TORSION CALCULATIONS IN KHOVANOV COHOMOLOGY

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Abstract. We obtain information on torsion in Khovanov cohomology by performing calculations directly over $\mathbb{Z}/p^k\mathbb{Z}$ for p prime and $k \geq 2$. In particular, we get that the torus knots $T(9,10)$ and $T(9,11)$ contain torsion of order 9 and 27 in their Khovanov cohomology.

1. Introduction

Khovanov cohomology [Kho00] has an abundance of 2-torsion, but no knot with at most 16 crossings contains torsion of odd order [Shu14]. Implementations of Bar-Natan's algorithm [BN07] have shown the existence of $\mathbb{Z}/p\mathbb{Z}$-summands for small odd primes in several torus knots, and this has recently been used in [MPS$^+$18] to create infinite families of knots which have 3, 5 or 7-torsion in their Khovanov cohomology.

Bar-Natan's algorithm works over any commutative ring, though implementations tend to be over $\mathbb{Z}$ or prime fields. Calculations over finite fields are much faster than over $\mathbb{Z}$, as in the latter case one can do fewer cancellations, the numbers in the boundary homomorphisms can get very large[1] and the necessary Smith normalisation may also be slow. Working over finite fields of varying characteristic detects torsion, but only of prime order. By working instead over $\mathbb{Z}/p^k\mathbb{Z}$ with p prime and $k \geq 2$ we can also detect torsion of prime power order. This requires a normalisation process for these finite rings, which turns out to be much faster than Smith normalisation.

Before we state results of our computations, let us write $\mathbb{Z}_n$ for $\mathbb{Z}/n\mathbb{Z}$, and we say that an abelian group G *contains* $\mathbb{Z}_n$*-torsion*, if $\mathbb{Z}_n$ is a direct summand of G.

Calculation 1.1. *The integral Khovanov cohomology of the torus knot* $T(9,11)$ *contains* $\mathbb{Z}_9$*-torsion in bi-degrees* (i,j) *equal to*

$$(26,115),\quad (31,117),\quad (32,121),\quad (35,123),\quad (37,125),\quad (39,127),\quad (40,131)$$

and $\mathbb{Z}_{27}$*-torsion in bi-degree* $(38,129)$.

The torus knot $T(9,10)$ *also contains* $\mathbb{Z}_9$*-torsion in several bi-degrees, and* $\mathbb{Z}_{27}$*-torsion in bi-degree* $(38,121)$.

We list more detailed information in Figures 1 to 5 below, in particular the $\mathbb{Z}_{27}$-torsion is part of $\mathbb{Z}_{756}$-torsion in the case of $T(9,10)$, and part of $\mathbb{Z}_{3780}$-torsion in the case of $T(9,11)$.

[1] Applying Bar-Natan's algorithm to $T(8,9)$ has led to integers larger than 10^{900} in the final chain complex before Smith normalisation.

Whether $T(9,10)$ and $T(9,11)$ contain $\mathbb{Z}_9$-torsion was asked in [MPS$^+$18, Problem 3.15(1)]. Earlier calculations for $T(m,m+1)$ with $m \in \{4,\dots,8\}$ suggested this existence. Our calculation for $T(9,11)$ can now be combined with [MPS$^+$18, Cor.3.2] to produce

Corollary 1.2. *There exists an infinite sequence of knots, each of which contains $\mathbb{Z}_9$-torsion and $\mathbb{Z}_{27}$-torsion in their integral Khovanov cohomology.*

We also note that the existence of $\mathbb{Z}_{27}$-torsion gives a counterexample with $p = 3$ for Conjecture (3') in [PS14], which states that a closed n-braid cannot have p^r-torsion for $p^r > n$ in its Khovanov cohomology. Counterexamples for $p = 2$ were produced in [MPS$^+$18].

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2. Homology over $\mathbb{Z}_{p^k}$

In this section we assume that we have a chain complex C_* over $\mathbb{Z}_{p^k}$ with p a prime and $k \geq 2$, such that each C_n is finitely generated free over $\mathbb{Z}_{p^k}$, and there are only finitely many $n \in \mathbb{Z}$ with C_n non-zero.

