 0$. So $f(x) = P_n(x)$ is the desired polynomial.

76. (a) Verify by direct computation that the Laguerre operator

$$\mathcal{L}_l = x\frac{d^2}{dx^2} + (1 - x)\frac{d}{dx}$$

on the space $\mathbb{R}[x]$ of polynomials in x is symmetric with respect to the inner product given by the formula $(f, g) = \int_0^{+\infty} f(x)g(x)e^{-x}dx$.

(b) Find the matrix and the characteristic polynomial of the Laguerre operator $\mathcal{L}_l$ on the space $\mathbb{R}[x]_2$ (use the basis $\{1,\, x,\, x^2\}$).

(c) What are all the eigenvalues of the Laguerre operator on $\mathbb{R}[x]$?

(d) Find all the eigenfunctions of the Laguerre operator on the space $\mathbb{R}[x]_2$.

(e) Find the Laguerre polynomial of degree 5. (For simplicity use the convention in which Laguerre polynomials have leading coefficient 1, even if this is not compatible with them having unit norm.)

Solution: (a) For any $f, g \in \mathbb{R}[x][0, +\infty)$, apply integration by parts

$$\begin{aligned}
(\mathcal{L}_l(f), g) &= \int_0^{+\infty} \left(e^{-x} x f'\right)' g dx \\
&= e^{-x} x f' g|_0^{+\infty} - \int_0^{+\infty} e^{-x} x f' g' dx \\
&= -\int_0^{+\infty} e^{-x} x f' g' dx.
\end{aligned}$$

Similarly,

$$\begin{aligned}
(f, \mathcal{L}_l(g)) &= \int_0^{+\infty} f \left(e^{-x} x g'\right)' dx \\
&= f e^{-x} x g'|_0^{+\infty} - \int_0^{+\infty} f' e^{-x} x g' dx \\
&= -\int_0^{+\infty} e^{-x} x f' g' dx.
\end{aligned}$$

So, $\mathcal{L}_l$ satisfies the definition of symmetric operator.

(b) The matrix is

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & -1 & 4 \\ 0 & 0 & -2 \end{pmatrix}$$

and its characteristic polynomial is $p_A(\lambda) = \det(A - \lambda I) = -\lambda(\lambda+1)(\lambda+2)$.

(c) $\mathcal{L}_l x^k = -k x^k +$ lower-order terms. So the matrix of $\mathcal{L}_l$ acting on $\mathbb{R}[x]_N$ with basis $\{1, x, x^2, \ldots, x^N\}$ is an upper triangular matrix with elements $0, -1, \ldots, -k, \ldots, -N$ on its principal diagonal. Now if P is any eigenfunction in $\mathbb{R}[x]$, with eigenvalue λ, then P is a polynomial of some degree N, and so $\lambda \in \{0, -1, \ldots, -k, \ldots, -N\}$. Hence the eigenvalues of $\mathcal{L}_l$ on $\mathbb{R}[x]$ are $\{-k \mid k \in \mathbb{Z}, k \geqslant 0\}$.

(d) Use (b) above. For the eigenvalue $\lambda = 0$, the corresponding eigenfunction is 1, for $\lambda = -1$, it is $x - 1$, for $\lambda = -2$, it is $x^2 - 4x + 2$.

(e) This polynomial can be written as $l_5(x) = x^5 + ax^4 + bx^3 + cx^2 + dx + e$ with some unknown coefficients $a, b, c, d, e \in \mathbb{R}$. Then $l_5(x)$ is an eigenfunction for $\mathcal{L}_l$ with the eigenvalue -5. Therefore,

$$\begin{aligned}
& x(20x^3 + 12ax^2 + 6bx + 2c) + (1 - x)(5x^4 + 4ax^3 + 3bx^2 + 2cx + d) \\
= \; & -5(x^5 + ax^4 + bx^3 + cx^2 + dx + e).
\end{aligned}$$

Comparing coefficients for x^4, x^3, x^2, x and 1 we obtain

$$\begin{aligned}
x^4 : \quad & 20 + 5 - 4a = -5a; \\
x^3 : \quad & 12a + 4a - 3b = -5b; \\
x^2 : \quad & 6b + 3b - 2c = -5c; \\
x : \quad & 2c + 2c - d = -5d; \\
1 : \quad & d = -5e.
\end{aligned}$$

Therefore, $a = -25$, $b = 200$, $c = -600$, $d = 600$, $e = -120$, i.e.

$$l_5(x) = x^5 - 25x^4 + 200x^3 - 600x^2 + 600x - 120.$$

77. (a) Verify by direct computation that the Hermite operator

$$\mathcal{L}_H = \frac{d^2}{dx^2} - 2x\frac{d}{dx}$$

on the space $\mathbb{R}[x]$ is symmetric with respect to the inner product given by the formula $(f, g) = \int_{-\infty}^{+\infty} f(x)g(x)e^{-x^2}dx$.

