KNOTJOB DOCUMENTATION

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1. Introduction

KnotJob is a Java programme designed to calculate Steenrod Squares for the stable Khovanov homotopy type of Lipshitz-Sarkar [2] and the $\mathfrak{sl}_n$-stable homotopy type for matched diagrams of Jones-Lobb-Schütz [1].

2. Usage

The command

```
java KnotJob
```

will calculate the invariant $\mathrm{St}(0,-6,3)$ of the pretzel link $P(-2,2,2)$. This is also achieved using

```
java KnotJob -st 0 -6 3 -dg -1,2.2,3.3,-1.
```

The command `-st` is followed by integers h, q and n for homological and quantum degrees in $\mathfrak{sl}_n$-Khovanov-Rozansky homology. The command

```
-dg -1,2.2,3.3,-1.
```

describes the diagram using the 'Double-Gauss'-code, see Section 4.

Other commands are

```
-a n
```

which computes the Khovanov homology with $\mathbb{Z}/2$ coefficients for $n = 2$, and the $\mathfrak{sl}_n$-Khovanov-Rozansky homology with $\mathbb{Z}/2$ coefficients for $n > 2$.

Specific calculation of a given group with homological degree h and quantum degree q is done by

```
-h h q n
```

Other commands are

```
-s1 h q n
-s2 h q n
-s3 h q n
-s21 h q n
-s22 h q n
```

which calculate the image of the Steenrod Square homomorphism

$$\mathrm{Sq}^j\,\mathrm{Sq}^i\colon H^{h,q} \to H^{h+i+j,q}$$

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for

$i = 0,\, j = 1$	`-s1`
$i = 0,\, j = 2$	`-s2`
$i = 0,\, j = 3$	`-s3`
$i = 2,\, j = 1$	`-s21`
$i = 2,\, j = 2$	`-s22`

The somewhat cryptic output after these commands is explained in Section 5.

Calculations with coefficients in $\mathbb{Z}/4$ can also be done. Using

-a4n+

will computes the Khovanov homology with $\mathbb{Z}/4$ coefficients for $n = 2$, and the $\mathfrak{sl}_n$-Khovanov-Rozansky homology with $\mathbb{Z}/4$ coefficients for $n > 2$ (note that for $n > 2$ this may not be well defined, so the result should be taken with caution).

Specific calculation of a given group with homological degree h and quantum degree q with coefficients in $\mathbb{Z}/4$ is done using

```
-h4 h q n
```

The Bockstein homomorphisms

$$\beta\colon H^{h,q}(K;\mathbb{Z}/2) \to H^{h+1,q}(K;\mathbb{Z}/4)$$
$$\beta\colon H^{h,q}(K;\mathbb{Z}/4) \to H^{h+1,q}(K;\mathbb{Z}/2)$$
$$\beta\colon H^{h,q}(K;\mathbb{Z}/4) \to H^{h+1,q}(K;\mathbb{Z}/4)$$

are done using the commands

```
-b24 h q n
-b42 h q n
-b44 h q n
```

respectively.

In case of problems, recompiling with

```
javac KnotJob.java
```

may help.

3. Link diagrams

At the moment there exist two ways to encode link diagrams in KnotJob. Consider the diagram for the pretzel knot $P(-2, 3, 3)$ in Figure 1. There are two tangles with 3 crossings, each contributing a $+3$ to the writhe, and one tangle with 2 crossings, also contributing $+2$ to the writhe. Notice that the orientation of the knot is so that at the 3-tangles the knot always enters the tangle from either below or above, while at the 2-tangle one entrance is from above and the other from below.

To encode the information, we require three tangles which have to be ordered in some way, and six paths between these tangles, which are numbered by 1 to 6. For $P(-2, 3, 3)$ we then enter

```
-n -2,3,3 1,2,4,5,3,6,2,1,5,4,6,3
```

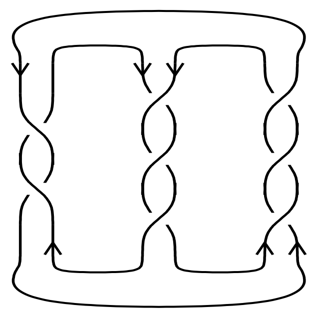

FIGURE 1. The pretzel knot $P(-2,3,3)$.

where the first three numbers list the number of crossings in each tangle, and the twelve following numbers encode where the paths enter the tangle. The first four numbers `1,2,4,5` mean that path 1 enters the tangle at the right bottom, path 2 enters at the right top, 4 enters at the left top and 5 enters at the left bottom.

The first tangle is encoded with -2 where the minus sign indicates that starting at the right bottom will begin with an overcross.

To calculate the $\mathbb{Z}/2$ Khovanov homology of $P(-2,3,3)$ one can therefore use the command

```
java KnotJob -a 2 -n -2,3,3 1,2,4,5,3,6,2,1,5,4,6,3
```

The output given is

```
( t^5 ) q^17
( t^5 ) q^15
( t^3 + t^4 ) q^13
( t^2 + t^3 + t^4 ) q^11
( t^2 ) q^9
( t^0 ) q^7
( t^0 ) q^5
```

which means, for example, that the $\mathbb{Z}/2$-Khovanov homology in quantum degree $q = 11$ is nontrivial in homological degrees 2, 3 and 4, with Betti number 1 each time. A larger Betti number, say 3 in homological degree 5 and quantum degree 17, would be expressed as `( 3t^5 ) q^17`.

