ON THE LUSTERNIK-SCHNIRELMAN THEORY OF A REAL COHOMOLOGY CLASS

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Abstract. Farber developed a Lusternik-Schnirelman theory for finite CW-complexes X and cohomology classes $\xi \in H^1(X;\mathbb{R})$. This theory has similar properties as the classical Lusternik-Schnirelman theory. In particular in [7] Farber defines a homotopy invariant $\mathrm{cat}(X,\xi)$ as a generalization of the Lusternik-Schnirelman category. If X is a closed smooth manifold this invariant relates to the number of zeros of a closed 1-form ω representing ξ. Namely, a closed 1-form ω representing ξ which admits a gradient-like vector field with no homoclinic cycles has at least $\mathrm{cat}(X,\xi)$ zeros. In this paper we define an invariant $F(X,\xi)$ for closed smooth manifolds X which gives the least number of zeros a closed 1-form representing ξ can have such that it admits a gradient-like vector field without homoclinic cycles and give estimations for this number.

1. Introduction

In [5, 6, 7, 8] Farber developed a Lusternik-Schnirelman theory for finite CW-complexes X and cohomology classes $\xi \in H^1(X;\mathbb{R})$. This theory has similar properties as the classical Lusternik-Schnirelman theory. In particular in [7] Farber defines a homotopy invariant $\mathrm{cat}(X,\xi)$ as a generalization of the Lusternik-Schnirelman category. If X is a closed smooth manifold this invariant relates to the number of zeros of a closed 1-form ω representing ξ.

To describe this relation we need the notion of a gradient-like vector field for ω. We say a vector field v is gradient-like for ω, if $v(x) = 0$ if and only if $\omega_x = 0$ and $\omega(v) > 0$ everywhere else. By a homoclinic orbit of v we mean a nontrivial trajectory γ of v with $\lim_{t\to\pm\infty}\gamma(t) = p$, where $v(p) = 0$. Slightly more general we define a homoclinic cycle to be a finite sequence of trajectories between zeros of v which form a closed cycle. The main connection between $\mathrm{cat}(X,\xi)$ and the zeros of closed 1-forms representing ξ is now summarized in

Theorem 1.1. *Let ω be a closed 1-form on the closed smooth manifold M and let $\xi \in H^1(M;\mathbb{R})$ denote its cohomology class. If ω admits a gradient-like vector field v with no homoclinic cycles, then ω has at least $\mathrm{cat}(M,\xi)$ zeros.*

This theorem was proved by Farber [7, Th.4.1] under slightly stronger assumptions on v which we remove in Section 2.

In general it is possible to represent a nonzero cohomology class on a connected manifold by a closed 1-form ω having at most one zero, see [7, Th.2.1], so for $\mathrm{cat}(M,\xi) \geq 2$ it is not possible to find gradient-like vector fields for such an ω

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which do not admit homoclinic orbits.

Gradient-like vector fields of closed 1-forms which do not have homoclinic cycles are of interest since it is possible to collect information about their closed orbit structure in form of a zeta function. This is shown in [16] which also discusses the properties of such zeta functions.

In this paper we define an invariant $F(M, \xi)$ which gives the least number of zeros a closed 1-form on M representing $\xi \in H^1(M;\mathbb{R})$ can have such that it admits a gradient-like vector field without homoclinic cycles. In view of Theorem 1.1 we get

$$\operatorname{cat}(M,\xi) \leq F(M,\xi). \tag{1}$$

If we represent the cohomology class ξ by a closed 1-form ω such that all of its zeros are of Morse type, then gradient-like vector fields have generically no homoclinic cycles. On the other hand a lot of zeros can be necessary when ξ is represented by such a Morse form. As an example we have the orientable surfaces M_g of genus g. Any Morse form ω representing a real cohomology class will have at least $2g-2$ zeros while we show in Section 4 that $F(M_g, \xi) = 1$ for $\xi \in H^1(M_g, \mathbb{Z}) - \{0\}$.

Our method to find closed 1-forms with a small number of zeros while still admitting gradient-like vector fields without homoclinic cycles is inspired by Farber's proof of [7, Th.2.1]. There one starts with an arbitrary Morse form and collides the nondegenerate zeros into degenerate zeros, a technique which originates from the work of Takens [17], who used this to get upper bounds for the original Lusternik-Schnirelman category. While it is possible for a nonzero integer valued cohomology class to collide all nondegenerate zeros into a single degenerate zero it is not possible to avoid homoclinic cycles using this method. But we refine this technique so that we can keep track of the vector field and by just colliding zeros of a common index we are able to avoid homoclinic cycles. This works for all Morse zeros except for those having index 1 or $n-1$, where $n = \dim M$. To avoid this difficulty we have to assume that the cohomology class ξ admits a certain group theoretic property which gives the existence of a Morse closed 1-form ω representing ξ without zeros of index $0, 1, n-1$ and n. We then obtain the following theorem.

Theorem 1.2. *Let M be a closed connected smooth manifold with $n = \dim M \geq 5$ and let $\xi \in H^1(M;\mathbb{R})$ be CC^0 at $-\infty$ and ∞. Then $F(M,\xi) \leq n-3$.*

Cohomology classes $\xi \in H^1(M;\mathbb{R})$ are in one-to-one correspondence with homomorphism $\pi_1(M) \to \mathbb{R}$. The condition that ξ is CC^0 at $-\infty$ and ∞ translates to a group theoretic condition on the corresponding homomorphism $\xi : \pi_1(M) \to \mathbb{R}$ explained in Section 3. A sufficient condition for ξ to be CC^0 at $-\infty$ and ∞ is that ξ has finitely generated kernel. In the case where the homomorphism ξ has discrete image in $\mathbb{R}$ this condition is also necessary. The technique above then allows us to collide all zeros of a fixed index into one degenerate zero which explains the upper bound $n-3$ in Theorem 1.2.

Another question is whether the inequality (1) is sharp. This is not the case since counterexamples can already be found for $\operatorname{cat}(M,\xi) = 0$. The corresponding condition $F(M,\xi) = 0$ requires the existence of a closed 1-form ω representing ξ without any zeros. This problem has been studied for $\dim M \geq 6$ by Latour [9] who gives two conditions necessary and sufficient for the existence of such a closed 1-form. It turns out that $\operatorname{cat}(M,\xi) = \operatorname{cat}(M,-\xi) = 0$ is equivalent to the first, homotopy theoretic condition used by Latour in [9, Th.1]. If this condition is satisfied one still has a K-theoretic obstruction for obtaining $F(M,\xi)=0$. But we can get the following result.

Theorem 1.3. *Let M be a closed connected smooth manifold with $n=\dim M\geq 5$ and let $\xi\in H^1(M;\mathbb{R})$ satisfy $\operatorname{cat}(M,\xi)=\operatorname{cat}(M,-\xi)=0$. Then $F(M,\xi)\leq 2$.*

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2. Recollection of the Lusternik-Schnirelman theory of a real cohomology class

Let us recall the Lusternik-Schnirelman theory for a finite CW-complex X and a cohomology class $\xi\in H^1(X;\mathbb{R})$ developed by Farber in [7]. We can identify the cohomology class with a homomorphism $\xi:\pi_1(X)\to\mathbb{R}$ where we think of $\pi_1(X)$ as the direct sum of the fundamental groups of the components of X. This homomorphism defines an action of $G=\pi_1(X)$ on $\mathbb{R}$ given by $g\cdot r=r+\xi(g)$. Let $\rho:\tilde{X}\to X$ be the universal cover of X. Since G acts freely on $\tilde{X}$ and $\mathbb{R}$ is contractible there exists an equivariant map $f:\tilde{X}\to\mathbb{R}$. If $A\subset X$, we denote $\tilde{A}=f^{-1}(A)\subset\tilde{X}$.

Definition 2.1. [7, Df.3.1] Let X be a finite CW-complex, $\xi\in H^1(X;\mathbb{R})$ and $f:\tilde{X}\to\mathbb{R}$ equivariant with respect to ξ. We define $\operatorname{cat}(X,\xi)$ to be the least integer k such that for any $N>0$ there exists an open cover $X=U\cup U_1\cup\ldots\cup U_k$ such that

1. Each inclusion $U_i\to X$ is nullhomotopic.
2. There exists an equivariant homotopy $\tilde{h}:\tilde{U}\times[0,1]\to\tilde{X}$ such that $\tilde{h}_0:\tilde{U}\to\tilde{X}$ is inclusion and for every $x\in\tilde{U}$ $f(\tilde{h}_1(x))-f(x)\leq -N$.

Note that this definition does not depend on the equivariant map f since if f_1 and f_2 are different equivariant maps there is a constant $C>0$ with $|f_1(x)-f_2(x)|\leq C$ for all $x\in\tilde{X}$ by cocompactness of G on $\tilde{X}$. Also if $\xi=0$ we get that U has to be empty for large N and we recover the classical Lusternik-Schnirelman category.

For basic properties and examples of $\operatorname{cat}(X,\xi)$ we refer the reader to Farber [7]. In particular it is shown in [7, §3.4] that $\operatorname{cat}(X,\xi)$ is a homotopy invariant.

If M is a closed smooth manifold, cohomology classes $\xi\in H^1(M;\mathbb{R})$ are represented by closed 1-forms ω. Furthermore ω pulls back to an exact form df on the universal cover $\tilde{M}$, i.e. there is a smooth map $f:\tilde{M}\to\mathbb{R}$ with $\rho^*\omega=df$. It is easily seen that f is equivariant with respect to ξ. We will only be interested in the case where ω has only finitely many zeros. For example this is the case if ω is a Morse form by which we mean a closed 1-form locally represented by differentials of real valued functions with nondegenerate critical points only.

Definition 2.2. Let ω be a closed 1-form.

1. A smooth vector field v is called a *gradient-like vector field for* ω, if $v=0$ if and only if $\omega=0$, and $\omega(v)>0$ if $v\neq 0$.
2. A smooth vector field v is called a *ω-gradient*, if there exists a Riemannian metric g such that v is dual to ω with respect to g.

