Linear Algebra 1, Solutions to exercises 79 to 91.
Epiphany 21/22.

79. (a) Verify by direct computation that the Chebyshev-I operator

$$\mathcal{L}_I = (1-x^2)\frac{d^2}{dx^2} - x\frac{d}{dx}$$

on the space $C[-1,1]$ is symmetric with respect to the inner product given by the formula $(f,g) = \int_{-1}^{1} f(x)g(x)(1-x^2)^{-1/2}dx$.

(b) Find the matrix and the characteristic polynomial of the Chebyshev-I operator $\mathcal{L}_I$ on the space $\mathbb{R}[x]_3$ (use the basis $\{1, x, x^2, x^3\}$).

(c) Hence find the Chebyshev-I polynomials of degree 2 and 3.

(d) What is the set of all eigenvalues of $\mathcal{L}_I$ as an operator on the space of all polynomials $\mathbb{R}[x]$?

(e) Find the Chebyshev-I polynomial of degree 5. (For simplicity use the convention in which Chebyshev-I polynomials have leading coefficient 1, even if this is not compatible with them having unit norm.)

Solution: (a) For any $f, g \in C[-1,1]$, apply integration by parts

$$\begin{aligned} (\mathcal{L}_I(f), g) &= \int_{-1}^{1} \left(\sqrt{1-x^2}f'\right)' g dx \\ &= \sqrt{1-x^2}f'g|_{-1}^{1} - \int_{-1}^{1} \sqrt{1-x^2}f'g'dx \\ &= -\int_{-1}^{1} \sqrt{1-x^2}f'g'dx. \end{aligned}$$

This is symmetric in f and g, so $(\mathcal{L}_I(f), g) = (\mathcal{L}_I(g), f)$, which equals $(f, \mathcal{L}_I(g))$ because of the symmetry of the inner product. So $\mathcal{L}_I$ is a symmetric operator.

(b) Our operator transforms 1 to 0, x to $-x$, x^2 to $-4x^2+2$, x^3 to $-9x^3+6x$. The corresponding matrix is

$$\begin{pmatrix} 0 & 0 & 2 & 0 \\ 0 & -1 & 0 & 6 \\ 0 & 0 & -4 & 0 \\ 0 & 0 & 0 & -9 \end{pmatrix}.$$

The characteristic polynomial is $p_A(\lambda) = \det(A-\lambda I) = \lambda(\lambda+1)(\lambda+4)(\lambda+9)$.

(c) The eigenvector corresponding to -4 is $(-1/2, 0, 1, 0)^t$, which gives the eigenfunction $f_2(x) = x^2 - 1/2$ (or $f_2(x) = 2x^2 - 1$). The eigenvector corresponding to -9 is $(0, -3/4, 0, 1)^t$, which gives the eigenfunction $f_3(x) = x^3 - 3x/4$ (or $f_3(x) = 4x^3 - 3x$).

(d) $\mathcal{L}_I x^k = -k^2 x^k +$ lower-order terms. So the matrix of $\mathcal{L}_I$ acting on $\mathbb{R}[x]_N$ with basis $\{1, x, x^2, \ldots, x^N\}$ is an upper triangular matrix with elements $0, -1, \ldots, -k^2, \ldots, -N^2$ on its principal diagonal. Now if P is any eigenfunction in $\mathbb{R}[x]$, with eigenvalue λ, then P is a polynomial of some degree N, and so $\lambda \in \{0, -2, \ldots, -k^2, \ldots, -N^2\}$. Hence the eigenvalues of $\mathcal{L}_I$ on $\mathbb{R}[x]$ are $\{-k^2 \mid k \in \mathbb{Z}, k \geqslant 0\}$.

