LOCAL EQUIVALENCE AND REFINEMENTS OF RASMUSSEN'S S-INVARIANT

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ABSTRACT. Inspired by the notions of local equivalence in monopole and Heegaard Floer homology, we introduce a version of local equivalence that combines odd Khovanov homology with equivariant even Khovanov homology into an algebraic package called a local even-odd (LEO) triple. We get a homomorphism from the smooth concordance group $\mathcal{C}$ to the resulting local equivalence group $\mathcal{C}_{LEO}$ of such triples. We give several versions of the s-invariant that descend to $\mathcal{C}_{LEO}$, including one that completely determines whether the image of a knot K in $\mathcal{C}_{LEO}$ is trivial. We discuss computer experiments illustrating the power of these invariants in obstructing sliceness, both statistically and for some interesting knots studied by Manolescu-Piccirillo. Along the way, we explore several variants of this local equivalence group, including one that is totally ordered.

CONTENTS

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1. Introduction

Over the last two decades, knot homologies—knot Floer homology, Khovanov homology, and related invariants—have led to a wealth of new concordance invariants. One of the first of these was Ozsváth-Szabó's homomorphism $\tau\colon \mathcal{C} \to \mathbb{Z}$ from the smooth concordance group, defined using knot Floer homology [OS03]. Rasmussen imitated the construction of τ using the Lee deformation of Khovanov homology to obtain another concordance homomorphism $s\colon \mathcal{C} \to \mathbb{Z}$, and he used it to give the first combinatorial proof of Milnor's conjecture on the slice genus of torus knots [Ras10]. The homomorphism s was soon shown to be different from τ [HO08] and continues to play a somewhat special role in the subject (e.g., [Pic20, HMP21]). Other examples include Manolescu-Owens's homomorphism δ coming from the correction term for the branched double cover [MO07], Hom's invariant $\varepsilon \in \{-1, 0, 1\}$ [Hom14], Ozsváth-Stipsicz-Szabó's Υ [OSS17], and many others.

Another class of concordance invariants was inspired by Manolescu's disproof of the Triangulation Conjecture using $Pin(2)$-equivariant Seiberg-Witten theory [Man16]. There, Manolescu introduced homology cobordism invariants α, β, and γ of homology 3-spheres. While studying them, Stoffregen [Sto20] abstracted the key features of Seiberg-Witten theory underlying their existence: for all homology 3-spheres, the Seiberg-Witten Floer spectrum has the same $Pin(2)$-fixed set (corresponding to the reducible connections), and the maps associated to homology cobordisms induce homotopy equivalences of these fixed sets. So, the local behavior near the fixed set gives a homology cobordism invariant. This behavior can be captured at the level of cochain complexes by using the Atiyah-Bott localization theorem, leading to Stoffregen's notion of "chain local equivalence."

Partial analogues of Stoffregen's construction work for Heegaard Floer homology of closed 3-manifolds, using Hendricks-Manolescu's involutive Heegaard Floer homology [HM17]. For example, Hendricks, Manolescu, and Zemke built a group whose elements are chain complexes over a particular ring, modulo an equivalence relation they called local equivalence, so that involutive Heegaard Floer homology induces a homomorphism from the 3-dimensional homology cobordism group to this group [HMZ18]. Using a variant of their definition, whose group structure is easier to analyze, Dai, Hom, Stoffregen, and Truong were able to prove that the homology cobordism group has a $\mathbb{Z}^\infty$ summand [DHST23]. Analogous constructions also work for the concordance group of knots, using knot Floer homology [Zem19, DHST21].

The first goal of this paper is to give an analogous construction using Khovanov homology. Given a knot K, we consider the pair $(CKh_o(K), CKh_h(K))$ of the odd Khovanov complex of K and the Bar-Natan deformation of the even Khovanov complex of K. These complexes come with quotient maps to $CKh(K; \mathbb{F}_2)$, the even Khovanov complex with coefficients in $\mathbb{F}_2$. We call the data of (complexes like) $CKh_o(K)$ and $CKh_h(K)$, and a homotopy equivalence between their mod-2 reductions, a *local even-odd triple* or *LEO triple* (Definition 3.1). Using the fact that knot cobordisms induce isomorphisms on CKh_h after suitably localizing, we define a notion of local equivalence for LEO triples (Definition 3.2) and prove:

Theorem 1.1. *The local equivalence classes of LEO triples form an abelian group $\mathcal{C}_{LEO}$, and the Khovanov complexes induce a group homomorphism from the smooth concordance group to this local equivalence group.*

(This is stated more precisely as Theorem 3.8.)