In both cases we call ω a Lyapunov form of v. We note that a gradient-like vector field v for ω is an ω-gradient if and only if there is a neighborhood U of the zeros of ω such that v is dual to ω with respect to a Riemannian metric on U, see [14, Lm.5.2]. But in general a gradient-like vector field need not be a gradient.

Definition 2.3. Let ω be a closed 1-form with finitely many zeros and let v be a gradient-like vector field for ω. A *homoclinic cycle* is an ordered k-tuple $(\gamma_1, \ldots, \gamma_k)$ of nonconstant trajectories γ_i of v such that the limits $\lim_{t\to\pm\infty} \gamma_i(t)$ exist for $i = 1, \ldots, k$ with $\lim_{t\to\infty} \gamma_i(t) = \lim_{t\to-\infty} \gamma_{i+1}(t)$ for $i = 1, \ldots, k-1$ and $\lim_{t\to\infty} \gamma_k(t) = \lim_{t\to-\infty} \gamma_1(t)$. Here k is a positive integer.

We will often write $\gamma = (\gamma_1, \ldots, \gamma_k)$ for a homoclinic cycle to simplify notation. We will also write

$$\xi(\gamma) \quad = \quad \sum_{i=1}^{k} \int_{\gamma_i} \omega$$

for a homoclinic cycle as this coincides with the image of the loop determined by γ under the homomorphism ξ.

If ω is a Morse form it is always possible to find ω-gradients without homoclinic cycles, in fact this is a generic property of ω-gradients. This follows from a version of the Kupka-Smale theorem, see e.g. Pajitnov [12, Lm.5.1]. On the other hand it is possible to collide the zeros of a Morse form ω into a single degenerate zero as is shown by Farber [7, Th.2.1]. But it might not be possible to find a gradient for such a closed 1-form which does not have homoclinic cycles because of the following theorem of Farber.

Theorem 2.4. [7, Th.4.1] *Let ω be a closed 1-form on a smooth closed manifold M and let $\xi = [\omega] \in H^1(M;\mathbb{R})$ denote the cohomology class of ω. If ω admits an ω-gradient with no homoclinic cycles, then ω has at least* $\mathrm{cat}(M, \xi)$ *geometrically distinct zeros.*

As is pointed out in Farber [8], the proof gives a slightly stronger theorem.

Theorem 2.5. [8, Th.2.7] *Let ω be a closed 1-form on a closed smooth manifold M having less than* $\mathrm{cat}(M, \xi)$ *zeros, where $\xi = [\omega] \in H^1(M;\mathbb{R})$ denotes the cohomology class of ω. Then there exists an integer $N > 0$ such that any ω-gradient v has a homoclinic cycle γ with $\xi(\gamma) \leq N$.*

These two theorems motivate the following definition.

Definition 2.6. Let M be a smooth closed manifold and $\xi \in H^1(M;\mathbb{R})$. Then let $F(M, \xi)$ be the least integer k such that there exists a closed 1-form ω representing ξ with k zeros and a gradient-like vector field v for ω which has no homoclinic cycles.

We have $F(M, \xi) < \infty$ since we can always find a Morse form representing ξ. Furthermore $F(M, \xi)$ is symmetric in that $F(M, \xi) = F(M, -\xi)$, while $\mathrm{cat}(X, \xi) \neq \mathrm{cat}(X, -\xi)$ in general, see Farber [7, Ex.3.4] for such an example. Because of this we define a symmetrized version of $\mathrm{cat}(X, \xi)$.

Definition 2.7. Let X be a finite CW-complex, $\xi \in H^1(X;\mathbb{R})$ and $f : \tilde{X} \to \mathbb{R}$ equivariant with respect to ξ. Then $\mathrm{cat}_s(X, \xi)$ is the least integer k such that for any $N > 0$ there exists an open cover $X = U \cup U_1 \cup \ldots \cup U_k$ such that

(1) Each inclusion $U_i \to X$ is nullhomotopic.

(2) There exists an equivariant homotopy $\tilde{h} : \tilde{U} \times [-1,1] \to \tilde{X}$ such that $\tilde{h}_0 : \tilde{U} \to \tilde{X}$ is inclusion and for every $x \in \tilde{U}$

$$\begin{aligned} f(\tilde{h}_1(x)) - f(x) &\leq -N \\ f(\tilde{h}_{-1}(x)) - f(x) &\geq N. \end{aligned}$$

Clearly $\mathrm{cat}_s(X,\xi) = \mathrm{cat}_s(X,-\xi) \geq \max\{\mathrm{cat}(X,\xi), \mathrm{cat}(X,-\xi)\}$. The proof of [7, Lm.3.6] carries over to show that $\mathrm{cat}_s(X,\xi)$ is a homotopy invariant.

Example 2.8. Let $\varphi : S^2 \to S^2$ be a map of degree 2 and $X = S^2 \times [0,1]/(x,0) = (\varphi(x),1)$, i.e. X is the mapping torus of φ. Define $g : X \to S^1$ by $g([x,t]) = \exp 2\pi i t$. This g defines an element $\xi \in [X, S^1] = H^1(X;\mathbb{Z}) \subset H^1(X;\mathbb{R})$ and it is shown in Farber [7, Ex.3.4] that $\mathrm{cat}(X,\xi) = 0$ and $\mathrm{cat}(X,-\xi) \geq 1$.

Now define $X' = S^2 \times [0,1]/(x,1) = (\varphi(x),0)$, $g' : X' \to S^1$ by $g'([x,t]) = \exp 2\pi i t$ and $\xi' = [g'] \in [X', S^1]$. Furthermore let $Y = X \sqcup X'$ be the disjoint union. We claim that $\mathrm{cat}_s(Y, \xi \oplus \xi') > \max\{\mathrm{cat}(Y,\xi\oplus\xi'), \mathrm{cat}(Y, -(\xi\oplus\xi'))\}$.

We have

$$\mathrm{cat}(Y, -(\xi\oplus\xi')) = \mathrm{cat}(Y, \xi\oplus\xi') = \mathrm{cat}(X',\xi') = \mathrm{cat}(X,-\xi).$$

The first and third equality are clear by definition of X' and Y. To see the second equality let $N > 0$ and $X' = U \cup U_1 \ldots \cup U_k$ be a cover as in Defintion 2.1. Then $Y = (X \cup U) \cup U_1 \cup \ldots \cup U_k$ and since $\mathrm{cat}(X,\xi) = 0$ we get $\mathrm{cat}(Y, \xi\oplus\xi') = \mathrm{cat}(X',\xi')$.

Now let $Y = U \cup U_1 \cup \ldots \cup U_k$ be a cover as in Definition 2.7. Since the U_i are contractible in Y we either have $U_i \subset X$ or $U_i \subset X'$. Let $U_1, \ldots, U_j$ be the ones contained in X and $U_{j+1}, \ldots, U_k$ be the ones contained in X'. Now $(U \cap X) \cup U_1 \cup \ldots \cup U_j$ satisfies the conditions for $\mathrm{cat}(X,-\xi)$ and $(U \cap X) \cup U_{j+1} \cup \ldots \cup U_k$ satisfies the conditions for $\mathrm{cat}(X',\xi')$. It follows that $\mathrm{cat}_s(Y, \xi\oplus\xi') = 2\,\mathrm{cat}(X,-\xi) > \mathrm{cat}(X,-\xi)$.

In [8], Farber defines another invariant $\mathrm{Cat}(X,\xi)$. More precisely it is the least integer k such that there exists an open cover $X = U \cup U_1 \cup \ldots \cup U_k$ with the following properties:

(1) Each inclusion $U_i \to X$ is nullhomotopic.
(2) There exists an equivariant homotopy $\tilde{h} : \tilde{U} \times (-\infty,\infty) \to \tilde{X}$ such that $\tilde{h}_0 : \tilde{U} \to \tilde{X}$ is inclusion and for every $N > 0$ there exists a $t_N > 0$ with

$$\begin{aligned} f(\tilde{h}(x,t)) - f(x) &\leq -N \\ f(\tilde{h}(x,-t)) - f(x) &\geq N \end{aligned}$$

for all $x \in \tilde{U}$ and $t > t_N$.

Here X, ξ and f are as in Definition 2.7. It follows immediately that $\mathrm{cat}_s(X,\xi) \leq \mathrm{Cat}(X,\xi)$. The author does not know of an example where $\mathrm{cat}_s(X,\xi) < \mathrm{Cat}(X,\xi)$, but for our purposes it will be more convenient to work with $\mathrm{cat}_s(X,\xi)$.

Theorem 2.9. *Let ω be a closed 1-form on the closed smooth manifold M and let $\xi \in H^1(M;\mathbb{R})$ denote the cohomology class of ω. If ω admits a gradient-like vector field v with no homoclinic cycles, then ω has at least $\mathrm{cat}_s(M,\xi)$ zeros.*

In other words we have

$$\mathrm{cat}_s(M,\xi) \leq F(M,\xi). \tag{2}$$

Note that this is more general than Theorem 2.4 in that we allow more vector fields and that we have a sharper lower bound. Since the proof in [7] uses properties of ω-gradients near the zeros of ω we do not generalize it directly. Nevertheless we reduce the problem to Theorem 2.5.

Let $S(\omega) = \{x \in M \mid \omega_x = 0\}$. We can assume that $S(\omega)$ is a finite set and we then denote the cardinality of $S(\omega)$ by $|S(\omega)|$.

Definition 2.10. Let ω be a closed 1-form on the closed smooth manifold M with finitely many zeros. An *exact cover* U of $S(\omega)$ is an open neighborhood U of $S(\omega)$ such that the inclusion $S(\omega) \to U$ is a homotopy equivalence.

It follows that an exact cover U has exactly $|S(\omega)|$ components, each of which is contractible. In particular we get that $\omega|_U$ is exact.

Given an exact cover U and a gradient-like vector field v for ω we want to look at trajectories of v which start and end in U. Let $\gamma : [a, b] \to M$ be a trajectory of v. Then v can be lifted to a trajectory $\tilde{\gamma} : [a, b] \to \tilde{M}$ of $\tilde{v}$, the lift of v to the universal cover $\tilde{M}$ of M.