Even if the ranks of the C_n are at most 1 one can get unusual chain complexes such as

$$\cdots \longrightarrow \mathbb{Z}_{p^k} \xrightarrow{p^{l_{n+1}}} \mathbb{Z}_{p^k} \xrightarrow{p^{l_n}} \mathbb{Z}_{p^k} \longrightarrow \cdots$$

as long as successive powers $l_{n+1} + l_n \geq k$. Clearly such a complex is not of the form $D_* \otimes \mathbb{Z}_{p^k}$ with D_* a finitely generated free chain complex over $\mathbb{Z}$, if there is n with $1 \leq l_{n+1}, l_n \leq k-1$.

Definition 2.1. A chain complex C_* over $\mathbb{Z}_{p^k}$ is called *elementary*, if there is $n \in \mathbb{Z}$ with $C_n = \mathbb{Z}_{p^k}$, $C_{n-1} = \mathbb{Z}_{p^k}$ or 0, and all other chain groups are 0.

We allow all possible $\partial_n \colon C_n \to C_{n-1}$ including 0 and multiplication by a unit, but we are mainly interested in the case where this boundary is multiplication by a non-zero element of the maximal ideal

$$I_p = \langle p \rangle \subset \mathbb{Z}_{p^k}.$$

Up to chain homotopy we only need to consider multiplication by p^l for $l = 0, \dots, k$.

If our chain complex C_* is of the form $C_* = D_* \otimes \mathbb{Z}_{p^k}$ for some free chain complex D_* over $\mathbb{Z}$, then C_* is chain homotopy equivalent to a direct sum of elementary chain complexes, simply because an analogous statement holds for finitely generated and bounded chain complexes over $\mathbb{Z}$.

If we perform Bar-Natan's algorithm over $\mathbb{Z}_{p^k}$, the resulting chain complex C_* need not be of this form, but it is certainly chain homotopy equivalent to such a complex, namely with D_* the Khovanov chain complex over $\mathbb{Z}$. We want to have an algorithm which turns C_* into a direct sum of elementary chain complexes, analogous to the Smith Normalization.

Before we do this, recall the various Bockstein homomorphisms we can associate to free chain complexes over $\mathbb{Z}_{p^k}$. Indeed, for every $2 \leq u \leq k$ and $1 \leq l \leq u-1$ there is a short exact sequence

$$0 \longrightarrow \mathbb{Z}_{p^l} \longrightarrow \mathbb{Z}_{p^u} \longrightarrow \mathbb{Z}_{p^{u-l}} \longrightarrow 0$$

and a corresponding Bockstein homomorphism that we denote by

$$\beta_i^{u-l,u}\colon H_i(C;\mathbb{Z}_{p^{u-l}}) \to H_{i-1}(C;\mathbb{Z}_{p^l}).$$

Lemma 2.2. *Let C_* be chain homotopy equivalent to a direct sum of elementary chain complexes over $\mathbb{Z}_{p^k}$. Then any composition of Bockstein homomorphisms $\beta_{i-1}^{l,v-l}\circ\beta_i^{u-l,l} = 0$, where $1 \leq l \leq u-1, v-1 < k$.* □

So now assume that we have a finitely generated free and bounded chain complex C_* over $\mathbb{Z}_{p^k}$ which is chain homotopy equivalent to a chain complex $D_* \otimes \mathbb{Z}_{p^k}$ with D_* finitely generated free and bounded over $\mathbb{Z}$. We also assume that all the entries in the matrices representing the boundary maps are from the maximal ideal $I_p \subset \mathbb{Z}_{p^k}$, for otherwise we can perform Gauss Elimination [BN07, Lm.3.2] on C_* until this is the case. Given a basis for C_*, we denote the matrix entries for the boundary maps by $[a:b] \in I_p$, where $a \in C_n$ and $b \in C_{n-1}$ are basis elements.

Let $b \in C_n$ be a generator such that there exists a generator $c \in C_{n-1}$ such that $[b:c] \in I_p - I_p^2$. Since all the entries in the matrix are in I_p, we can perform elementary row and column operations until $[b:c'] = 0$ for all $c' \neq c$ and $[b':c] = 0$ for all $b' \neq b$.