We also give several versions of the construction modeled on reduced Khovanov homology. A *reduced LEO triple* (C, D, f) consists of finitely generated, free, bigraded, cochain complexes C over $\mathbb{Z}$ and D over $\mathbb{Z}[h]$, and a bigraded homotopy equivalence

$$f\colon C\otimes_{\mathbb{Z}}\mathbb{Z}/(2)\to D\otimes_{\mathbb{Z}[h]}\mathbb{Z}[h]/(2,h),$$

so that $D\otimes_{\mathbb{Z}[h]}\mathbb{Z}[h,h^{-1}]$ is homotopy equivalent to a free graded module of rank 1 over $\mathbb{Z}[h,h^{-1}]$. Local equivalence classes of these form an abelian group $\widetilde{\mathcal{C}}_{LEO}$. There is also a variant where D is merely defined over $\mathbb{F}_2[h]$, giving a group $\widetilde{\mathcal{C}}^o_{LEO}$, and here the analogy with Heegaard Floer homology goes farther:

Theorem 1.2. *There is an invariant $\tilde{s}_o\in\mathbb{Z}$ of reduced LEO triples so that $[(C,D,f)]$ is trivial in $\widetilde{\mathcal{C}}^o_{LEO}$ if and only if $\tilde{s}_o(C,D,f)=\tilde{s}_o((C,D,f)^*)=0$, where $(C,D,f)^*$ is the inverse to (C,D,f) in $\widetilde{\mathcal{C}}^o_{LEO}$. Further, the relation that $[(C,D,f)]\geq 0$ if and only if $\tilde{s}_o(C,D,f)\geq 0$ makes $\widetilde{\mathcal{C}}^o_{LEO}$ into a totally ordered abelian group.*

This is re-stated and proved as Theorems 5.1 and 5.6. There are partial analogues for some other variants of local equivalence as well: Theorem 5.1 for $\widetilde{\mathcal{C}}_{LEO}$ and Theorem 6.5 for the subgroup of $\mathcal{C}_{LEO}$ coming from knots. Theorem 1.2 is reminiscent of Hom's work on the ε-invariant, with $\widetilde{\mathcal{C}}^o_{LEO}$ being analogous to the group $\mathcal{CFK}$ of knot Floer-like complexes modulo those with $\varepsilon=0$ [Hom15].

The groups $\mathcal{C}_{LEO}$ and $\widetilde{\mathcal{C}}_{LEO}$ contain $\mathbb{Z}^\infty$-summands, though we do not know if these are realized by knots (Section 3.2); $\widetilde{\mathcal{C}}^o_{LEO}$ has at least a $\mathbb{Z}$-summand and is not isomorphic to $\mathbb{Z}$. Beyond this, the structure of these groups is open; see Remark 6.11. We do show that the images of the concordance group in $\mathcal{C}_{LEO}$ and $\widetilde{\mathcal{C}}_{LEO}$ are isomorphic (Corollary 6.6). (The image in $\widetilde{\mathcal{C}}^o_{LEO}$ is smaller.) Understanding the structure of these groups further might lead to new concordance information.