(e) This polynomial can be written as $C_5(x) = x^5 + ax^4 + bx^3 + cx^2 + dx + e$ with unknown coefficients $a, b, c, d, e \in \mathbb{R}$. Then $C_5(x)$ is an eigenfunction for $\mathcal{L}_L$ with the eigenvalue -25. Therefore,

$$\begin{aligned} & (1-x^2)(20x^3 + 12ax^2 + 6bx + 2c) - x(5x^4 + 4ax^3 + 3bx^2 + 2cx + d) \\ = \;& -25(x^5 + ax^4 + bx^3 + cx^2 + dx + e). \end{aligned}$$

Comparing coefficients for x^4, x^3, x^2, x and 1 we obtain

$$\begin{aligned}
x^4: &\quad -12a - 4a = -25a;\\
x^3: &\quad 20 - 6b - 3b = -25b;\\
x^2: &\quad 12c - 2c - 2c = -25c;\\
x: &\quad 6b - d = -25d;\\
1: &\quad 2c = -25e.
\end{aligned}$$

Therefore, $a = c = e = 0$, $b = -5/4$ and $d = 5/16$, i.e. $L_5(x) = x^5 - \frac{5}{4}x^3 + \frac{5}{16}x$.

80. (a) Verify by direct computation that the Chebyshev-II operator

$$\mathcal{L}_{II} = (1 - x^2)\frac{d^2}{dx^2} - 3x\frac{d}{dx}$$

on the space $C[-1, 1]$ is symmetric with respect to the inner product given by the formula $(f, g) = \int_{-1}^{1} f(x)g(x)(1 - x^2)^{1/2}dx$.

(b) Find the matrix and the characteristic polynomial of the Chebyshev-II operator $\mathcal{L}_{II}$ on the space $\mathbb{R}[x]_4$ (use the basis $\{1, x, x^2, x^3, x^4\}$).

(c) What is the set of all eigenvalues of $\mathcal{L}_{II}$ as an operator on the space of all polynomials $\mathbb{R}[x]$?

(d) Find the Chebyshev-II polynomial of degree 5. (For simplicity use the convention in which Chebyshev-II polynomials have leading coefficient 1, even if this is not compatible with them having unit norm.)

Solution: (a) For any $f, g \in C[-1, 1]$, apply integration by parts

$$\begin{aligned}
(\mathcal{L}_{II}(f), g) &= \int_{-1}^{1} \left((1 - x^2)^{3/2} f'\right)' g dx\\
&= (1 - x^2)^{3/2} f'g|_{-1}^{1} - \int_{-1}^{1} (1 - x^2)^{3/2} f'g'dx\\
&= -\int_{-1}^{1} (1 - x^2)^{3/2} f'g'dx.
\end{aligned}$$

Similarly,

$$\begin{aligned}
(f, \mathcal{L}_{II}(g)) &= \int_{-1}^{1} f\left((1 - x^2)^{3/2} g'\right)' dx\\
&= f(1 - x^2)^{3/2} g'|_{-1}^{1} - \int_{-1}^{1} f'(1 - x^2)^{3/2} g'dx\\
&= -\int_{-1}^{1} (1 - x^2)^{3/2} f'g'dx.
\end{aligned}$$

So, $\mathcal{L}_{II}$ satisfies the definition of symmetric operator.

(b) In the basis $\{1, x, x^2, x^3, x^4\}$ the matrix of our operator is

$$\begin{pmatrix}
0 & 0 & 2 & 0 & 0\\
0 & -3 & 0 & 6 & 0\\
0 & 0 & -8 & 0 & 12\\
0 & 0 & 0 & -15 & 0\\
0 & 0 & 0 & 0 & -24
\end{pmatrix}.$$

The characteristic polynomial is $p_A(\lambda) = \det(A - \lambda I) = -\lambda(\lambda+3)(\lambda+8)(\lambda+15)(\lambda+24)$.