We define $\mathcal{T}_U(v)$ to be the set of trajectories $\gamma : [a, b] \to M$ of v satisfying the following properties:

(1) $\gamma(a), \gamma(b) \in U$.
(2) $\tilde{\gamma}(a)$ and $\tilde{\gamma}(b)$ are not in the same component of $\tilde{U} \subset \tilde{M}$ for a lift $\tilde{\gamma}$ of γ.

Note that condition (2) does not depend on the choice of $\tilde{\gamma}$.

Definition 2.11. Let ω be a closed 1-form on the closed smooth manifold M with finitely many zeros. Let U be an exact cover of $S(\omega)$, v a gradient-like vector field for ω and k a positive integer. Then a *U-cycle of length k* is an ordered k-tuple $(\gamma_1, \ldots, \gamma_k)$ with $\gamma_j : [a_j, b_j] \to M \in \mathcal{T}_U(v)$ for $j = 1, \ldots, k$ such that $\gamma_j(b_j)$ and $\gamma_{j+1}(a_{j+1})$ are in the same component of U for $j = 1, \ldots, k-1$ and also $\gamma_1(a_1)$ and $\gamma_k(b_k)$ are in the same component of U.

As with homoclinic cycles we will write $\gamma = (\gamma_1, \ldots, \gamma_k)$ for a U-cycle. Given γ, we can choose paths $\delta_j : [0, 1] \to U$ for $j = 1, \ldots, k$ such that $\delta_j(0) = \gamma_j(b_j)$ and $\delta_j(1) = \gamma_{j+1}(a_{j+1})$ for $j = 1, \ldots, k-1$ and $\delta_k(0) = \gamma_k(b_k)$, $\delta_k(1) = \gamma_1(a_1)$. The trajectories and the δ_j's can be combined to a map $S^1 \to M$ and the image under ξ of this loop does not depend on the choice of the δ_j. Therefore we denote the image by $\xi(\gamma) \in \mathbb{R}$.

There exist exact covers U which admit U-cycles with $\xi(\gamma) \leq 0$ but we can avoid this by choosing the exact cover small enough as we will see.

Lemma 2.12. *There exists an $\varepsilon > 0$ and an exact cover U such that $\int_\gamma \omega \geq \varepsilon$ for every $\gamma \in \mathcal{T}_U(v)$.*

Proof. Let U be a union of small open balls around $S(\omega)$ such that the components of $\overline{U}$, the closure of U, are still disjoint. We claim that there is a $\delta > 0$ such that for every $\gamma : [a, b] \to M \in \mathcal{T}_U(v)$ there is a subinterval $[c, c + \delta] \subset [a, b]$ with $\gamma([c, c + \delta]) \subset M - U$. To see this equip M with a Riemannian metric. Since $\gamma \in \mathcal{T}_U(v)$, γ has to travel a positive minimal distance in $M - U$. Here distance refers to the path metric coming from the Riemannian metric. But since M is compact, this distance cannot be travelled in arbitrary small time intervals by a trajectory of v. Therefore the required $\delta > 0$ exists.

Since $M-U$ is compact there is a $d>0$ such that $\omega_x(v(x)) \geq d$ for all $x \in M-U$. Now

$$\begin{aligned} \int_\gamma \omega &= \int_a^b \omega_{\gamma(t)}(v(\gamma(t)))\,dt \\ &\geq \int_c^{c+\delta} \omega_{\gamma(t)}(v(\gamma(t)))\,dt \;\geq\; \delta \cdot d \end{aligned}$$

for all $\gamma \in \mathcal{T}_U(v)$. Therefore $\varepsilon = \delta \cdot d$ will work. □

If U is an exact cover we define

$$l_U(v) \;=\; \sup\{t \in \mathbb{R} \,|\, t \leq \int_\gamma \omega \text{ for all } \gamma \in \mathcal{T}_U(v)\}.$$

Notice that if $V \subset U$ are both exact covers, then $l_U(v) \leq l_V(v)$. Let us also define

$$b_U^\xi(v) \;=\; \inf\{\xi(\gamma) \in \mathbb{R} \,|\, \gamma \text{ is a } U\text{-cycle}\}.$$

Here we interpret the infimum of a set without lower bound as $-\infty$. To see that there exist exact covers U with $b_U^\xi(v) > 0$ choose an exact cover V with $l_V(v) > 0$ by Lemma 2.12. Now choose an exact cover $U \subset V$ with $|\int_\delta \omega| < \frac{l_V(v)}{2}$ for every piecewise smooth path $\delta : [0,1] \to U$. Then $b_U^\xi(v) \geq \frac{l_U(v)}{2} > 0$.

Lemma 2.13. *Let $R > 0$. If v has no homoclinic cycles there exists an exact cover U with $b_U^\xi(v) > R$.*

Proof. Assume not, then there exists a sequence U_i of exact covers with $\bigcap_{i=1}^\infty U_i = S(\omega)$ and $b_\xi^{U_i}(v) \leq R$ for all i. We can assume that $|\int_\delta \omega| < \frac{l_{U_1}(v)}{2}$ for every piecewise smooth path $\delta : [0,1] \to U_1$ by the discussion above. There exists a sequence (γ_i) with γ_i a U_i-cycle and $\xi(\gamma_i) \leq R + \frac{1}{i}$. We have $\gamma_i = (\gamma_{i1}, \ldots, \gamma_{ik_i})$ with $\gamma_{ij} \in \mathcal{T}_{U_i}(v)$. Since $l_{U_i}(v)0$ we can assume that every γ_{ij} is minimal in the sense that it cannot be written as a concatenation $\gamma_{ij} = \gamma' * \gamma''$ of trajectories $\gamma', \gamma'' \in \mathcal{T}_{U_i}(v)$. For otherwise we replace γ_{ij} by γ' and γ'' thus increasing the length of γ_i. Also since $l_{U_1}(v) > 0$ we get that the lengths of the γ_i are bounded. By passing to a subsequence we can therefore assume that the lengths are constant, so there is a positive integer k such that $\gamma_i = (\gamma_{i1}, \ldots, \gamma_{ik})$ for every i.

Note that every exact cover U can be written as the disjoint union $U = \coprod_{x \in S(\omega)} U(x)$, where $U(x)$ is the component containing $x \in S(\omega)$. By passing successively to subsequences we can assume that there exist $x_1, \ldots, x_k \in S(\omega)$ such that for all i and every $j = 1, \ldots, k$ γ_{ij} is a trajectory starting in $U_i(x_j)$ and ending in $U_i(x_{j+1})$. Here we identified $x_1 = x_{k+1}$. Now choose a point $y_{ij} \in M - U_1$ on the trajectory γ_{ij}. Since $M - U_1$ is compact we can assume that the y_{ij} converge to $y_j \in M - U_1$ for $j = 1, \ldots, k$. It follows that the trajectory $\gamma_{\infty,j} : (-\infty, \infty) \to M$ with $\gamma_{\infty,j}(0) = y_j$ connects x_j and x_{j+1}. Now $(\gamma_{\infty,1}, \ldots, \gamma_{\infty,k})$ is a homoclinic cycle contradicting the nonexistence of homoclinic cycles. □

Proof of Theorem 2.9. Let v be a gradient-like vector field and $R > 0$. By Lemma 2.13 there is an exact cover U with $b_U^\xi(v) > R$. Now change v on U only to get an ω-gradient v'. Since v' agrees with v outside of U we get $b_\xi^U(v') > R$. This means we can find for every $R > 0$ an ω-gradient v' with no homoclinic cycles γ satisfying

$\xi(\gamma) \leq R$. By Theorem 2.5 ω has at least $\mathrm{cat}(M,\xi)$ zeros. The proof of Theorem 2.5 in [7, §4] can easily be adapted to see that one has at least $\mathrm{cat}_s(M,\xi)$ zeros. □

We will see in the next section that the inequality (2) is not an equality in general.

3. Spaces of category zero

In this section we want to analyze the case when $\mathrm{cat}(X,\xi)$ or $\mathrm{cat}_s(X,\xi)$ are equal to 0. By the inequality (2) we know that this is the case for manifolds M and cohomology classes $\xi \in H^1(M;\mathbb{R})$ which admit a nowhere vanishing closed 1-form ω with $[\omega] = \xi$. This raises the question whether $\mathrm{cat}_s(M,\xi) = 0$ is sufficient for the existence of such a closed 1-form. Latour [9] has given necessary and sufficient conditions for the existence of a nowhere vanishing closed 1-form and we will show how the Lusternik-Schnirelman category of ξ fits into that framework.

Lemma 3.1. *Let X be a finite CW-complex, $\xi \in H^1(X;\mathbb{R})$ and $f : \tilde{X} \to \mathbb{R}$ equivariant with respect to ξ.*

1. $\mathrm{cat}(X,\xi) = 0$ *if and only if there is an $\varepsilon > 0$ and an equivariant homotopy $\tilde{h} : \tilde{X} \times [0,1] \to \tilde{X}$ with $\tilde{h}_0 = \mathrm{id}_{\tilde{X}}$ and $f \circ \tilde{h}(x,1) - f(x) \leq -\varepsilon$ for all $x \in \tilde{X}$.*
2. $\mathrm{cat}_s(X,\xi) = 0$ *if and only if there is an $\varepsilon > 0$ and an equivariant homotopy $\tilde{h} : \tilde{X} \times [-1,1] \to \tilde{X}$ with $\tilde{h}_0 = \mathrm{id}_{\tilde{X}}$ and $f \circ \tilde{h}(x,1) - f(x) \leq -\varepsilon$ and $f \circ \tilde{h}(x,-1) - f(x) \geq \varepsilon$ for all $x \in \tilde{X}$.*

Proof. $\mathrm{cat}(X,\xi) = 0$ means $U = X$ in Definition 2.1. If the homotopy exists for $\varepsilon > 0$, it exists for every $N > 0$ by iterating the homotopy. □

Let us recall some definitions and results of Latour [9]. We will assume that X is connected since we can always look at the components.