The local equivalence class associated to a knot determines its s-invariant, as well as the refinements using the even and odd Sq^1-operations (cases of refinements studied earlier [LS14, SSS20]). We give further integer invariants of knots that are determined by the local equivalence class of their Khovanov complexes. Certain of these invariants completely determine whether a knot K gives rise to the trivial element of $\mathcal{C}_{LEO}$, $\widetilde{\mathcal{C}}_{LEO}$, and, $\widetilde{\mathcal{C}}^o_{LEO}$, see Theorems 5.1 and 6.5. Even the most subtle of our invariants can be computed in practice for a given knot K with 20 crossings using `KnotJob` [Sch23], allowing us to determine when such K gives rise to the trivial element of $\mathcal{C}_{LEO}$. We demonstrate their efficacy by computing them for many knots where the s-invariant and its even and odd Sq^1-refinements vanish. First, we looked at the roughly 18,000 prime knots with at most 19 crossings whose (smooth) slice status has not been resolved by Dunfield and Gong [DGon] using a wide range of techniques; as there are 352 million prime knots in this range [Bur20], these 18,000 knots are very unusual in their difficulty to analyze. The integer invariants of Section 6 obstruct sliceness for 890 of these knots, reducing the number of mystery knots by 5%. Second, we use these invariants to give an alternate proof that the five intriguing knots of Manolescu-Piccirillo [MP] are not slice; this was originally shown by Nakamura [Nak22] using 0-surgery homeomorphisms to stably relate slice properties of two knots.

After posting this paper, we learned that Lewark has independently studied a notion of local equivalence for Khovanov homology [Lew23, Lew24], although without odd Khovanov homology.

This paper is organized as follows. Section 2 collects the results on Khovanov homology needed for the rest of the paper. In particular, Section 2.2 recalls Putyra's Künneth theorem for the odd Khovanov homology of connected sums (which is more subtle than the even case). Section 3 introduces our notion of local equivalence and develops its basic properties. Section 4 gives integer invariants of local equivalence classes coming from refinements of s, and uses them to further study the local equivalence group. The ties between these invariants and the structure of the local equivalence groups is studied in Section 5; in particular, Theorem 1.2 is proved there. These invariants have somewhat simpler behavior for LEO triples coming from knots, and we study properties specific to knots in Section 6. We conclude, in Section 7, with results of computer computations of these invariants, including the applications to certain difficult knots mentioned above.

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2. Background

2.1. The even and odd Khovanov complexes and the Bar-Natan deformation.

In this section, we introduce the notation we will use for the Khovanov complex and its variants, and specify our grading conventions.

Given a commutative ring R, we will denote the Khovanov complex for a knot K with coefficients in R by $CKh(K;R)$. We largely follow the conventions for Khovanov homology from the first papers in the subject [Kho00, BN02, Ras10]. The Khovanov complex is constructed by applying the Frobenius algebra $R[X]/(X^2)$ with comultiplication

$$\Delta(1) = 1 \otimes X + X \otimes 1 \qquad \Delta(X) = X \otimes X$$

(over R) to the cube of resolutions for K, and then taking either an iterated mapping cone or the total complex. Specifically, in $CKh(K;R)$ the differential goes from the 0-resolution of a crossing to the 1-resolution, and the homological and quantum gradings of a generator x lying over the vertex $v \in \{0,1\}^n$ are given by:

$$\begin{aligned}\mathrm{gr}_h(x) &= |v| - n_- \\ \mathrm{gr}_q(x) &= |v| + \#\{x(C) = 1\} - \#\{x(C) = X\} + n_+ - 2n_-,\end{aligned}$$

where $n_\pm$ are the number of positive and negative crossings of K, $|v|$ is the number of entries of v which are 1, and where $\#\{x(C) = 1\}$ and $\#\{x(C) = X\}$ are the number of circles in the resolution K_v that x labels 1 and X respectively. We will write $CKh^{i,j}(K;R)$ for the summand of $CKh(K;R)$ spanned by generators x with $(\mathrm{gr}_h(x), \mathrm{gr}_q(x)) = (i,j)$; the differential is a map $CKh^{i,j}(K;R) \to CKh^{i+1,j}(K;R)$. The homology of $CKh(K;R)$ is $Kh(K;R) = \bigoplus_{i,j} Kh^{i,j}(K;R)$.

We will use $\{n\}$ to denote an upwards shift in the quantum grading by n, so

$$\big(CKh(K;R)\{1\}\big)^{i,j} = CKh^{i,j-1}(K;R).$$

At the level of graded Euler characteristic, $\{1\}$ corresponds to multiplying by q.

Choosing a basepoint p on K, not at any of the crossings, makes $CKh(K;R)$ into a module over $R[X]/(X^2)$: X acts by multiplying the label of the circle containing p in each resolution by X. Equivalently, this module structure corresponds to merging a 0-crossing unknot at p.