We claim that $[a:b] = 0$ and $[c:d] = 0$ for all basis elements $a \in C_{n+1}$ and $d \in C_{n-2}$. Note that $[a:b], [c:d] \in I_p^{k-1}$, for otherwise $\partial^2 \neq 0$. We have $H_*(C;\mathbb{Z}_p) \cong C_* \otimes \mathbb{Z}_p$, and if $x \in C_*$ is a basis element, we write $\bar{x} \in H_*(C;\mathbb{Z}_p)$ for the corresponding generator. Also, $H_n(C;\mathbb{Z}_{p^{k-1}})$ has a direct summand $\mathbb{Z}_p$ coming from $p^{k-2}\cdot b$, and we denote the homology generator also by $\bar{b}$.

From the definition of the Bockstein homomorphism we have $\beta_n^{k-1,1}(\bar{b}) = \bar{c}$, and if there is a d with $[c:d] \neq 0$, then $\beta_{n-1}^{1,k-1}(\bar{c}) \neq 0$. But $\beta_{n-1}^{1,k-1}\circ\beta_n^{k-1,1} \neq 0$ is not possible by Lemma 2.2, since C_* is chain homotopy equivalent to a direct sum of elementary complexes.

Using a dual argument using the cohomological Bockstein homomorphisms we see that $[a:b] = 0$ for all basis elements $a \in C_{n+1}$. This implies that b and c form an elementary complex which is a direct summand of C_*. After repeating this argument finitely many times we get that C_* is the direct summand of elementary complexes and a finitely generated free complex C'_* such that all the entries in the boundary matrices of C'_* are in I_p^2.

Inductively, if we have a chain complex $C_*^{(i)}$ with all entries in I_p^{i+1} for some i, and there are generators b, c with $[b:c] \in I_p^{i+1} - I_p^{i+2}$, we can perform elementary row and column operations and split off an elementary complex as a direct summand, using the Bockstein homomorphisms $\beta_n^{i+1,k-i-1}$ and $\beta_{n-1}^{k-i-1,i+1}$ to ensure we do have a direct summand.

This gives an effective algorithm for calculating the (co)homology of a finitely generated free and bounded chain complex C_* over $\mathbb{Z}_{p^k}$, which is chain homotopy equivalent to $D_* \otimes \mathbb{Z}_{p^k}$ with D_* finitely generated free and bounded over $\mathbb{Z}$. In particular, we can use this algorithm after the Bar-Natan algorithm to calculate the Khovanov cohomology of a link with coefficients in $\mathbb{Z}_{p^k}$.

3. Calculations

The algorithm described in Section 2 has been implemented in the programme `TKnotJob` available at [Sch19]. Calculations for $T(9, 10)$ were done on a standard PC with 32 GB RAM, which took about two hours per prime power. For $T(9, 11)$ a similar machine, but with 64 GB RAM, was used. The calculation for a single prime power was done in less than twelve hours.

For $T(9, 10)$ we computed for the prime powers $p^k = 2^3, 3^4, 5^2, 7^2, 11^2, 13^2$ and 17^2, while for $T(9, 11)$ we computed for $p^k = 2^5, 3^4, 5^2, 7^2$ and 11^2. We also made computations for $T(9, 9)$, but found no $\mathbb{Z}_9$- or $\mathbb{Z}_{27}$-torsion in its unreduced Khovanov cohomology. Unlike $T(9, 10)$ and $T(9, 11)$ it does however contain $\mathbb{Z}_8$-torsion.

The programme `TKnotJob` has been extensively tested and compared to calculations from other programmes. In particular, we get the same results for the torus knot $T(8, 9)$ as Lewark, who computed its integral Khovanov cohomology using both `JavaKh` [BNMea07] and `Khoca` [Lew16]. Furthermore, we calculated the $\mathbb{Z}_3$-cohomology of $T(9, 10)$ using `JavaKh` and obtained the same $\mathbb{Z}_3$-cohomology. This calculation took about half a day on the 32GB machine described above.

Our computations can be pieced together to obtain an *approximate* integral Khovanov cohomology by the Universal Coefficient Theorem. The result is only approximate, since

- there may be more torsion for larger primes where we did not do computations,
- two copies of $\mathbb{Z}$ in adjacent homological degrees may mask additional torsion in the larger of the two homological degrees. Note that this would give torsion of order at least as large as the product of all computed prime powers.