For $\mathfrak{sl}_n$ with $n \geq 3$ it is important to note that a bipartite diagram with matching orientations (meaning the orientation enters the tangle on one strand from above and on one strand from below) is needed. KnotJob does not check for this, so one can enter the above line with $n = 3$, but one should not confuse the result with a knot invariant.

Changing the odd numbers to even numbers will make the above diagram valid for $n \geq 3$, in particular, changing them to 2 will lead to the pretzel link $P(-2,2,2)$.

4. Double-Gauss code

The Double-Gauss code is a simple way to encode bipartite link diagrams inspired by the (extended) Gauss code. It only works for 2-tangles, so to encode $2n$-tangles one has to split this tangle into n 2-tangles. Enumerate the tangles arbitrarily, then pick a point on the link away from the tangles and follow the orientation. Write down the number of the first tangle that is entered, and use a minus sign in front of the number of the tangle in case it starts with an overcrossing. Continue with the next tangle in the same way. If a tangle is entered for the second time, use a minus sign in front of the number of the tangle in case the writhe of the tangle is positive. Separate the numbers by commas, and use a full-stop if after the tangle the component of a link is being closed. Continue then with the next component, if there are more.

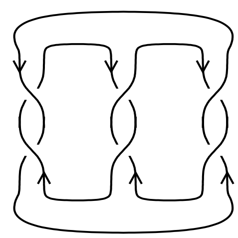

FIGURE 2. The Pretzel link $P(-2, 2, 2)$.

In KnotJob the Double-Gauss code is entered via `-dg -1,2.2,3.3,-1.` where we start at the right lower corner of the leftmost tangle. The tangles are enumerated from left to right. As we start with an overcrossing, the first number is negative. Likewise, the second time we get to tangle 1 we denote it by -1 as the writhe of that tangle is $+2$. The '.' symbolizes the end of a component and needs to be entered.

For another example, consider Figure 3.

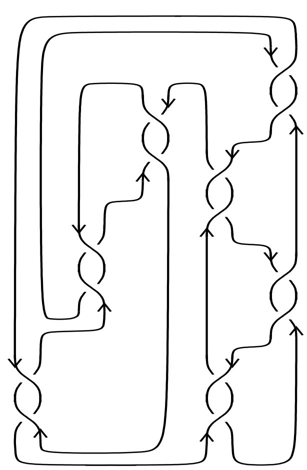

FIGURE 3. A bipartite knot diagram.

Its $\mathfrak{sl}_3$-Khovanov-Rozansky homology can be calculated by

```
java KnotJob -a 3 -dg -1,-2,3,-2,4,-5,6,-7,6,4,-1,7,5,-3.
```

The $\mathfrak{sl}_4$-Khovanov-Rozansky homology with coefficients in $\mathbb{Z}/4$ can be calculated with

```
java KnotJob -a4 4 -dg -1,-2,3,-2,4,-5,6,-7,6,4,-1,7,5,-3.
```

The output consists of one line for each quantum degree, and for example the line

```
( 2 t^-2 + [2] t^-1 + (2 + [2])t^0 ) q^1
```

means that in quantum degree $q = 1$ we have non-trivial groups in homological degrees -2, -1 and 0, with

$$H^{-2,1} = (\mathbb{Z}/4)^2, \qquad H^{-1,1} = \mathbb{Z}/2, \qquad H^{0,1} = (\mathbb{Z}/4)^2 \oplus \mathbb{Z}/2.$$

5. Direct calculation of Steenrod Squares

The command

```
java KnotJob -s2 2 11 2 -n -2,3,3 1,2,4,5,3,6,2,1,5,4,6,3
```

computes the second Steenrod square of the torus knot $T_{3,4}$ with the output

```
Cocycle : 6 5 3 2 1 0
Steenrod : 5 1 2 7
Represents : 7
```

This means there is one cocycle, and the numbers indicate which generators in the cochain complex are needed for this cocycle. The numbers following `Steenrod` list the generators needed for the cocycle representing the Steenrod square of the previous cocycle. The number following `Represents` indicates the generator in the cohomology group that the Steenrod square represents. If no numbers would follow `Represents` then the Steenrod square would be trivial.

If the cohomology group H^i has more than one generator, several cocycles would be listed which form a basis of the cohomology. Also, their Steenrod squares would be listed in the same order.

The output for other Steenrod square commands follows the same principle.

The command

```
java KnotJob -b24 -2 -1 4 -dg -1,-2,3,-2,4,-5,6,-7,6,4,-1,7,5,-3.
```

calculates the Bockstein homomorphism $\beta\colon H^{-2,-1}(\mathbb{Z}/2) \to H^{-1,-1}(\mathbb{Z}/4)$ and the output

```
Represents (mod4): 2473(2)
Represents (mod4):
```

means that one of the generators of $H^{-2,-1}(\mathbb{Z}/2)$ has non-trivial image with the `(2)` indicating that it represents the element of order 2 in a $\mathbb{Z}/4$ summand.

References

[1] D. Jones, A. Lobb, D. Schütz, *An $\mathfrak{sl}_n$-stable homotopy type for matched diagrams*, preprint.
[2] R. Lipshitz, S. Sarkar, *A Steenrod square on Khovanov homology*, J. Topol. 7 (2014), 817–848.

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