(b) Find the matrix and the characteristic polynomial of the Hermite operator $\mathcal{L}_H$ on the space $\mathbb{R}[x]_3$ (use the basis $\{1, x, x^2, x^3\}$). What is the set of all eigenvalues of $\mathcal{L}_H$ as an operator on the space $\mathbb{R}[x]_3$?

(c) What are all the eigenvalues of the Hermite operator on $\mathbb{R}[x]$?

(d) Find all eigenfunctions of the Hermite operator on the space $\mathbb{R}[x]_3$.

(e) Find the Hermite polynomial of degree 5. (For simplicity use the convention in which Hermite polynomials have leading coefficient 1, even if this is not compatible with them having unit norm.)

Solution: (a) For any $f, g \in \mathbb{R}[x]$, apply integration by parts

$$\begin{aligned}(\mathcal{L}_H(f), g) &= \int_{-\infty}^{+\infty} \left(e^{-x^2} f'\right)' g dx \\ &= e^{-x^2} f' g|_{-\infty}^{+\infty} - \int_{-\infty}^{+\infty} e^{-x^2} f' g' dx \\ &= -\int_{-\infty}^{+\infty} e^{-x^2} f' g' dx.\end{aligned}$$

Similarly,

$$\begin{aligned}(f, \mathcal{L}_H(g)) &= \int_{-\infty}^{+\infty} f\left(e^{-x^2} g'\right)' dx \\ &= f e^{-x^2} g'|_{-\infty}^{+\infty} - \int_{-\infty}^{+\infty} f' e^{-x^2} g' dx \\ &= -\int_{-\infty}^{+\infty} e^{-x^2} f' g' dx.\end{aligned}$$

So, $\mathcal{L}_H$ satisfies the definition of a symmetric operator.

(b) Our operator transforms 1 to 0, x to $-2x$, x^2 to $2-4x^2$, x^3 to $6x-6x^3$. The corresponding matrix is

$$\begin{pmatrix} 0 & 0 & 2 & 0 \\ 0 & -2 & 0 & 6 \\ 0 & 0 & -4 & 0 \\ 0 & 0 & 0 & -6 \end{pmatrix}.$$

It has the eigenvalues $\lambda_1 = 0$, $\lambda_2 = -2$, $\lambda_3 = -4$, $\lambda_4 = -6$. The characteristic polynomial $p_A(t) = \det(A - \lambda I) = \lambda(\lambda + 2)(\lambda + 4)(\lambda + 6)$.

(c) $\mathcal{L}_H x^k = -2kx^k +$ lower-order terms. So the matrix of $\mathcal{L}_H$ acting on $\mathbb{R}[x]_N$ with basis $\{1, x, x^2, \ldots, x^N\}$ is an upper triangular matrix with elements $0, -2, \ldots, -2k, \ldots, -2N$ on its principal diagonal. Now if P is any eigenfunction in $\mathbb{R}[x]$, with eigenvalue λ, then P is a polynomial of some degree N, and so $\lambda \in \{0, -2, \ldots, -2k, \ldots, -2N\}$. Hence the eigenvalues of $\mathcal{L}_H$ on $\mathbb{R}[x]$ are $\{-2k \mid k \in \mathbb{Z}, k \geqslant 0\}$.

(d) Use (b) above. The corresponding eigenfunctions are $1, x, x^2 - \frac{1}{2}, x^3 - \frac{3}{2}x$.

(e) This polynomial can be written as $H_5(x) = x^5 + ax^4 + bx^3 + cx^2 + dx + e$ with some unknown coefficients $a, b, c, d, e \in \mathbb{R}$. Then $H_5(x)$ is an eigenfunction for $\mathcal{L}_H$ with the eigenvalue -10. Therefore,

$$20x^3+12ax^2+6bx+2c-2x(5x^4+4ax^3+3bx^2+2cx+d) = -10(x^5+ax^4+bx^3+cx^2+dx+e)$$

Comparing coefficients for x^4, x^3, x^2, x and 1 we obtain

$$\begin{aligned} x^4: &\quad -8a = -10a; \\ x^3: &\quad 20 - 6b = -10b; \\ x^2: &\quad 12a - 4c = -10c; \\ x: &\quad 6b - 2d = -10d; \\ 1: &\quad 2c = -10e. \end{aligned}$$

Therefore, $a = c = e = 0$, $b = -5$ and $d = 15/4$, i.e. $H_5(x) = x^5 - 5x^3 + \frac{15}{4}x$.