Definition 3.2. Let X be a connected finite CW-complex, $\xi \in H^1(X;\mathbb{R})$ and $f : \tilde{X} \to \mathbb{R}$ equivariant with respect to ξ. Then

$$\mathcal{C}_\xi(X) = \{\gamma : [0,\infty) \to X \mid \lim_{t\to\infty} f \circ \tilde{\gamma}(t) = -\infty\}$$

where $\tilde{\gamma} : [0,\infty) \to \tilde{X}$ is a lift of γ. We give $\mathcal{C}_\xi(X)$ the topology generated by the subbasis consisting of

$$W(a,b;U) = \{\gamma \in \mathcal{C}_\xi(X) \mid \gamma([a,b]) \subset U\}$$

for $a,b \in [0,\infty)$ and U open in X and

$$W(a,A) = \{\gamma \in \mathcal{C}_\xi(X) \mid \forall\, t \geq a \;\, f \circ \tilde{\gamma}(0) - f \circ \tilde{\gamma}(t) > A\}$$

for $a, A \in [0,\infty)$.

Remark 3.3. We changed the sign in the definition of $\mathcal{C}_\xi(X)$, i.e. our $\mathcal{C}_\xi(X)$ equals $\mathcal{C}_{-\xi}(X)$ in Latour [9, Def.1.2]. This is done to be consistent with the signs in Section 2.

It follows that the evaluation map $e : \mathcal{C}_\xi(X) \to X$ given by $e(\gamma) = \gamma(0)$ is a fibration and we denote the fiber by $\mathcal{M}_\xi(X) = \{\gamma \in \mathcal{C}_\xi(X) \mid \gamma(0) = x_0\}$ for some basepoint $x_0 \in X$.

Proposition 3.4. *Let X be a finite CW-complex and $\xi \in H^1(X;\mathbb{R})$. Then we have* $\mathrm{cat}(X,\xi) = 0$ *if and only if there is a section $\sigma : X \to \mathcal{C}_\xi(X)$.*

Proof. If $\operatorname{cat}(X,\xi)=0$ there is an equivariant homotopy $\tilde{h}:\tilde{X}\times[0,1]\to\tilde{X}$ with $\tilde{h}_0=\mathrm{id}_{\tilde{X}}$ and $f\tilde{h}(x,1)-f(x)\leq -1$. We can iterate h to define $h:X\times[0,\infty)\to X$ such that for every $x\in X$ we have $h(x,\cdot)\in\mathcal{C}_\xi(X)$ with $h(x,0)=x$. It is easy to see that $\sigma(x)=h(x,\cdot)$ defines a continuous map.

Now if $\sigma:X\to\mathcal{C}_\xi(X)$ is a section we define $h:X\times[0,\infty)\to X$ by $h(x,t)=\sigma(x)(t)$. It is easy to see that h can be used to define $\tilde{h}$ as required in Lemma 3.1.1. giving $\operatorname{cat}(X,\xi)=0$. □

It is shown in Latour [9, Prop.1.4] that the existence of a section $\sigma:X\to\mathcal{C}_\xi(X)$ is equivalent to $e:\mathcal{C}_\xi(X)\to X$ being a homotopy equivalence. In particular $\operatorname{cat}_s(X,\xi)=0$ is equivalent to the homotopy theoretical condition used in Latour [9, Th.1]. To conclude $F(M,\xi)=0$ from $\operatorname{cat}_s(M,\xi)=0$ we do require the vanishing of a K-theoretic obstruction. To define this obstruction we need a certain ring.

Let A be an $n\times n$ matrix over $\mathbb{Z}G$, where $G=\pi_1(X)$. The matrix A is called *ξ-negative*, if we have $A_{ij}(g)=0$ for every $g\in G$ with $\xi(g)\geq 0$ and every entry of A. Note that we consider elements of the group ring $\mathbb{Z}G$ as functions $G\to\mathbb{Z}$ which vanish for all but finitely many $g\in G$. Let

$$\Sigma_\xi \quad = \quad \{I-A\in M_{n\times n}(\mathbb{Z}G)\,|\,n \text{ is a positive integer}, A \text{ is } \xi\text{-negative}\}.$$

Then we can form the Cohn localization $\Sigma_\xi^{-1}\mathbb{Z}G$ which is a ring together with a ring homomorphism $\rho_\xi:\mathbb{Z}G\to\Sigma_\xi^{-1}\mathbb{Z}G$ such that $\rho_\xi(\Sigma_\xi)\subset GL(\Sigma_\xi^{-1}\mathbb{Z}G)$ and which has the following universal property: If $\pi:\mathbb{Z}G\to R$ is any ring homomorphism with $\pi(\Sigma_\xi)\subset GL(R)$, then there is a unique ring homomorphism $\rho:\Sigma_\xi^{-1}\mathbb{Z}G\to R$ with $\pi=\rho\circ\rho_\xi$. This ring exists by Cohn [4], see also Schofield [13].

Another ring with the property that all matrices in Σ_ξ become invertible is the Novikov ring. It is defined by

$$\widehat{\mathbb{Z}G}_\xi \quad = \quad \{\lambda:G\to\mathbb{Z}\,|\,\forall\, R\in\mathbb{R}\ \#\{g\in G\,|\,\lambda(g)\neq 0,\xi(g)\geq R\}<\infty\}.$$

Multiplication is the same as in the group ring. By the universal property of the Cohn localization the inclusion $\mathbb{Z}G\subset\widehat{\mathbb{Z}G}_\xi$ factors as $\rho\circ\rho_\xi$ with $\rho:\Sigma_\xi^{-1}\mathbb{Z}G\to\widehat{\mathbb{Z}G}_\xi$. In particular ρ_ξ is injective.

If $\pi:\mathbb{Z}G\to R$ is a ring homomorphism, denote by $C_*(X;R)=R\otimes_{\mathbb{Z}G}C_*(\tilde{X})$, where $C_*(\tilde{X})$ is the finitely generated free cellular $\mathbb{Z}G$ complex of $\tilde{X}$.

Proposition 3.5. *If* $\operatorname{cat}(X,\xi)=0$*, then* $C_*(X;\Sigma_\xi^{-1}\mathbb{Z}G)$ *is acyclic.*

Proof. Let $f:\tilde{X}\to\mathbb{R}$ be equivariant with respect to ξ. For every $N>0$ there is a homotopy $h^N:\tilde{X}\times[0,1]\to\tilde{X}$ with $h^N(x,0)=x$ and $f\circ h^N(x,1)-f(x)<-N$ for all $x\in\tilde{X}$. This homotopy induces a chain homotopy $H^N:C_*(\tilde{X})\to C_{*+1}(\tilde{X})$ with $\partial H^N+H^N\partial=\mathrm{id}_*-h^N_{1*}$. Choose a basis of $C_*(\tilde{X})$. By choosing N large enough we get a matrix representation $\partial H^N+H^N\partial=I-A$ with A ξ-negative. Therefore the chain map $\mathrm{id}_*-h^N_{1*}$ is an isomorphism over $C_*(X;\Sigma_\xi^{-1}\mathbb{Z}G)$ and chain homotopic to 0. □

It follows that $C_*(X;\widehat{\mathbb{Z}G}_\xi)$ is also acyclic if $\operatorname{cat}(X,\xi)=0$.

Let

$$\mathrm{Wh}(G;\xi)=K_1(\widehat{\mathbb{Z}G}_\xi)/\langle\tau(\pm g),\tau(1-a)\,|\,g\in G,a\in\widehat{\mathbb{Z}G}_\xi^-\rangle$$

where $\widehat{\mathbb{Z}G}_\xi^- = \{\lambda \in \widehat{\mathbb{Z}G}_\xi \,|\, \lambda(g) = 0 \text{ for } g \in G \text{ with } \xi(g) \geq 0\}$. Thus if $\mathrm{cat}(X, \xi) = 0$ we have a well defined element $\tau(X, \xi) = \tau(C_*(X; \widehat{\mathbb{Z}G}_\xi)) \in \mathrm{Wh}(G; \xi)$. Latour's theorem now reads as

Theorem 3.6. [9, Th.1] *Let M^n be a closed connected smooth manifold with $n \geq 6$ and $\xi \in H^1(M; \mathbb{R})$. There exists a nowhere vanishing closed 1-form ω representing ξ if and only if* $\mathrm{cat}_s(M, \xi) = 0$ *and* $\tau(M, \xi) = 0$.

In other words $F(M, \xi) = 0$ if and only if $\mathrm{cat}_s(M, \xi) = 0$ and $\tau(M, \xi) = 0$, provided $\dim M \geq 6$. In particular we get that the inequality (2) is not an equality in general. Notice that $\tau(X, \xi) = \rho_* \tau(C_*(X; \Sigma_\xi^{-1}\mathbb{Z}G))$. Also the natural map

$$\rho_* : K_1(\Sigma_\xi^{-1}\mathbb{Z}G)/\langle \tau(\pm g), \tau(I - A) \,|\, g \in G, I - A \in \Sigma_\xi \rangle \to \mathrm{Wh}(G; \xi)$$

can be seen to be an isomorphism using [15, Lm.6.2] and Schofield [13, Th.4.3]. Thus the K-theoretic obstruction $\tau(M, \xi)$ in Theorem 3.6 can be expressed in terms of the noncommutative localization $\Sigma_\xi^{-1}\mathbb{Z}G$ as well.

We now want to give a group theoretic condition which together with the acyclicity of $C_*(X; \Sigma_\xi^{-1}\mathbb{Z}G)$ implies $\mathrm{cat}(X, \xi) = 0$.

Definition 3.7. Let G be a finitely presented group, $\xi : G \to \mathbb{R}$ a nonzero homomorphism, X a connected finite CW-complex with $\pi_1(X) = G$ and $f : \tilde{X} \to \mathbb{R}$ a map with $f(gx) = f(x) + \xi(g)$ for every $x \in \tilde{X}$.