The Bar-Natan deformation of the Khovanov Frobenius algebra is the Frobenius algebra $\mathcal{R} = R[h,X]/(X^2 - hX)$ over $R[h]$ with comultiplication

$$\Delta(1) = 1 \otimes X + X \otimes 1 - h \otimes 1 \qquad \Delta(X) = X \otimes X.$$

(This is a signed version of a construction of Bar-Natan's [BN05, Section 9.3]. The tensor products are over $R[h]$, so $h \otimes 1 = 1 \otimes h$.) We view h as having quantum grading -2. The unit is still 1, and the counit to $R[h]$ sends $1 \mapsto 0$ and $X \mapsto 1$. Applying this Frobenius algebra to the cube of resolutions gives the Bar-Natan complex $CKh_h^{i,j}(K;R)$, which is again a bigraded complex, but now over the graded ring $R[h]$.

Fixing a basepoint p on K makes $CKh_h(K;R)$ into a module over $\mathcal{R}$, where X acts on a vertex of the cube by multiplying the label of the circle containing p by X. Again, this can also be thought of as merging in an unknot at p.

A fundamental property of the Bar-Natan complex is that

$$h^{-1}Kh_h(K;R) \coloneqq Kh_h(K;R) \otimes_{R[h]} R[h,h^{-1}] \cong R[h,h^{-1}] \oplus R[h,h^{-1}]$$

(e.g., [LS22, Proposition 2.1]); the two summands correspond to the two orientations of K, and both lie in homological grading 0. Since localization is exact, we could equivalently localize $CKh_h^{i,j}(K;R)$ and then take homology. In fact, this identification holds at the chain level, and respects the action of X as well: there is a chain homotopy equivalence, over $\mathcal{R}$,

$$h^{-1}CKh_h(K;R) \simeq h^{-1}\mathcal{R} = \mathcal{R} \otimes_{R[h]} R[h,h^{-1}] \tag{2.1}$$

[LS22, Proof of Proposition 2.1]. We will often abuse notation and write $h^{-1}Kh_h^{i,j}(K;R)$ to mean the part of $h^{-1}Kh_h(K;R)$ in bigrading (i,j).

Ozsváth, Rasmussen, and Szabó constructed another variant of Khovanov homology, odd Khovanov homology $Kh_o(K;R) = \bigoplus_{i,j} Kh_o^{i,j}(K;R)$ [ORS13]. (To make the distinction clear, we henceforth refer to the original Khovanov homology as the *even Khovanov homology*.) The underlying odd Khovanov complex $CKh_o(K;R)$ is defined similarly to the even Khovanov complex, except that instead of viewing the space associated to a vertex K_v in the cube of resolutions as a tensor product of copies of $R[X]/(X^2)$ over the circles in K_v, it is viewed as the exterior algebra on the set of circles in K_v. As an R-module, if one chooses an ordering of the circles in K_v, there is an identification between the groups, sending $C_{i_1} \wedge \cdots \wedge C_{i_k} \in CKh_o(K;R)$ to the generator $x \in CKh(K;R)$ which labels each C_{i_j} by X and the other circles by 1. The exterior algebra leads to different signs, and hence a different invariant if $\mathrm{char}(R) \neq 2$. On the other hand, $Kh_o(K;\mathbb{F}_2)$ and $Kh(K;\mathbb{F}_2)$ are canonically isomorphic.

Putyra observed that the complex $CKh_o(K;R)$ associated to a based knot (or link) is a bimodule over $R[X]/(X^2)$ [Put16]. We review the construction of this action in Section 2.2. The left and right actions on $CKh_o(K;\mathbb{Z})$ have the same mod-2 reduction, and agree with the usual action on $CKh(K;\mathbb{F}_2)$. (In fact, they are related by an automorphism of the complex; see Lemma 2.13.)