Both cases seem rather unlikely so we conjecture that we indeed obtain the integral Khovanov cohomology of $T(9, 10)$ and $T(9, 11)$ from our computations. We list our computations for $T(9, 10)$ in Figures 3, 4 and 5, while Figures 1 and 2 list a few quantum degrees for $T(9, 11)$ only. The torsion summands listed there are in fact direct summands in the integral Khovanov cohomology.

Full results of calculations, including reduced Khovanov cohomology for $T(9, 9)$, $T(9, 10)$ and $T(9, 11)$ can be found with `TKnotJob`. Conjecture (3) of [PS14] would imply that there is no p-torsion with $p \geq 11$ for these links, and we would get the integral Khovanov cohomology. This conjecture was recently shown to be false [Muk19], but it may still hold for torus links.

Finally, we note that the reduced Khovanov cohomology of $T(9, 10)$ and $T(9, 11)$ contains $\mathbb{Z}_7$-torsion. In particular the family mentioned in Corollary 1.2 also contains $\mathbb{Z}_7$-torsion in their reduced cohomology.

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$h \backslash q$	125	127	129	131
42				$\mathbb{Z}$
41				$\mathbb{Z}^2 \oplus (\mathbb{Z}_2)^2$
40		$\mathbb{Z}$	$\mathbb{Z}^3$	$\mathbb{Z} \oplus (\mathbb{Z}_2)^5 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_9$
39		$(\mathbb{Z}_2)^2 \oplus \mathbb{Z}_9$	$\mathbb{Z}^5 \oplus (\mathbb{Z}_2)^3$	$\mathbb{Z}^4 \oplus (\mathbb{Z}_2)^5$
38	$\mathbb{Z}^2$	$\mathbb{Z}^2 \oplus (\mathbb{Z}_2)^2$	$(\mathbb{Z}_2)^7 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_7 \oplus \mathbb{Z}_{27}$	$\mathbb{Z}^5 \oplus (\mathbb{Z}_2)^4$
37	$\mathbb{Z} \oplus (\mathbb{Z}_2)^2 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_9$	$\mathbb{Z}^7 \oplus (\mathbb{Z}_2)^5$	$\mathbb{Z}^3 \oplus (\mathbb{Z}_2)^4$	$(\mathbb{Z}_2)^2 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_7$
36	$\mathbb{Z}^2 \oplus (\mathbb{Z}_2)^4$	$\mathbb{Z} \oplus (\mathbb{Z}_2)^{10} \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_4 \oplus (\mathbb{Z}_5)^2 \oplus \mathbb{Z}_7 \oplus \mathbb{Z}_9$	$\mathbb{Z}^7 \oplus (\mathbb{Z}_2)^2$	0
35	$\mathbb{Z}^{10} \oplus (\mathbb{Z}_2)^4$	$\mathbb{Z} \oplus (\mathbb{Z}_2)^5$	$\mathbb{Z} \oplus (\mathbb{Z}_2)^3$	$\mathbb{Z}$
34	$\mathbb{Z}^4 \oplus (\mathbb{Z}_2)^7 \oplus \mathbb{Z}_4$	$\mathbb{Z}^6 \oplus \mathbb{Z}_2$	0	
33	$(\mathbb{Z}_2)^4$	$\mathbb{Z}^2 \oplus \mathbb{Z}_2$	$\mathbb{Z}$	
32	$\mathbb{Z}^5$			
31	$\mathbb{Z}^2 \oplus \mathbb{Z}_2$			

FIGURE 1. The approximate integral Khovanov cohomology of $T(9,11)$ in quantum degrees $q = 125$ to 131.