(c) $\mathcal{L}_{II}x^k = -k(k+2)x^k$+lower-order terms. So the matrix of $\mathcal{L}_{II}$ acting on $\mathbb{R}[x]_N$ with basis $\{1, x, x^2, \dots, x^N\}$ is an upper triangular matrix with elements $0, -3, \dots, -k(k+2), \dots, -N(N+2)$ on its principal diagonal. Now if P is any eigenfunction in $\mathbb{R}[x]$, with eigenvalue λ, then P is a polynomial of some degree N, and so $\lambda \in \{0, -2, \dots, -k(k+2), \dots, -N(N+2)\}$. Hence the eigenvalues of $\mathcal{L}_{II}$ on $\mathbb{R}[x]$ are $\{-k(k+2) \mid k \in \mathbb{Z}, k \geqslant 0\}$.

(d) This polynomial can be written as $C_5^*(x) = x^5 + ax^4 + bx^3 + cx^2 + dx + e$ with some unknown coefficients $a, b, c, d, e \in \mathbb{R}$. Then $C_5^*(x)$ is an eigenfunction for $\mathcal{L}_{II}$ with the eigenvalue -35. Therefore

$$\begin{aligned} & (1-x^2)(20x^3 + 12ax^2 + 6bx + 2c) - 3x(5x^4 + 4ax^3 + 3bx^2 + 2cx + d) \\ = \; & -35(x^5 + ax^4 + bx^3 + cx^2 + dx + e). \end{aligned}$$

Comparing coefficients for x^4, x^3, x^2, x and 1 we obtain

$$\begin{aligned} x^4: \quad & -12a - 12a = -35a; \\ x^3: \quad & 20 - 6b - 9b = -35b; \\ x^2: \quad & 12c - 2c - 6c = -35c; \\ x: \quad & 6b - 3d = -35d; \\ 1: \quad & 2c = -35e. \end{aligned}$$

Therefore, $a = c = e = 0$, $b = -1$ and $d = 3/16$, i.e. $L_5(x) = x^5 - x^3 + \frac{3}{16}x$.

81. Let $F[2\pi]$ be the vector space of all real 2π-periodic infinitely differentiable functions in one variable t with the inner product $(f, g) = \int_{-\pi}^{\pi} f(t)g(t)dt$.

(a) Prove that the operator $L = d^2/dt^2$ on $F[2\pi]$ is symmetric.

(b) Find all eigenvalues and eigenfunctions of the above operator L on $F[2\pi]$.

Solution: (a) For $f, g \in F[2\pi]$ we have

$$(Lf, g) = \int_{-\pi}^{\pi} f''(t)\, g(t)\, dt = \left[f'(t)\, g(t)\right]_{-\pi}^{\pi} - \int_{-\pi}^{\pi} f'(t)\, g'(t)\, dt.$$

The first term is zero owing to the periodicity, and the second term is symmetric in f and g. So $(Lf, g) = (Lg, f) = (f, Lg)$, where the second equality follows from the symmetry of the inner product. Thus the operator L is symmetric.

(b) The eigenvalue equation $Lf = \lambda f$ is a differential equation with the general solution $f(t) = A\exp(\sqrt{\lambda}t) + B\exp(-\sqrt{\lambda}t)$. (or $f(t) = At + B$ if $\lambda = 0$). This is periodic iff $\lambda = -n^2$ with $n = 0, 1, 2, \dots$. For each $n > 0$ we have two independent eigenfunctions, namely $\cos(nt)$ and $\sin(nt)$. For $\lambda = 0$ there is one eigenfunction, namely $f = \text{constant}$.

82. Let $F[2\pi]$ be the vector space from the above problem **??** and consider the operator $L_1 = d/dt$ on $F[2\pi]$.

(a) Prove that L_1 is skew-symmetric, i.e. for any $f, g \in F[2\pi]$, we have $(L_1 f, g) = -(f, L_1 g)$.

(b) Deduce that the only eigenfunctions for L_1 in $F[2\pi]$ are constant functions (with the zero eigenvalue).

(c) Let $F_{\mathbb{C}}[2\pi]$ be the complexification of the above $F[2\pi]$. Prove that the only eigenvalues for L_1 on $F_{\mathbb{C}}[2\pi]$ are complex numbers $\{ni \mid n \in \mathbb{Z}\}$ and for each such number $\lambda_n = ni$, there is only one (up to a non-zero scalar factor) eigenfunction e^{int} in $F_{\mathbb{C}}[2\pi]$.