(1) $\xi : G \to \mathbb{R}$ is called *controlled 0-connected (CC^0) over $-\infty$*, if for every $s \in \mathbb{R}$ there is a $\lambda(s) \geq 0$ such that every map $g : S^0 \to f^{-1}((-\infty, s])$ extends to a map $\overline{g} : D^1 \to f^{-1}((-\infty, s + \lambda(s)])$ and $s + \lambda(s) \to -\infty$ as $s \to -\infty$.
(2) $\xi : G \to \mathbb{R}$ is called *controlled 1-connected (CC^1) over $-\infty$*, if x is CC^0 over $-\infty$ and for every $s \in \mathbb{R}$ there is a $\lambda(s) \geq 0$ such that every map $g : S^1 \to f^{-1}((-\infty, s])$ extends to a map $\overline{g} : D^2 \to f^{-1}((-\infty, s + \lambda(s)])$ and $s + \lambda(s) \to -\infty$ as $s \to -\infty$.
(3) $\xi : G \to \mathbb{R}$ is called CC^0 *over* ∞ (respectively CC^1 *over* ∞), if $-\xi$ is CC^0 over $-\infty$ (respectively CC^1 over $-\infty$).

These conditions only depend on the homomorphism and not on X or f and they are closely related to the Σ-invariants of the Bieri-Neumann-Strebel-Renz theory [2, 3]. For the exact connections see Bieri and Geoghegan [1, §10.6].

Let us collect conditions equivalent to $\mathrm{cat}(X, \xi) = 0$, most of which are due to Latour [9].

Proposition 3.8. *Let X be a finite CW-complex and $\xi \in H^1(X; \mathbb{R})$. The following are equivalent:*

(1) $\mathrm{cat}(X, \xi) = 0$.
(2) *ξ is CC^1 at $-\infty$ and $C_*(X; \Sigma_\xi^{-1}\mathbb{Z}G)$ is acyclic.*
(3) *ξ is CC^1 at $-\infty$ and $C_*(X; \widehat{\mathbb{Z}G}_\xi)$ is acyclic.*
(4) *$\mathcal{M}_\xi(X)$ is contractible.*
(5) *$e : \mathcal{C}_\xi(X) \to X$ is a homotopy equivalence.*
(6) *There is a section $\sigma : X \to \mathcal{C}_\xi(X)$.*

Proof. (1) $\Rightarrow$ (2) By Proposition 3.5 it remains to show that ξ is CC^1 at $-\infty$. Let $g : S^i \to f^{-1}((-\infty, s])$ be any map with $i = 0$ or 1. This extends to a map

$g' : D^{i+1} \to \tilde{X}$, since $\tilde{X}$ is simply connected. Using the homotopy $h : \tilde{X} \times [0,1] \to \tilde{X}$ coming from $\mathrm{cat}(X, \xi) = 0$ it is easy to change g' to a map $\overline{g} : D^{i+1} \to f^{-1}((-\infty, s])$.

(2) $\Rightarrow$ (3) is trivial.

(3) $\Leftrightarrow$ (4) is Latour [9, Cor.5.23].

(4) $\Leftrightarrow$ (5) $\Leftrightarrow$ (6) is Latour [9, Prop.1.4].

(6) $\Rightarrow$ (1) is Proposition 3.4. □

4. Orientable surfaces

In this section we calculate $F(M, \xi)$ and $\mathrm{cat}_s(M, \xi)$ for closed orientable surfaces M and $\xi \in H^1(M; \mathbb{Z})$. It is clear that for $M = S^2$ we only have $\xi = 0$ and $F(S^2, 0) = \mathrm{cat}_s(S^2, 0) = 2$.

Since S^1 admits a nonzero closed 1-form we get $F(T^2, \xi) = \mathrm{cat}_s(M, \xi) = 0$ for all $\xi \in H^1(M; \mathbb{R})$, where $T^2 = S^1 \times S^1$ is the standard 2-torus.

Now let M_2 be the orientable surface of genus 2. Let $f : M_2 \to S^1$ be a circle-valued Morse function representing $\xi \in H^1(M_2; \mathbb{Z})$. We can assume that f has no critical points of index 0 and 2. Since the Novikov complex recovers the Euler characteristic we have 2 critical points of index 1. Let $\bar{M}_2$ be the connected infinite cyclic covering space corresponding to $\ker f_\# : \pi_1(M_2) \to \mathbb{Z}$ and let $\bar{f} : \bar{M}_2 \to \mathbb{R}$ be a lifting of f such that $0 \in \mathbb{R}$ is a regular value. Then $\bar{f}^{-1}(\{0\})$ has finitely many components and we can change f so that we can assume $\bar{f}^{-1}(\{0\})$ to be connected. Then the fundamental domain $\bar{f}^{-1}([0,t])$, where $t \in \mathbb{Z}$ is the generator of $\mathrm{im}\, f_\#$, looks as in Figure 1.

If v is a transverse gradient of f we can change f to f' so that v is also a gradient

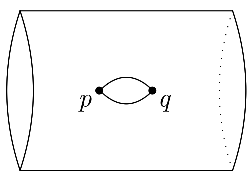

Figure 1.

of f' and f' has exactly one critical value. This is done by the techniques of Milnor [10, §4]. Rename f' back to f. We assume that $t/2$ is the critical value of $\bar{f}$ in $[0,t]$. Now change the vector field v in $\bar{f}^{-1}((0, t/2))$ and $\bar{f}^{-1}((t/2, t))$ so that $\bar{f}^{-1}([0,t])$ with the stable and unstable manifolds of the two critical points p and q looks as in Figure 2. That means we push the stable manifolds of p and q down and the

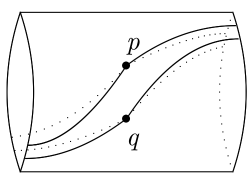

Figure 2.

unstable manifolds up. We can assume that there is a trajectory on the top and one on the bottom of the cobordism which give rise to two closed orbits on M_2. We can push the unstable manifolds in Figure 2 as close as we like to the top. It is easy to see that we can connect p and q within $\bar{f}^{-1}(\{t/2\})$ without getting close to the two closed orbits on the top and bottom. Call the image of that path $I \subset M$. Now as in Farber [7], or see Takens [17] for the original technique, we can change f near I to a smooth function $g : M_2 \to S^1$ which has only one critical point. To get a gradient we only have to change v near I. Call the new gradient of g by w. If $x \in M_2$ is on the stable manifold of the critical point of w, it is either near a stable manifold of v or the trajectory of v through x contains points close to I. A similar statement holds for the unstable manifolds. Because of the way we chose v we now get that w has no homoclinic cycles. Therefore $F(M_2, \xi) \leq 1$. By the same argument we also get $F(M_g, \xi) \leq 1$ for every orientable surface of genus $g \geq 2$ and $\xi \in H^1(M_g; \mathbb{Z}) - \{0\}$.

Since the Euler characteristic of M_g is nonzero for $g \geq 2$ we get that $\operatorname{cat}_s(M_g, \xi) \geq 1$ for every $\xi \in H^1(M_g; \mathbb{R}) - \{0\}$. Hence we get

Proposition 4.1. *Let M_g be the orientable surface of genus $g \geq 2$ and $\xi \in H^1(M_g; \mathbb{Z}) - \{0\}$. Then $\operatorname{cat}_s(M_g, \xi) = F(M_g, \xi) = 1$.*

We believe that this is true also for every $\xi \in H^1(M_g; \mathbb{R}) - \{0\}$ but do not attempt a proof. Since any ω-gradient v is also an ω'-gradient for ω' agreeing with ω near the zeros and close enough everywhere else we get $F(M_g, \xi) = 1$ for an open dense subset of $H^1(M; \mathbb{R}) - \{0\}$.

The philosophy in the above construction is that if we collide two critical points p, q into one critical point along the path I, then every point which flows into p or q or through I under v will flow into the new critical point under w. Notice that the unstable manifold of a critical point of index 1 is $(n-1)$-dimensional and so it might not be possible for the unstable manifold to avoid I. In our very graphic situation above we were able to avoid this difficulty but in general we cannot deal with critical points of index 1 and $n-1$. But for critical points of index $2 \leq i \leq n-2$ we show in the next section how to collide such critical points.

5. Colliding critical points in $\mathbb{R}^n$

Takens [17] has given a general technique to reduce the number of critical points of a smooth function on a manifold M. This was used by Farber [7, Th.2.1] to show the existence of a closed 1-form ω representing a given nonzero cohomology class $\xi \in H^1(M; \mathbb{Z})$ which has at most one zero. Provided that $\operatorname{cat}_s(M, \xi) > 1$ we get a homoclinic cycle for every gradient-like vector field v of ω by Theorem 2.9. In this section we refine the techniques of [17] and [7] so that we can keep track of the gradient-like vector fields when colliding critical points.

Let us give the model for colliding two critical points into one. For simplicity we think of $\mathbb{R}$ as a subset of $\mathbb{R}^n$ by $\mathbb{R} = \mathbb{R} \times \{0\} \subset \mathbb{R}^n$. If $A \subset \mathbb{R}$ or $x \in \mathbb{R}$ we will then write $A \subset \mathbb{R}^n$ and $x \in \mathbb{R}^n$ instead of $A \times \{0\} \subset \mathbb{R}^n$ and $(x, 0) \in \mathbb{R}^n$.

Lemma 5.1. *Let $f_0 : \mathbb{R}^n \to \mathbb{R}$ be a smooth function with $[-1, 1] \subset f^{-1}(\{0\})$ and such that -1 and $1 \in \mathbb{R}^n$ are the only two critical points of f_0. Let v_0 be a gradient-like vector field of f_0. Given a neighborhood $U_0 \subset \mathbb{R}^n$ of $[-1, 1]$ there is a smooth function $f : \mathbb{R}^n \to \mathbb{R}$ which has exactly one critical point at $0 \in \mathbb{R}^n$ and which agrees with f_0 outside of U_0. Furthermore there is a gradient-like vector field v of*

f which agrees with v_0 outside of U_0 and whose stable and unstable manifold at the critical point 0 satisfies the following condition. We have $x \in \mathbb{R}^n$ is in the stable, respectively unstable, manifold of v at 0 if and only if either

- *x is in the stable, respectively unstable, manifold of v_0 at -1.*
- *x is in the stable, respectively unstable, manifold of v_0 at 1.*
- *the trajectory of v_0 through x passes through $[-1,1]$.*

For $x \in \mathbb{R}^n - U_0$ let $\gamma_0 \subset \mathbb{R}^n$ be the image of the trajectory of v_0 through x. If γ_0 satisfies $\overline{\gamma_0} \cap [-1,1] = \emptyset$ then the image γ of the trajectory of v through x satisfies $\gamma_0 - U_0 = \gamma - U_0$.