$h \backslash q$	115	117	119	121	123
36				$\mathbb{Z}$	$\mathbb{Z}^3$
35				$\mathbb{Z}_2$	$\mathbb{Z}^2 \oplus (\mathbb{Z}_2)^5 \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_9$
34			$\mathbb{Z}$	$\mathbb{Z}^4 \oplus \mathbb{Z}_2$	$\mathbb{Z} \oplus (\mathbb{Z}_2)^6$
33			$\mathbb{Z}_2$	$\mathbb{Z}^5 \oplus (\mathbb{Z}_2)^7$	$\mathbb{Z}^{11} \oplus (\mathbb{Z}_2)^4$
32		$\mathbb{Z}^2$	$\mathbb{Z}^6 \oplus \mathbb{Z}_2$	$(\mathbb{Z}_2)^9 \oplus \mathbb{Z}_9$	$\mathbb{Z}^7 \oplus (\mathbb{Z}_2)^7$
31		$(\mathbb{Z}_2)^4 \oplus \mathbb{Z}_9$	$\mathbb{Z}^9 \oplus (\mathbb{Z}_2)^5$	$\mathbb{Z}^8 \oplus (\mathbb{Z}_2)^3$	$(\mathbb{Z}_2)^5$
30	$\mathbb{Z}^4$	$\mathbb{Z}^4 \oplus (\mathbb{Z}_2)^3$	$(\mathbb{Z}_2)^9 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_4$	$\mathbb{Z}^8 \oplus (\mathbb{Z}_2)^4$	$\mathbb{Z}^3$
29	$\mathbb{Z} \oplus (\mathbb{Z}_2)^4 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_4$	$\mathbb{Z}^{11} \oplus (\mathbb{Z}_2)^5$	$\mathbb{Z}^6 \oplus (\mathbb{Z}_2)^3$	$(\mathbb{Z}_2)^3 \oplus \mathbb{Z}_5$	$\mathbb{Z}$
28	$\mathbb{Z}^3 \oplus (\mathbb{Z}_2)^3$	$\mathbb{Z} \oplus (\mathbb{Z}_2)^{11} \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_5$	$\mathbb{Z}^8 \oplus (\mathbb{Z}_2)^3$	$\mathbb{Z}$	
27	$\mathbb{Z}^{12} \oplus (\mathbb{Z}_2)^3$	$\mathbb{Z}^3 \oplus (\mathbb{Z}_2)^2$	$(\mathbb{Z}_2)^2 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_5$	$\mathbb{Z}$	
26	$\mathbb{Z} \oplus (\mathbb{Z}_2)^8 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_9$	$\mathbb{Z}^6 \oplus \mathbb{Z}_2$			
25	$\mathbb{Z} \oplus (\mathbb{Z}_2)^2$	$\mathbb{Z}_2 \oplus \mathbb{Z}_5$			
24	$\mathbb{Z}^5 \oplus \mathbb{Z}_2$				
23	$\mathbb{Z}_2$				

FIGURE 2. Quantum degrees $q = 115$ to 123 for $T(9,11)$.

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$h \backslash q$	71	73	75	77	79	81	83	85	87	89	91	93	95	97	99
24															$\mathbb{Z}^2$
23															$(\mathbb{Z}_2)^2$
22														$\mathbb{Z}^2$	$\mathbb{Z}^4 \oplus \mathbb{Z}_2$
21														$(\mathbb{Z}_2)^2$	$\mathbb{Z}^5 \oplus (\mathbb{Z}_2)^4$
20													$\mathbb{Z}^3$	$\mathbb{Z}^4$	$(\mathbb{Z}_2)^4$
19													$(\mathbb{Z}_2)^3 \oplus \mathbb{Z}_3$	$\mathbb{Z}^6 \oplus (\mathbb{Z}_2)^3$	$\mathbb{Z}^5$
18												$\mathbb{Z}^3$	$\mathbb{Z}^3 \oplus (\mathbb{Z}_2)^2$	$(\mathbb{Z}_2)^4$	$\mathbb{Z}^2 \oplus \mathbb{Z}_2$
17												$\mathbb{Z} \oplus (\mathbb{Z}_2)^3$	$\mathbb{Z}^6 \oplus (\mathbb{Z}_2)^2$	$\mathbb{Z}^3$	0
16										$\mathbb{Z}$	$\mathbb{Z}^4$	$\mathbb{Z}^2$	$(\mathbb{Z}_2)^2$	$\mathbb{Z}^2 \oplus \mathbb{Z}_2$	$\mathbb{Z}$
15										$\mathbb{Z}_2$	$\mathbb{Z} \oplus (\mathbb{Z}_2)^3$	$\mathbb{Z}^5 \oplus \mathbb{Z}_2$	$\mathbb{Z}^2$		
14									$\mathbb{Z}$	$\mathbb{Z}^3$	$\mathbb{Z} \oplus \mathbb{Z}_2$	$(\mathbb{Z}_2)^2$	$\mathbb{Z}$		
13									0	$\mathbb{Z} \oplus (\mathbb{Z}_2)^2$	$\mathbb{Z}^4 \oplus \mathbb{Z}_2$	$\mathbb{Z}$			
12								$\mathbb{Z}$	$\mathbb{Z}^3$	$\mathbb{Z}$	$\mathbb{Z}_2 \oplus \mathbb{Z}_5$	$\mathbb{Z}$			
11								$\mathbb{Z}_2$	$\mathbb{Z} \oplus (\mathbb{Z}_2)^2$	$\mathbb{Z}^3$					
10							$\mathbb{Z}$	$\mathbb{Z}^2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$					
9							0	$\mathbb{Z} \oplus \mathbb{Z}_2$	$\mathbb{Z}^2$						
8						$\mathbb{Z}$	$\mathbb{Z}^2$	0							
7						$\mathbb{Z}_2$	$\mathbb{Z} \oplus \mathbb{Z}_2$	$\mathbb{Z}$							
6					$\mathbb{Z}$	$\mathbb{Z}$	0								
5					0	$\mathbb{Z}$	$\mathbb{Z}$								
4				$\mathbb{Z}$	$\mathbb{Z}$										
3				$\mathbb{Z}_2$	$\mathbb{Z}$										
2			$\mathbb{Z}$												
1															
0	$\mathbb{Z}$	$\mathbb{Z}$													