78. (a) Verify by direct computation that the Legendre operator

$$\mathcal{L}_L = (1 - x^2)\frac{d^2}{dx^2} - 2x\frac{d}{dx}$$

on the space $C[-1, 1]$ is symmetric with respect to the inner product given by the formula $(f, g) = \int_{-1}^{1} f(x)g(x)dx$.

(b) Find the characteristic polynomial of the Legendre operator $\mathcal{L}_L$ on the space $\mathbb{R}[x]_4$.

(c) What is the set of all eigenvalues of $\mathcal{L}_L$ as an operator on $\mathbb{R}[x]$?

(d) Find the Legendre polynomial of degree 5. (For simplicity use the convention in which Legendre polynomials have leading coefficient 1, even if this is not compatible with them having unit norm.)

Solution: (a) For any $f, g \in C[-1, 1]$, apply integration by parts

$$\begin{aligned} (\mathcal{L}_L(f), g) &= \int_{-1}^{1} \left((1 - x^2)f'\right)' g dx \\ &= (1 - x^2)f'g|_{-1}^{1} - \int_{-1}^{1} (1 - x^2)f'g'dx \\ &= -\int_{-1}^{1} (1 - x^2)f'g'dx. \end{aligned}$$

Similarly,

$$\begin{aligned} (f, \mathcal{L}_L(g)) &= \int_{-1}^{1} f\left((1 - x^2)g'\right)' dx \\ &= f(1 - x^2)g'|_{-1}^{1} - \int_{-1}^{1} f'(1 - x^2)g'dx \\ &= -\int_{-1}^{1} (1 - x^2)f'g'dx. \end{aligned}$$

So, $\mathcal{L}_L$ satisfies the definition of symmetric operator.

(b) Our operator transforms 1 to 0, x to $-2x$, x^2 to $-6x^2 + 2$, x^3 to $-12x^3 + 6x$, x^4 to $-20x^4 + 12x^2$. The corresponding matrix is

$$\begin{pmatrix} 0 & 0 & 2 & 0 & 0 \\ 0 & -2 & 0 & 6 & 0 \\ 0 & 0 & -6 & 0 & 12 \\ 0 & 0 & 0 & -12 & 0 \\ 0 & 0 & 0 & 0 & -20 \end{pmatrix}$$

and the corresponding characteristic polynomial is
$p_A(\lambda) = \det(A - \lambda I) = -\lambda(\lambda+2)(\lambda+6)(\lambda+12)(\lambda+20)$.

(c) $\mathcal{L}_L x^k = -k(k+1)x^k +$lower-order terms. So the matrix of $\mathcal{L}_L$ acting on $\mathbb{R}[x]_N$ with basis $\{1, x, x^2, \ldots, x^N\}$ is an upper triangular matrix with elements $0, -2, \ldots, -k(k+1), \ldots, -N(N+1)$ on its principal diagonal. Now if P is any eigenfunction in $\mathbb{R}[x]$, with eigenvalue λ, then P is a polynomial of some degree N, and so $\lambda \in \{0, -2, \ldots, -k(k+1), \ldots, -N(N+1)\}$. Hence the eigenvalues of $\mathcal{L}_L$ on $\mathbb{R}[x]$ are $\{-k(k+1) \mid k \in \mathbb{Z}, k \geqslant 0\}$.

(d) This polynomial can be written as $L_5(x) = x^5 + ax^4 + bx^3 + cx^2 + dx + e$ with unknown coefficients $a, b, c, d, e \in \mathbb{R}$. Then $L_5(x)$ is an eigenfunction for $\mathcal{L}_L$ with the eigenvalue -30. Therefore we have

$$\begin{aligned} & (1-x^2)(20x^3 + 12ax^2 + 6bx + 2c) - 2x(5x^4 + 4ax^3 + 3bx^2 + 2cx + d) \\ = \quad & -30(x^5 + ax^4 + bx^3 + cx^2 + dx + e). \end{aligned}$$

Comparing coefficients for x^4, x^3, x^2, x and 1 we obtain

$$\begin{aligned} x^4: &\quad -12a - 8a = -50a; \\ x^3: &\quad 20 - 6b - 6b = -30b; \\ x^2: &\quad 12c - 2c - 4c = -30c; \\ x: &\quad 6b - 2d = -30d; \\ 1: &\quad 2c = -30e. \end{aligned}$$

Therefore, $a = c = e = 0$, $b = -10/9$ and $d = 5/21$, i.e. $L_5(x) = x^5 - \frac{1}{9}x^3 + \frac{5}{21}x$.