Proof. Choose open subsets $U_m \subset \mathbb{R}^n$ for positive integers m with $[-1,1] \subset U_{m+1} \subset \overline{U}_{m+1} \subset U_m \subset U_0$ and with $\bigcap_{m=1}^{\infty} U_m = [-1,1]$. Let φ_1 be a diffeomorphism of $\mathbb{R}^n$ which is the identity outside of U_0 and which maps $[-\frac{1}{2}, \frac{1}{2}]$ linearly onto $[-1,1]$ and which satisfies $\varphi_1^{-1}(U_m) \subset U_m$ for all m. This can be done by choosing nice sets U_m and nice φ_1. Let $V_1 = \varphi_1^{-1}(U_1)$, a neighborhood of $[-\frac{1}{2}, \frac{1}{2}]$. Also let

$$f_1 = f_0 \circ \varphi_1 \text{ and } v_1 = d\varphi_1^{-1} \circ v_0 \circ \varphi_1.$$

Then f_1 has two critical points, $-\frac{1}{2}$ and $\frac{1}{2}$ and v_1 is gradient-like with respect to f_1. Furthermore φ_1^{-1} maps the trajectories of v_0 onto the trajectories of v_1.

Now let φ_k be a diffeomorphism of $\mathbb{R}^n$ sending $[-\frac{1}{2^k}, \frac{1}{2^k}]$ linearly onto $[-\frac{1}{2^{k-1}}, \frac{1}{2^{k-1}}]$ which is the identity outside of V_{k-1} and which satisfies $\varphi_k^{-1}(\varphi_{k-1}^{-1} \circ \ldots \circ \varphi_1^{-1}(U_m)) \subset \varphi_{k-1}^{-1} \circ \ldots \circ \varphi_1^{-1}(U_m)$ for all m. Again we set $V_k = \varphi_k^{-1} \circ \ldots \circ \varphi_1^{-1}(U_k)$,

$$f_k = f_{k-1} \circ \varphi_k \text{ and } v_k = d\varphi_k^{-1} \circ v_{k-1} \circ \varphi_k.$$

Again v_k is gradient-like for f_k and f_k has the critical points $-\frac{1}{2^k}$ and $\frac{1}{2^k}$.

This can be done for every positive integer k inductively. We claim that $\bigcap_{m=1}^{\infty} V_m = \{0\}$. This follows because for every k we have $\bigcap_{m=k}^{\infty} \varphi_k^{-1} \circ \ldots \circ \varphi_1^{-1}(U_m) = [-\frac{1}{2^k}, \frac{1}{2^k}]$. and $V_m = \varphi_m^{-1} \circ \ldots \circ \varphi_1^{-1}(U_m) \subset \varphi_k^{-1} \circ \ldots \circ \varphi_1^{-1}(U_m)$ for $m \geq k$ by assumption. Since φ_m is the identity outside of V_k for $m \geq k$ we get that for $x \in \mathbb{R}^m - \{0\}$

$$g(x) = \lim_{m \to \infty} f_m(x)$$

defines a smooth function on $\mathbb{R}^n - \{0\}$ which extends to a continuous function on $\mathbb{R}^n$ by setting $g(0) = 0$. Note that for $x \neq 0$ $f_m(x)$ does not depend on m for large m. With the same argument

$$w(x) = \lim_{m \to \infty} v_m(x)$$

defines a smooth vector field on $\mathbb{R}^n - \{0\}$, but which might not extend continuously to $\mathbb{R}^n$. But on $\mathbb{R}^n - \{0\}$ it is gradient-like with respect to g. Furthermore the trajectories of w lead exactly into 0 if there is a point on the trajectory outside of U_0 whose trajectory under v_0 passes through $[-1,1]$ or converges to a critical point of f_0.

Now we can smooth g to a smooth function f on $\mathbb{R}^n$ with exactly one critical point 0 such that f agrees with g outside of U_0 by using Takens [17, Th.2.1]. By examining the proof of [17, Th.2.1] we get that w is gradient-like with respect to f on $\mathbb{R}^n - \{0\}$. It remains to change w to a gradient-like vector field of f on $\mathbb{R}^n$.

We note that if v is any vector field on $\mathbb{R}^n$ and $k : \mathbb{R}^n \to (0, \infty)$ is a smooth map, the trajectories of v and $k \cdot v$ agree as sets. This follows because if $\gamma : (a,b) \to \mathbb{R}^n$

satisfies $\gamma'(t) = v(\gamma(t))$, then $\delta(t) = \gamma(l(t))$ satisfies $\delta'(t) = k(\delta(t))\cdot v(\delta(t))$, provided $l'(t) = k \circ \gamma(l(t))$.

So now we want to define a function $k : \mathbb{R}^n \to [0,\infty)$ which is nonzero for $x \neq 0$ and such that $v(x) = k(x) \cdot w(x)$ and $v(0) = 0$ defines a smooth vector field on $\mathbb{R}^n$. For $R > 0$ let $B_R(0)$ be the closed Euclidean ball of radius R around $0 \in \mathbb{R}^n$. Let i_0 be a positive integer such that $B_{\frac{1}{i_0}}(0) \subset U_0$. For positive integers $i \geq i_0$ let $\lambda_i : \mathbb{R}^n \to [0,1]$ be a smooth function with $\lambda_i(x) = 1$ for $x \notin U_0$ and $\lambda_i^{-1}(\{0\}) = B_{\frac{1}{i+1}}$. Also let $(a_i)_{i\geq i_0}$ be a sequence with $\sum_{i=i_0}^{\infty} a_i = 1$ and let b_i be the maximum of the norms of the first i derivatives of w on the closure of $B_1(0) - B_{\frac{1}{i+1}}(0)$. Let $c_i : \mathbb{R}^n \to [0,1]$ be a smooth function which is constant 1 outside of U_0 and constant $\min\{1, \frac{1}{b_i}\}$ on V_1. Then

$$v(x) = \sum_{i=i_0}^{\infty} \lambda_i(x) \cdot c_i(x) \cdot a_i \cdot w(x)$$

converges uniformly on $\mathbb{R}^n$ with all derivatives and is of the form $v(x) = k(x)\cdot w(x)$ with $k : \mathbb{R}^n \to [0,1]$ a smooth function with $k(0) = 0$, $k(x) > 0$ for $x \neq 0$ and $k(x) = 1$ for $x \notin U_0$.

We conclude that v is the required gradient-like vector field of f, because the images of trajectories of v are the same as for w and for an image of a trajectory γ_0 of v_0 whose closure misses $[-1,1]$ there is a diffeomorphism $\varphi = \varphi_k^{-1} \circ \ldots \circ \varphi_1^{-1}$ which maps γ_0 onto γ, the image of a trajectory of v. Since φ is the identity outside of U_0 we get all the requirements for the trajectories of v. □

In Lemma 5.1 we distinguish between three types of trajectories of the original vector field v_0. First there are those that either start or end in a critical point of f_0. If x is a point on such a trajectory outside of U_0, then the trajectory of v through x also starts or ends in the critical point $0 \in \mathbb{R}^n$ of f.

Then there are trajectories that pass through $[-1,1] \subset \mathbb{R}^n$. Those trajectories split into two trajectories of v, one which ends in 0 and one which starts in 0.

The remaining trajectories of v_0 correspond to trajectories of v that avoid 0 and which only differ on U_0.

6. Colliding critical points of closed 1-forms

Let ω be a closed 1-form on the closed connected smooth manifold M with finitely many zeros and let v be a gradient-like vector field of ω. Let $\bar{\rho} : \bar{M} \to M$ be the regular covering space of M corresponding to $\ker \xi$, where $\xi : \pi_1(M) \to \mathbb{R}$ is the homomorphism induced by ω. We get $\bar{\rho}^*\omega = df_\omega$ for a smooth map $f_\omega : \bar{M} \to \mathbb{R}$ and the critical points of f_ω correspond to liftings of the zeros of ω.

Now assume that $p, q \in S(\omega)$ are different zeros of ω which have lifts $\bar{p}, \bar{q} \in \bar{M}$ with $f_\omega(\bar{p}) = f_\omega(\bar{q}) = 0 \in \mathbb{R}$. Furthermore assume that there is a smoothly embedded path I in $f_\omega^{-1}(\{0\})$ between $\bar{p}$ and $\bar{q}$. It is easy to see that $\bar{\rho}$ restricts to an injective map $f_\omega^{-1}(\{0\}) \to M$; in the case where $\operatorname{im}\xi \subset \mathbb{R}$ is discrete, $f_\omega^{-1}(\{0\})$ is compact, but if $\operatorname{im}\xi \subset \mathbb{R}$ is not discrete, $\bar{\rho}(f_\omega^{-1}(\{0\}))$ is dense in M. In particular we get that $J = \bar{\rho}(I)$ is a smoothly embedded path between p and q in M. Since J is compact we can find a small neighborhood U of J in M which is diffeomorphic to $\mathbb{R}^n$. We can furthermore assume that ω restricted to U looks as the model in Lemma 5.1

with J corresponding to $[-1,1]$, i.e. ω restricts to the differential of f_0 on U.