FIGURE 3. Approximate integral Khovanov cohomology of $T(9,10)$.

$h \backslash q$	101	103	105	107	109	111	113	115
36							$\mathbb{Z}$	$\mathbb{Z}^2$
35							$\mathbb{Z}_2$	$\mathbb{Z}^2 \oplus (\mathbb{Z}_2)^2 \oplus \mathbb{Z}_9$
34						$\mathbb{Z}$	$\mathbb{Z}^2 \oplus \mathbb{Z}_2$	$(\mathbb{Z}_2)^4$
33						$\mathbb{Z}_2$	$\mathbb{Z}^4 \oplus (\mathbb{Z}_2)^3$	$\mathbb{Z}^5 \oplus (\mathbb{Z}_2)^2$
32					$\mathbb{Z}^2$	$\mathbb{Z}^3 \oplus \mathbb{Z}_2$	$(\mathbb{Z}_2)^5 \oplus \mathbb{Z}_9$	$\mathbb{Z}^4 \oplus (\mathbb{Z}_2)^3 \oplus \mathbb{Z}_3$
31					$(\mathbb{Z}_2)^3 \oplus \mathbb{Z}_9$	$\mathbb{Z}^6 \oplus (\mathbb{Z}_2)^2$	$\mathbb{Z}^4 \oplus (\mathbb{Z}_2)^2$	$(\mathbb{Z}_2)^4$
30				$\mathbb{Z}^3$	$\mathbb{Z}^2 \oplus (\mathbb{Z}_2)^2$	$(\mathbb{Z}_2)^6 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_4$	$\mathbb{Z}^5 \oplus (\mathbb{Z}_2)^3$	$\mathbb{Z}^2$
29				$\mathbb{Z} \oplus (\mathbb{Z}_2)^2 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_4$	$\mathbb{Z}^8 \oplus (\mathbb{Z}_2)^3$	$\mathbb{Z}^4 \oplus (\mathbb{Z}_2)^2$	$(\mathbb{Z}_2)^2 \oplus \mathbb{Z}_5$	$\mathbb{Z}$
28		$\mathbb{Z}$	$\mathbb{Z}^4$	$\mathbb{Z}^2 \oplus (\mathbb{Z}_2)^3$	$\mathbb{Z} \oplus (\mathbb{Z}_2)^8 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_5$	$\mathbb{Z}^6 \oplus (\mathbb{Z}_2)^2$	$\mathbb{Z}$	
27		$\mathbb{Z}_2$	$\mathbb{Z}^2 \oplus (\mathbb{Z}_2)^5 \oplus \mathbb{Z}_3$	$\mathbb{Z}^9 \oplus (\mathbb{Z}_2)^2$	$\mathbb{Z}^2 \oplus (\mathbb{Z}_2)^2$	$(\mathbb{Z}_2)^2 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_5$	$\mathbb{Z}$	
26	$\mathbb{Z}$	$\mathbb{Z}^4 \oplus \mathbb{Z}_2$	$\mathbb{Z} \oplus (\mathbb{Z}_2)^4$	$\mathbb{Z} \oplus (\mathbb{Z}_2)^6 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_9$	$\mathbb{Z}^5 \oplus \mathbb{Z}_2$			
25	$\mathbb{Z}_2$	$\mathbb{Z}^3 \oplus (\mathbb{Z}_2)^4 \oplus \mathbb{Z}_3$	$\mathbb{Z}^9 \oplus (\mathbb{Z}_2)^2$	$\mathbb{Z} \oplus (\mathbb{Z}_2)^2$	$\mathbb{Z}_2 \oplus \mathbb{Z}_5$			
24	$\mathbb{Z}^5$	$\mathbb{Z} \oplus (\mathbb{Z}_2)^4$	$\mathbb{Z}^2 \oplus (\mathbb{Z}_2)^7 \oplus \mathbb{Z}_5$	$\mathbb{Z}^4 \oplus \mathbb{Z}_2$				
23	$\mathbb{Z}^4 \oplus (\mathbb{Z}_2)^5$	$\mathbb{Z}^8 \oplus \mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$				
22	$(\mathbb{Z}_2)^4$	$\mathbb{Z}^2 \oplus (\mathbb{Z}_2)^4 \oplus \mathbb{Z}_5$	$\mathbb{Z}^3$					
21	$\mathbb{Z}^7 \oplus (\mathbb{Z}_2)^2$	$\mathbb{Z}_2$						
20	$\mathbb{Z}^3 \oplus (\mathbb{Z}_2)^3$	$\mathbb{Z}^2$						
19	0							
18	$\mathbb{Z}$							