(d) Prove that the functions $1, e^{it}, e^{-it}, e^{2it}, e^{-2it}, \ldots, e^{nit}, e^{-int}, \ldots$ are orthogonal in $F_{\mathbb{C}}[2\pi]$. What are the norms of those functions?

Solution: (a) For $f, g \in F[2\pi]$ we have

$$(L_1 f, g) = \int_{-\pi}^{\pi} f'(t)\, g(t)\, dt = [f(t)\, g(t)]_{-\pi}^{\pi} - \int_{-\pi}^{\pi} f(t)\, g'(t)\, dt = -(f, L_1 g).$$

(b) If $L_1 f = \lambda f$ with $f \neq 0$, then $(L_1 f, f) = -(f, L_1 f)$ gives $\lambda \|f\|^2 = -\lambda \|f\|^2$ and hence $\lambda = 0$. So we get $L_1 f = f' = 0$ and thus f is constant.

(c) In the complex case, the inner product is $\langle f, g \rangle = \int_{-\pi}^{\pi} f(t)\, \overline{g(t)}\, dt$. The same calculation as in (b) now gives $\lambda \|f\|^2 = -\bar{\lambda} \|f\|^2$, and hence λ has the form $\lambda = in$ with n real. Thus the eigenfunction equation $L_1 f = \lambda f$ says $f' = inf$, which has the solution $f(t) = A \exp(int)$. But for this to be 2π-periodic, we need n to be an integer.

(d) Use $\int_{-\pi}^{\pi} \exp(ipt) \exp(iqt)\, dt = 2\pi$ if $q = -p$, and zero if $q \neq -p$. So the given functions are mutually-orthogonal, and each has norm $\sqrt{2\pi}$.

83. Prove that each of the following sets forms a group under ordinary multiplication.

(a) $\{2^k \mid k \in \mathbb{Z}\}$.

(b) $\{\frac{1+2m}{1+2n} \mid m, n \in \mathbb{Z}\}$.

(c) $\{\cos\theta + i \sin\theta \mid \theta\}$ where θ runs over all rational numbers.

Solution: Associativity of the multiplication in the real or complex numbers implies associativity in each case.

(a) $2^k 2^l = 2^{k+l}$ implies closure. The identity is $1 = 2^0$ and the inverse of 2^k is 2^{-k}.

(b) Closure follows from the observation that the product of two odd numbers is odd. The identity has $m = n = 0$. The inverse swaps the role of m and n.

(c) We are dealing with complex numbers of the form $e^{i\theta}$ where θ is rational. The multiplication then just adds the relevant θ values. That the sum of two rationals is rational gives closure. The identity has $\theta = 0$ and the inverse for θ requires the value $-\theta$ which is also rational.

84. Think of the integers $\mathbb{Z}$ as points equally spaced along the real line. Define two kinds of transformations on $\mathbb{Z}$:

(1) Translations of the form T_a (where a is an integer) which have the effect of translating $\mathbb{Z}$ a places to the right (if $a \geq 0$; or $-a$ places to the left if $a < 0$) using the formula $n \mapsto n + a$.

(2) Reflections of the form R_c (where c is an integer) which have the effect of reflecting $\mathbb{Z}$ in the point $\frac{c}{2}$ using the formula $n \mapsto c - n$.

Work out the effect of composing the following pairs of transformations: (a) $T_b T_a$, (b) $R_d T_a$, (c) $T_b R_c$, (d) $R_d R_c$. [In each case, because these are *functions* the compositions have to be evaluated from right to left; e.g., $T_b T_a$ means first do T_a and then do T_b.]

Now let A be the set of all such T_a and R_c. Show that A is a group and that we can find examples of elements $g,\ h \in A$ such that $gh \neq hg$, $g^2 = h^2 = e$ and $\forall s > 0,\ (gh)^s \neq e$.