Thus we can change f_0 to f and get a vector field w gradient-like with respect to f on U using Lemma 5.1. Now $df - df_0$ extends to an exact form dg on M such that $\omega' = \omega + dg$ agrees with ω outside a small neighborhood of J and has one zero less than ω. Call the remaining zero $r \in M$. Furthermore the vector field w extends to a gradient-like vector field v' of ω' which agrees with v outside of a small neighborhood of J. Since we can keep track of the trajectories of the vector field in Lemma 5.1 we can now tell whether v' has homoclinic cycles. The following possibilities for homoclinic cycles involving r exist:

(1) There exists a homoclinic cycle of v passing through J.
(2) There exists a trajectory (possibly broken) of v or $-v$ from p to q.
(3) There exists a trajectory (possibly broken) of v or $-v$ from a point on J to p or q.
(4) There exists a trajectory (possibly broken) of v or $-v$ from a point on J to another point on J.

Notice that (2) and (3) are special cases of (4) but it seems to be useful to distinguish points in the interior of J to the endpoints since there is only one trajectory passing through a point in the interior while there can be many trajectories approaching the critical points p or q.

Homoclinic cycles arising from (1) can be avoided by assuming that v has no homoclinic cycles. But because of the possibilities (2)-(4) we have to deal with certain restrictions on which critical points we can collide without producing homoclinic cycles.

Lemma 6.1. *Let $\xi \in H^1(M;\mathbb{R})$, $n = \dim M$, ω' a Morse form representing ξ without critical points of index 0 and n and v a transverse ω'-gradient. Then there exists a Morse form ω representing ξ such that v is also an ω-gradient and with the following properties.*

(1) *For the pullback $f : \tilde{M} \to \mathbb{R}$ of ω there exist $c_i \in \mathbb{R}$ for $2 \leq i \leq n-2$ such that for every critical point of ω having index i there exists a lifting in $f^{-1}(\{c_i\})$.*
(2) *For any two critical points p and q of ω with $\operatorname{ind} p = \operatorname{ind} q = i$ with $2 \leq i \leq n-2$ there exists a smoothly embedded path $\gamma_{p,q} : [0,1] \to f^{-1}(\{c_i\})$ such that $\rho(\gamma_{p,q}(0)) = p$ and $\rho(\gamma_{p,q}(1)) = q$.*

We note that if ξ is nonzero it is always possible to find a Morse form ω' representing ξ without critical points of index 0 and n and a transverse ω'-gradient v.

Proof. If $\operatorname{im} \xi$ is discrete we can order the critical points by the usual method described in Milnor [10, §4]. If $\operatorname{im} \xi$ is not discrete we can find liftings of all critical points with index i in $(f')^{-1}(I_\varepsilon)$, where I_ε is an arbitrarily small interval of real numbers. Here $f' : \tilde{M} \to \mathbb{R}$ is the pullback of ω'. Assume $I_\varepsilon = (c, c+\varepsilon)$ for some $c \in \mathbb{R}$ and $\varepsilon > 0$. Choose $\varepsilon > 0$ so small that $W^s(\tilde{r}) \cap (f')^{-1}([f'(\tilde{r}) - \varepsilon, f'(\tilde{r})])$ and $W^s(\tilde{r}) \cap (f')^{-1}([f'(\tilde{r}), f'(\tilde{r}) + \varepsilon])$ are smoothly embedded discs for all critical points $\tilde{r}$ of f'. Then we can change ω' to ω'' by using [15, Lm.3.2] so that (1) is satisfied. For every critical point p of ω'' let $\tilde{p}$ be a lifting in $(f'')^{-1}(\{c_{\operatorname{ind} p}\})$. Again f'' is the pullback of ω''. Since $\tilde{M}$ is connected we can find for every pair $(\tilde{p}, \tilde{q})$ with $\operatorname{ind} \tilde{p} = \operatorname{ind} \tilde{q}$ a path in $\tilde{M}$. We want to use the flow of $\tilde{v}$, the lifting of v to $\tilde{M}$, to push the path into $(f'')^{-1}(\{c_{\operatorname{ind} p}\})$. The path can be chosen so that its interior avoids all stable and unstable manifolds except the unstable manifolds of critical points with index 1 and the stable manifolds of critical points with index $n-1$. So to push the path into $(f'')^{-1}(\{c_{\mathrm{ind}\,p}\})$ there exist only two obstacles. There can be points $x \in \tilde{M}$ on the path with $x \in W^u(\tilde{r})$ with $\mathrm{ind}\,\tilde{r} = 1$ and $f''(\tilde{r}) > c_{\mathrm{ind}\,p}$ or there can be points $y \in \tilde{M}$ on the path with $y \in W^s(\tilde{r})$ with $\mathrm{ind}\,\tilde{r} = n-1$ and $f''(\tilde{r}) < c_{\mathrm{ind}\,p}$, see Figure 3.

Choose paths $\gamma_{p,q}$ for every pair (p,q) with $\mathrm{ind}\,p = \mathrm{ind}\,q$ and let

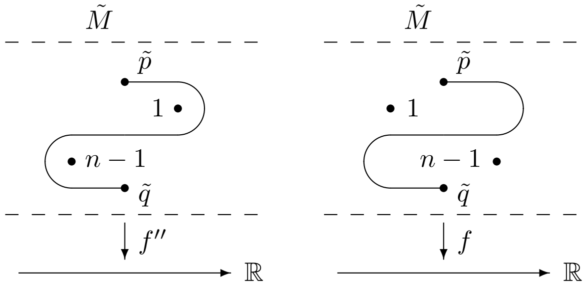

FIGURE 3.

$$R = \max\{|f''(\gamma_{p,q}(t_1)) - f''(\gamma_{p,q}(t_2))| \in [0,\infty) \mid (p,q) \text{ critical points with } \mathrm{ind}\,p = \mathrm{ind}\,q,\ t_1, t_2 \in [0,1]\}.$$

Now we can change ω to ω'' using [15, Lm.3.2] such that $f''(\tilde{r}) - f(\tilde{r}) = R$ for $\mathrm{ind}\,\tilde{r} = n-1$ and $f(\tilde{r}) - f''(\tilde{r}) = R$ for $\mathrm{ind}\,\tilde{r} = 1$ and $f(\tilde{p}) = f''(\tilde{p})$ for all other critical points.

Then all paths $\gamma_{p,q}$ can be pushed into $f^{-1}(c_i)$, where i is the index of p and q. Now we can find a smoothly embedded path in $f^{-1}(c_i)$ with the required properties. □

We now want to collide the critical points of common index i with $2 \leq i \leq n-2$ into degenerate critical points, one for each i. In order to avoid the creation of homoclinic cycles we have to check that none of the possibilities (1)-(4) from the beginning of Section 6 are fulfilled. We can exclude (1) and (2) for the vector field v used in Lemma 6.1 since it is assumed to be transverse. The next lemma shows that we can also exclude the possibilities (3) and (4) Before we state it let us give a name for closed 1-forms as in the conclusion of Lemma 6.1.

Definition 6.2. A Morse form ω on the closed connected smooth manifold M is called *almost nicely ordered* if ω has no critical points of index 0 and $n = \dim M$ and furthermore

(1) For the pullback $f : \tilde{M} \to \mathbb{R}$ of ω there exist $c_i \in \mathbb{R}$ for $2 \leq i \leq n-2$ such that for every critical point of ω having index i there exists a lifting in $f^{-1}(\{c_i\})$.
(2) For any two critical points p and q of ω with $\mathrm{ind}\,p = \mathrm{ind}\,q = i$ with $2 \leq i \leq n-2$ there exists a smoothly embedded path $\gamma_{p,q} : [0,1] \to f^{-1}(\{c_i\})$ such that $\rho(\gamma_{p,q}(0)) = p$ and $\rho(\gamma_{p,q}(1)) = q$.

Notice that the conditions (1) and (2) are trivial if $\dim M \leq 3$ so we will implicitly assume that $\dim M \geq 4$.

Lemma 6.3. *Let $\xi \in H^1(M;\mathbb{R})$, $n = \dim M$ and ω an almost nicely ordered Morse form representing ξ. Then there exists a transverse ω-gradient v such that for every pair (p,q) of critical points of ω with $\operatorname{ind} p = \operatorname{ind} q = i$ and $2 \leq i \leq n-2$ there is a smoothly embedded path $\gamma_{p,q} : [0,1] \to f^{-1}(\{c_i\})$ connecting p and q with the following properties.*

1. *We have $\gamma_{p,q}((0,1)) \cap W^u(\tilde{r}) = \emptyset$ for all critical points $\tilde{r}$ of f with $2 \leq \operatorname{ind} \tilde{r} \leq n-1$.*
2. *We have $\gamma_{p,q}((0,1)) \cap W^s(\tilde{r}) = \emptyset$ for all critical points $\tilde{r}$ of f with $1 \leq \operatorname{ind} \tilde{r} \leq n-2$.*
3. *We have $\gamma \cap \rho(\gamma_{p,q}([0,1])) = \emptyset$ if γ is the image of a closed orbit of v in M.*
4. *Any trajectory γ of v passes through at most one path $\rho(\gamma_{p,q}([0,1]))$ in at most one point.*

Here $f : \tilde{M} \to \mathbb{R}$ is a smooth function with $df = \rho^*\omega$ and the c_i are as in Definition 6.2.

Proof. Let v' be a transverse ω-gradient. Since $W^u(\tilde{r}) \cap f^{-1}(\{c_i\})$ is at most $(n-3)$-dimensional for $2 \leq \operatorname{ind} \tilde{r} \leq n-1$ and $\gamma_{p,q}$ is 1-dimensional we can choose smooth paths that satisfy (1) by transversality arguments. The same holds for (2).

To achieve (3) and (4) we have to be more careful with the closed orbit structure of the ω-gradient. For $0 < a \leq b < 1$ we look at smooth embeddings $\gamma_{p,q} : [0,1] \to f^{-1}(\{c_i\})$ and transverse ω-gradients v such that (1)-(4) holds for $\gamma_{p,q}([a,b])$. The existence of such $\gamma_{p,q}$ and v is given by standard transversality arguments. By letting $a \to 0$ and $b \to 1$ by a countable sequence we get the existence of the required v and $\gamma_{p,q}$ by Baire category type arguments. □

Because of the conditions (1)-(4) in Lemma 6.3 we can now collide all critical points of index i with $2 \leq i \leq n-2$ into one critical point without creating homoclinic cycles. Note that the paths change slightly when colliding two critical points since in Lemma 5.1 the critical points -1 and 1 are pushed to 0, but the properties of avoiding stable and unstable manifolds and trajectories through the paths remain intact.