FIGURE 4. Quantum degrees $q = 101$ to 115 for $T(9,10)$.

$h \backslash q$	117	119	121	123	125	127	129	131
43							$\mathbb{Z}_3$	$\mathbb{Z}_3$
42					$\mathbb{Z}_3$	$\mathbb{Z}_2 \oplus \mathbb{Z}_3$	$\mathbb{Z}_2 \oplus \mathbb{Z}_5$	$\mathbb{Z}_5$
41				$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}_3$	$\mathbb{Z}_3$	
40		$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}_2 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_9$	$\mathbb{Z} \oplus \mathbb{Z}_3$			
39		$\mathbb{Z}_2 \oplus \mathbb{Z}_9$	$\mathbb{Z}^2$	$\mathbb{Z}_2$	$\mathbb{Z}_2 \oplus \mathbb{Z}_7$			
38	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_7 \oplus \mathbb{Z}_{27}$	$\mathbb{Z} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_3$	$\mathbb{Z}_2$			
37	$\mathbb{Z} \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_9$	$\mathbb{Z}^3 \oplus \mathbb{Z}_2$	$(\mathbb{Z}_2)^2$	$\mathbb{Z}_2 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_7$	$\mathbb{Z}_3$			
36	$(\mathbb{Z}_2)^2$	$\mathbb{Z} \oplus (\mathbb{Z}_2)^4 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_7 \oplus \mathbb{Z}_9$	$\mathbb{Z}^2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_3$					
35	$\mathbb{Z}^4 \oplus (\mathbb{Z}_2)^2$	$(\mathbb{Z}_2)^3$	$\mathbb{Z}_2 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_7$					
34	$\mathbb{Z}^2 \oplus (\mathbb{Z}_2)^2 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_7$	$\mathbb{Z}^2$	0					
33	$(\mathbb{Z}_2)^2$	$\mathbb{Z} \oplus \mathbb{Z}_2$	$\mathbb{Z}$					
32	$\mathbb{Z}^3$							
31	$\mathbb{Z} \oplus \mathbb{Z}_2$							

FIGURE 5. Quantum degrees $q = 117$ to 131 for $T(9,10)$.