Solution: (a) The effect of $T_b T_a$ is $n \mapsto n + a \mapsto n + a + b$, so $T_b T_a = T_{a+b}$.

(b) The effect of $R_d T_a$ is $n \mapsto n + a \mapsto d - (n + a) = (d - a) - n$, so $R_d T_a = R_{d-a}$.

(c) $n \mapsto c - n \mapsto b + c - n$, so $T_b R_c = R_{b+c}$.

(d) $n \mapsto c - n \mapsto d - (c - n) = (d - c) + n$, so $R_d R_c = T_{d-c}$.

The above calculations show we have closure. The group law is composition of functions so associativity holds. The identity is T_0. The inverse of T_a is T_{-a} and that of R_c is R_c.

For the final bit, just take two different reflections. E.g., $g = R_1$ and $h = R_0$. Then $g^2 = h^2 = e$, $gh = T_1$, $hg = T_{-1}$, and $(gh)^k = T_k \neq e$.

85. Let G be the set of all 2×2 matrices of the form $\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$ where $a \in \mathbb{R}$. Show that G is an abelian group under matrix multiplication. What is it isomorphic to?

Solution: Since $\left(\begin{smallmatrix} 1 & a \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 & a+b \\ 0 & 1 \end{smallmatrix}\right)$, the product of two elements is precisely of the form $\begin{pmatrix} 1 & \tilde{a} \\ 0 & 1 \end{pmatrix}$ where $\tilde{a} = a + b$, so we have closure. Associativity comes from the fact that matrix multiplication is associative. I_2 is in G simply by choosing $a = 0$ and is the identity. Finally, the inverse of $\left(\begin{smallmatrix} 1 & a \\ 0 & 1 \end{smallmatrix}\right)$ is $\left(\begin{smallmatrix} 1 & -a \\ 0 & 1 \end{smallmatrix}\right)$.

Since $\left(\begin{smallmatrix} 1 & a \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 & a+b \\ 0 & 1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & a \\ 0 & 1 \end{smallmatrix}\right)$ the group is abelian and isomorphic to $\mathbb{R}$ as an additive group.

86. (a) Let G be the set of all 2×2 matrices of the form $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$ where $a,\ b,\ d \in \mathbb{R}$, and $ad \neq 0$. Show that G is a group under matrix multiplication.

(b) With G as in part (a), define $Z(G) = \{g \in G \,|\, \text{such that}, \forall h \in G,\ gh = hg\}$. Identify the elements of $Z(G)$ and show that it is also a group. [$Z(G)$ is called the *centre* of G.]

Solution: (a) Since $\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right)\left(\begin{smallmatrix} e & f \\ 0 & h \end{smallmatrix}\right) = \left(\begin{smallmatrix} ae & af+bh \\ 0 & dh \end{smallmatrix}\right)$ and $\mathbb{R}$ is closed under addition and multiplication we have that all entries are real and $aedh \neq 0$ since $ad, eh \neq 0$, we have closure. Associativity comes from the fact that matrix multiplication is associative (this is a case where you should *not* multiply out three example matrices two ways!). I_2 is in G and is the identity. Finally, the inverse of $\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right)$ is $\left(\begin{smallmatrix} a^{-1} & -a^{-1}bd^{-1} \\ 0 & d^{-1} \end{smallmatrix}\right)$ with entries once more in $\mathbb{R}$.