Definition 6.4. Let $\xi \in H^1(M;\mathbb{R})$ be nonzero, $n = \dim M$. Then let $mc(\xi)$ be the least integer k such that there exists a Morse form ω representing ξ without critical points of index 0 and n and with k critical points of index 1 or $n-1$.

Proposition 6.5. *Let $\xi \in H^1(M;\mathbb{R})$ be nonzero and $n = \dim M \geq 4$. Then $F(M,\xi) \leq mc(\xi) + n - 3$.*

Proof. Let ω' be a Morse form representing ξ without critical points of index 0 and n and with $mc(\xi)$ critical points of index 1 or $n-1$. By Lemma 6.1 we can assume that ω' is almost nicely ordered and by Lemma 6.3 we can find a transverse ω'-gradient v' and smoothly embedded paths $\gamma_{p,q}$ with the properties described there. Using Lemma 5.1 we can now collide all critical points of index i with $2 \leq i \leq n-2$ into one degenerate critical point without producing homoclinic cycles. Therefore we obtain a closed 1-form ω representing ξ with at most $mc(\xi)+n-3$ critical points and a gradient-like vector field v which has no homoclinic cycles. □

Example 6.6. Let M_g be the orientable surface of genus g and let $\xi \in H^1(M;\mathbb{R})$ be nonzero. Let $n \geq 4$. Then $\dim M_g \times S^{n-2} \geq 4$ and we can think of ξ as an element of $H^1(M_g \times S^{n-2};\mathbb{R})$. It is easy to see that we can represent ξ by a Morse form ω with $2g-2$ critical points of index 1 and $2g-2$ critical points of index $n-1$ while having no other critical points. Hence we get for the Novikov homology

$$H_*(M_g \times S^{n-2}; \widehat{\mathbb{Z}G}_\xi) = \begin{cases} \left(\widehat{\mathbb{Z}G}_\xi\right)^{2g-2} & * = 1 \text{ or } n-1 \\ 0 & \text{otherwise} \end{cases}.$$

Since there is a nonzero ring homomorphism from the Novikov ring to a field (for $\xi \in H^1(M_g \times S^{n-2}; \mathbb{Z})$ we can use $\mathbb{Q}((t))$ as such a field) we see that $mc(\xi) = 4g-4$. On the other hand we get $F(M_g \times S^{n-2}, \xi) \leq 2$ by Proposition 4.1 for $\xi \in H^1(M_g \times S^{n-2}; \mathbb{Z})$.

This shows that we cannot expect good results from Proposition 6.5 if $mc(\xi) \neq 0$. But for $\dim M \geq 5$ it is well understood under which conditions we get $mc(\xi) = 0$. Namely we have

Proposition 6.7. [15, Prop.4.4] *Let $\xi \in H^1(M; \mathbb{R})$ be nonzero and $\dim M \geq 5$. Then $mc(\xi) = 0$ if and only if ξ is CC^0 at $-\infty$ and ∞.*

Note that we consider the cohomology class ξ to be CC^0 at $-\infty$ or ∞ if the corresponding homomorphism $\xi : \pi_1(M) \to \mathbb{R}$ is. A sufficient condition for ξ to be CC^0 at $-\infty$ and ∞ is that $\ker \xi$ is finitely generated. In the case that ξ has discrete image in $\mathbb{R}$ this is also necessary. This follows from Bieri, Neumann and Strebel [2, Th.B1], see Bieri and Geoghegan [1] to relate the CC^0 condition to the language used in [2]. As a result we get the following theorem.

Theorem 6.8. *Let M be a closed connected smooth manifold with $n = \dim M \geq 5$ and let $\xi \in H^1(M; \mathbb{R})$ be CC^0 at $-\infty$ and ∞. Then $F(M, \xi) \leq n-3$.*

Corollary 6.9. *Let M be a closed connected smooth manifold with $n = \dim M \geq 5$ and let $\xi \in H^1(M; \mathbb{R})$ be CC^0 at $-\infty$ and ∞. Then $\operatorname{cat}_s(M, \xi) \leq n-3$.*

Proof. This follows directly from Theorem 6.8 and (2). □

Farber gives examples in [6, Ex.5.4] and [7, Ex.6.5] of manifolds M and $\xi \in H^1(M; \mathbb{Z})$ such that $\operatorname{cat}_s(M, \xi) \geq n-1$. It follows from Corollary 6.9 that ξ in these examples cannot be CC^0 (at least for $n \geq 5$). By looking at the above mentioned examples of Farber in [6, 7] it is easy to see that the corresponding homomorphisms do not have a finitely generated kernel.

Let us now sharpen the conditions on $\xi \in H^1(M; \mathbb{R})$. By assuming that the dimension of M is at least 6 and that ξ is CC^1 at $-\infty$ and ∞ we can invoke [15, Th.1.2] which gives the following theorem.

Theorem 6.10. *Let M be a closed connected smooth manifold with $n = \dim M \geq 6$ and let $\alpha \in H^1(M; \mathbb{R})$ be CC^1 at $-\infty$ and ∞. Let D_* be a finitely generated free based $\widehat{\mathbb{Z}G}_\chi$ complex with $D_i = 0$ for $i \leq 1$ and $i \geq n-1$ which is simple chain homotopy equivalent to $C^\Delta_*(M; \widehat{\mathbb{Z}G}_\xi)$. Then there is a Morse form ω such that ω has exactly $\operatorname{rank} D_i$ critical points of index i for $i = 0, \ldots, n$.*

For D_* to be simple chain homotopy equivalent to $C^\Delta_*(M; \widehat{\mathbb{Z}G}_\xi)$ we mean there exists a chain homotopy equivalence $\varphi : D_* \to C^\Delta_*(M; \widehat{\mathbb{Z}G}_\xi)$ with $\tau(\varphi) = 0 \in \mathrm{Wh}(G; \xi)$.

In view of Theorem 6.10 we make the following definition.

Definition 6.11. Let $\xi \in H^1(M;\mathbb{R})$ and $G = \pi_1(M)$. Then let $cl(\xi)$ be the least integer k such that there exists a finitely generated free chain complex D_* over $\widehat{\mathbb{Z}G}_\xi$ which is simple chain homotopy equivalent to $C_*^\Delta(M;\widehat{\mathbb{Z}G}_\xi)$ and with $k = \#\{i \in \mathbb{Z} \,|\, D_i \neq 0\}$.

Notice that we could have also used $\Sigma_\xi^{-1}\mathbb{Z}G$ instead of $\widehat{\mathbb{Z}G}_\xi$ in the definition of $cl(\xi)$. By the results in [15] this would make no difference in the next theorem.

Theorem 6.12. *Let M be a closed connected smooth manifold with $n = \dim M \geq 6$ and let $\xi \in H^1(M;\mathbb{R})$ be CC^1 at $-\infty$ and ∞. Then $F(M,\xi) \leq cl(\xi)$.*

Proof. Let D_* be a finitely generated free chain complex over $\widehat{\mathbb{Z}G}_\xi$ simple homotopy equivalent to $C_*^\Delta(M;\widehat{\mathbb{Z}G}_\xi)$ with $\#\{i \in \mathbb{Z} \,|\, D_i \neq 0\} = cl(\xi)$. Then we can assume that $D_i = 0$ for $i \leq 1$ and $i \geq n-1$. To see this note that unless $D_* = 0$ there is a minimal $i \in \mathbb{Z}$ with $D_i \neq 0$ since D_* is finitely generated. If for this i we have $i \leq 1$, then $H_j(D_*) = H_j(M;\widehat{\mathbb{Z}G}_\xi) = 0$ for $j \leq i$. Now it is easy to construct a chain complex E_* simple homotopy equivalent to D_* with $\#\{i \in \mathbb{Z} \,|\, E_i \neq 0\} \leq \#\{i \in \mathbb{Z} \,|\, D_i \neq 0\}$. A dual argument works for $i \geq n-1$, compare Pajitnov [11, Prop.7.14].

Now we can find a Morse form ω and a transverse ω-gradient v such that ω has $c_i = \operatorname{rank} D_i$ critical points of index i and the result follows just as in the proof of Proposition 6.5. □

Generally it is very hard to calculate $cl(\xi)$ or get estimates better than $cl(\xi) \leq n-3$, but let us state at least two corollaries of Theorem 6.12.

Theorem 6.13. *Let M be a closed connected smooth manifold with $n = \dim M \geq 6$ and $\pi_2(M) = 0$. Let $\xi \in H^1(M;\mathbb{R})$ be CC^1 at $-\infty$ and ∞. Then $F(M,\xi) \leq \max\{2, n-5\}$.*

Proof. Because of the condition $\pi_2(M) = 0$ we get that $H_i(M;\widehat{\mathbb{Z}G}_\xi) = 0$ for $i \leq 2$ and $i \geq n-2$ by [15, Prop.9.7]. Then we can trade generators in dimension 2 against generators in dimension 4 and generators in dimension $n-2$ against generators in dimension $n-4$. Therefore $cl(\xi) \leq n-5$ for $n \geq 7$ and $cl(\xi) \leq 2$ for $n = 6$. □

Theorem 6.14. *Let M be a closed connected smooth manifold with $n = \dim M \geq 5$ and let $\xi \in H^1(M;\mathbb{R})$ satisfy $\operatorname{cat}_s(M,\xi) = 0$. Then $F(M,\xi) \leq 2$.*

Proof. By Proposition 3.8 we get that ξ is CC^1 at $-\infty$ and ∞ and that $C_*^\Delta(M;\widehat{\mathbb{Z}G}_\xi)$ is acyclic. For $n = 5$ the theorem now follows directly from Theorem 6.8 and for $n \geq 6$ we get $cl(\xi) \leq 2$ since we can trade the generators of $C_*^\Delta(M;\widehat{\mathbb{Z}G}_\xi)$ into 2 dimensions. □

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