An oriented cobordism $\Sigma \subset [0,1]\times\mathbb{R}^3$ from $K_0 \subset \{0\}\times\mathbb{R}^3$ to $K_1 \subset \{1\}\times\mathbb{R}^3$, decomposed as a movie of elementary cobordisms (Reidemeister moves, births, saddles, and deaths), induces a chain map

$$\Sigma_*\colon CKh^{i,j}(K_0;R) \to CKh^{i,j+\chi(\Sigma)}(K_1;R),$$

as well as chain maps Σ_* on CKh_h and CKh_o, with the same grading shift by $\chi(\Sigma)$. (The constructions of the maps Σ_* are given by Khovanov, Bar-Natan, and Putyra [Kho00, BN05, Put14].) Up to homotopy and an overall sign, the maps on CKh and CKh_h are independent of the decomposition of Σ as a movie (or, equivalently, of isotopies of Σ), but the maps on CKh_o are not known to be; we will never use below that any of the maps are independent of the choice of movie. If none of the Reidemeister moves in the movie cross the basepoint (i.e., it is a *based movie*), then the map Σ_* commute with the X-action on the two sides.

For all of the variants of Khovanov homology, the complexes and maps with R coefficients are obtained from the corresponding complexes over $\mathbb{Z}$ by tensoring with R. So, for instance, given a ring homomorphism $f\colon R \to S$, the square

$$\begin{tikzcd} CKh(K_0;R) \arrow[r, "f_*"] \arrow[d, "\Sigma_*"] & CKh(K_0;S) \arrow[d, "\Sigma_*"] \\ CKh(K_1;R) \arrow[r, "f_*"] & CKh(K_1;S) \end{tikzcd}$$

commutes. The complex $CKh(K;R)$ is obtained from $CKh_h(K;R)$ by setting $h=0$, i.e.,

$$CKh(K;R) = CKh_h(K;R) \otimes_{R[h]} R$$

where $R = R[h]/(h)$. Again, this identification is functorial, in the sense that

$$\begin{tikzcd} CKh_h(K_0;R) \arrow[r, "/(h)"] \arrow[d, "\Sigma_*"] & CKh(K_0;R) \arrow[d, "\Sigma_*"] \\ CKh_h(K_1;R) \arrow[r, "/(h)"] & CKh(K_1;R) \end{tikzcd}$$

commutes.

There are reduced versions of the Khovanov complex, lifting the reduced Jones polynomial. If we view R as an $R[X]/(X^2)$-module where X acts by 0 then the reduced Khovanov complex is

$$\widetilde{CKh}(K;R) = CKh(K;R) \otimes_{R[X]/(X^2)} R\{-1\}.$$

Similarly, we can make $R[h]$ into a module over $\mathcal{R}$ by letting X act by 0, and

$$\widetilde{CKh}_h(K;R) = CKh_h(K;R) \otimes_{\mathcal{R}} R[h]\{-1\}.$$

Finally, using either the left or right module structure on $CKh_o(K;R)$, we have

$$\widetilde{CKh}_o(K;R) = CKh_o(K;R) \otimes_{R[X]/(X^2)} R\{-1\}.$$

By construction, these are all natural with respect to changes of ring R, and based movies induce maps on the reduced complexes.

2.2. **Künneth theorems.** The main reason we record the action of $R[X]/(X^2)$ on Khovanov homology is the well-known Künneth theorem for connected sums:

Lemma 2.2. *Given based knot (or link) diagrams K_1 and K_2, there is an isomorphism of chain complexes*

$$CKh(K_1\#K_2;R) \cong CKh(K_1;R) \otimes_{R[X]/(X^2)} CKh(K_2;R), \tag{2.3}$$

where $K_1\#K_2$ is the result of taking the connected sum at the basepoints. Moreover, if we choose a basepoint on $K_1\#K_2$ to be one of the two arcs that connect K_1 and K_2, then the isomorphism (2.3) intertwines the actions of $R[X]/(X^2)$ on the two sides. Further, given a ring homomorphism $R \to R'$, the isomorphism (2.3) commutes with the change-of-ring maps $CKh(K;R) \to CKh(K;R')$ in the obvious sense.