(b) Suppose $\left(\begin{smallmatrix} A & B \\ 0 & D \end{smallmatrix}\right)$ is in $Z(G)$, so $\left(\begin{smallmatrix} Aa & Ab+Bd \\ 0 & Dd \end{smallmatrix}\right) = \left(\begin{smallmatrix} A & B \\ 0 & D \end{smallmatrix}\right)\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right) = \left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right)\left(\begin{smallmatrix} A & B \\ 0 & D \end{smallmatrix}\right) = \left(\begin{smallmatrix} aA & aB+bD \\ 0 & dD \end{smallmatrix}\right)$ whenever $ad \neq 0$, or equivalently, $Ab + Bd = aB + bD$ whenever $ad \neq 0$. Taking $a = 2$, $d = 1$ and $b = 0$ shows $B = 0$; then taking $b = 1$ shows $A = D$. If $B = 0$ and $A = D$ the equation is satisfied, so $Z(G) = \{\left(\begin{smallmatrix} A & 0 \\ 0 & A \end{smallmatrix}\right) \text{ with } A \neq 0\}$. To see $Z(G)$ is a group we note that closure follows from $\left(\begin{smallmatrix} A & 0 \\ 0 & A \end{smallmatrix}\right)\left(\begin{smallmatrix} A' & 0 \\ 0 & A' \end{smallmatrix}\right) = \left(\begin{smallmatrix} AA' & 0 \\ 0 & AA' \end{smallmatrix}\right)$, matrix multiplication is associative, we get the identity for $A = 1$, and $\left(\begin{smallmatrix} A & 0 \\ 0 & A \end{smallmatrix}\right)$ has inverse $\left(\begin{smallmatrix} A^{-1} & 0 \\ 0 & A^{-1} \end{smallmatrix}\right)$.

87. (a) The modular group is defined by $SL(2, \mathbb{Z}) = \{A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \text{ with } a, b, c, d \in \mathbb{Z} \text{ and } \det A = 1\}$. Show that $SL(2, \mathbb{Z})$ is indeed a group under matrix multiplication (you may assume associativity).

(b) Show that $T = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ belongs to $SL(2, \mathbb{Z})$ and compute T^n with $n \in \mathbb{Z}$ (for negative integers T^{-n} means $(T^{-1})^n$). What is the connection with Exercise **??** ?

Solution: (a) Since $\mathbb{Z}$ is a group, $SL(2, \mathbb{Z})$ is closed under matrix multiplication and inversion. The identity matrix $I_2 \in SL(2, \mathbb{Z})$ so $SL(2, \mathbb{Z})$ is a group.

(b) T has clearly determinant $+1$ and all its entries are in $\mathbb{Z}$ so $T \in SL(2, \mathbb{Z})$. Furthermore $T^n = \left(\begin{smallmatrix} 1 & n \\ 0 & 1 \end{smallmatrix}\right)$ for all $n \in \mathbb{Z}$. This means that $G = \{T^n \text{ with } n \in \mathbb{Z}\}$ is a subgroup of $SL(2, \mathbb{Z})$ identical to the subgroup of Exercise **??** with $a \in \mathbb{Z}$. This subgroup is isomorphic to $\mathbb{Z}$ as an additive group.

88. Let G be a group such that for every element $g \in G$, $g^2 = e$. Show that G is abelian (i.e. $gf = fg$ for any $f, g \in G$).

Solution: Let g, h be any two elements in G. Then since G is a group we have closure under group operation which implies $gh \in G$, hence $(gh)^2 = e$ and expanding the left hand side out we have $ghgh = e$. Multiply this on the left by g and on the right by h to find $g^2hgh^2 = geh$, and since $g^2 = h^2 = e$ this simplifies to $hg = gh$, so the group is Abelian.

89. Show that the group $\mathbb{Z}_8^\times$ has order 4. Is it isomorphic either to $\mathbb{Z}_4$ or to the Klein group V?

Solution: The group table, from multiplication modulo 8, is

$\times$	$\overline{1}$	$\overline{3}$	$\overline{5}$	$\overline{7}$
$\overline{1}$	$\overline{1}$	$\overline{3}$	$\overline{5}$	$\overline{7}$
$\overline{3}$	$\overline{3}$	$\overline{1}$	$\overline{7}$	$\overline{5}$
$\overline{5}$	$\overline{5}$	$\overline{7}$	$\overline{1}$	$\overline{3}$
$\overline{7}$	$\overline{7}$	$\overline{5}$	$\overline{3}$	$\overline{1}$

The Klein group V has table

$\cdot$	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

and clearly $\mathbb{Z}_8^\times \cong V$.

90. Write down the group table of the multiplicative group $\mathbb{Z}_9^\times$. Is this group isomorphic to $\mathbb{Z}_n$ for any n?

Solution: The group table of $\mathbb{Z}_9^\times$ is

$\times$	$\overline{1}$	$\overline{2}$	$\overline{4}$	$\overline{5}$	$\overline{7}$	$\overline{8}$
$\overline{1}$	$\overline{1}$	$\overline{2}$	$\overline{4}$	$\overline{5}$	$\overline{7}$	$\overline{8}$
$\overline{2}$	$\overline{2}$	$\overline{4}$	$\overline{8}$	$\overline{1}$	$\overline{5}$	$\overline{7}$
$\overline{4}$	$\overline{4}$	$\overline{8}$	$\overline{7}$	$\overline{2}$	$\overline{1}$	$\overline{5}$
$\overline{5}$	$\overline{5}$	$\overline{1}$	$\overline{2}$	$\overline{7}$	$\overline{8}$	$\overline{4}$
$\overline{7}$	$\overline{7}$	$\overline{5}$	$\overline{1}$	$\overline{8}$	$\overline{4}$	$\overline{2}$
$\overline{8}$	$\overline{8}$	$\overline{7}$	$\overline{5}$	$\overline{4}$	$\overline{2}$	$\overline{1}$

The group table of $\mathbb{Z}_6$, with its elements re-arranged, is

$+$	$\overline{0}$	$\overline{1}$	$\overline{2}$	$\overline{5}$	$\overline{4}$	$\overline{3}$
$\overline{0}$	$\overline{0}$	$\overline{1}$	$\overline{2}$	$\overline{5}$	$\overline{4}$	$\overline{3}$
$\overline{1}$	$\overline{1}$	$\overline{2}$	$\overline{3}$	$\overline{0}$	$\overline{5}$	$\overline{4}$
$\overline{2}$	$\overline{2}$	$\overline{3}$	$\overline{4}$	$\overline{1}$	$\overline{0}$	$\overline{5}$
$\overline{5}$	$\overline{5}$	$\overline{0}$	$\overline{1}$	$\overline{4}$	$\overline{3}$	$\overline{2}$
$\overline{4}$	$\overline{4}$	$\overline{5}$	$\overline{0}$	$\overline{3}$	$\overline{2}$	$\overline{1}$
$\overline{3}$	$\overline{3}$	$\overline{4}$	$\overline{5}$	$\overline{2}$	$\overline{1}$	$\overline{0}$

The tables are the same, and so $\mathbb{Z}_9^\times \cong \mathbb{Z}_6$.

91. Write down the group table for $\mathbb{Z}_2 \times \mathbb{Z}_2$, the direct product of two copies of the cyclic group of order two, and compute its order. Is this group isomorphic to any group discussed during lectures?

Solution: Every element in $\mathbb{Z}_2 \times \mathbb{Z}_2$ can be written as (g_1, g_2) with $g_1, g_2 \in \mathbb{Z}_2$. The order of the direct product group is the product of the orders of each $\mathbb{Z}_2$ factors, i.e. it has order 4 and the elements are $\mathbb{Z}_2 \times \mathbb{Z}_2 = \{(0,0),(1,0),(0,1),(1,1)\}$ with group table given by

$\cdot$	$(0,0)$	$(1,0)$	$(0,1)$	$(1,1)$
$(0,0)$	$(0,0)$	$(1,0)$	$(0,1)$	$(1,1)$
$(1,0)$	$(1,0)$	$(0,0)$	$(1,1)$	$(0,1)$
$(0,1)$	$(0,1)$	$(1,1)$	$(0,0)$	$(1,0)$
$(1,1)$	$(1,1)$	$(0,1)$	$(1,0)$	$(0,0)$

which is clearly isomorphic to the Klein group.