Proof. It is immediate from the construction of Khovanov homology that

$$CKh(K_1 \amalg K_2;R) \cong CKh(K_1;R) \otimes_R CKh(K_2;R). \tag{2.4}$$

Let U be a 0-crossing unknot which is adjacent to the basepoints in K_1 and K_2. The two cobordisms

$$K_1 \amalg U \amalg K_2 \rightrightarrows K_1 \amalg K_2 \to K_1 \# K_2,$$

gotten by first merging U to K_1 and then to K_2, or first to K_2 and then to K_1, induce the same map of Khovanov complexes. Using the isomorphism (2.4) and the fact that $CKh(U;R) = R[X]/(X^2)$, we get a commutative diagram

$$\begin{aligned} CKh(K_1;R) \otimes_R R[X]/(X^2) \otimes_R CKh(K_2;R) &\rightrightarrows CKh(K_1;R) \otimes_R CKh(K_2;R) \\ &\to CKh(K_1 \# K_2;R), \end{aligned}$$

and hence a chain map $CKh(K_1;R) \otimes_{R[X]/(X^2)} CKh(K_2;R) \to CKh(K_1\#K_2;R)$. It is easy to see this map induces a bijection on generators, and hence is a chain isomorphism.

All of these maps commute with merging on an unknot somewhere else to K_1 or K_2, and hence respect the $R[X]/(X^2)$-module structure. It is also immediate from the construction that the maps are natural with respect to change of rings. □

Remark 2.5. Lemma 2.2 can be seen as a special case of Khovanov's tangle invariant [Kho02] for the 2-ended tangles obtained by deleting neighborhoods of the basepoints.

Lemma 2.6. *With notation as in Lemma 2.2, there is an isomorphism of chain complexes*

$$CKh_h(K_1\#K_2;R) \cong CKh_h(K_1;R) \otimes_{\mathcal{R}} CKh_h(K_2;R) \tag{2.7}$$

respecting the action of $\mathcal{R}$ and natural with respect to changes in ground ring. Moreover, reducing the isomorphism (2.7) *modulo h gives the isomorphism* (2.3).

Proof. The proof is the same as for Lemma 2.2, using the isomorphism

$$CKh_h(L_1 \amalg L_2;R) \cong CKh_h(L_1;R) \otimes_{R[h]} CKh_h(L_2;R)$$

in place of Formula (2.4). □

For odd Khovanov homology, the Künneth theorems for disjoint unions and connected sums were proved by Putyra [Put16], and are substantially more complicated. Putyra in fact proves more general results about unified Khovanov homology. For the reader's convenience, we give a streamlined explanation for just the case of odd Khovanov homology. To keep notation short, we state the results only for $R = \mathbb{Z}$; the general case follows by tensoring all of the complexes with R.

To define the odd Khovanov complex, one first picks *crossing orientations*. A generator of the complex $CKh_o(K)$ at a vertex v of the cube is an element of the exterior algebra $\Lambda(\pi_0 K_v)$ on the components of the complete resolution K_v. Reducing modulo 2, the generator $Z_{i_1} \wedge \cdots \wedge Z_{i_k}$ of $CKh_o(K;\mathbb{F}_2)$, where the Z_{i_j} are circles in the v-resolution of K, corresponds to the generator of $CKh(K;\mathbb{F}_2)$ which labels the circles Z_{i_j} by X and the other circles by 1. To get the differential, one also uses an *edge assignment* [ORS13, Definition 1.2], which assigns ± 1 to each edge in a way that makes each face anti-commute, and subject to an extra condition on so-called *type* X and *type* Y faces.

A key observation of Putyra's is that for each vertex v in the cube of resolutions of K, there is an integer $\mathrm{split}(v)$, the *number of splits before* v. Explicitly,

$$\mathrm{split}(v) = \frac{1}{2}\big(|\pi_0 K_v| - |\pi_0 K_{\vec{0}}| + |v|\big),$$

the difference in the number of circles in K_v and the number of circles in $K_{\vec{0}}$, plus the sum of the entries of v, divided by 2.

Fix a basepoint p on K. Given a complete resolution K_v of K, let $Z_p \subset K_v$ be the circle containing p. There is a right action of $\mathbb{Z}[X]/(X^2)$ on $CKh_o(K)$ by

$$(Z_{i_1} \wedge \cdots \wedge Z_{i_j}) \cdot X = Z_{i_1} \wedge \cdots \wedge Z_{i_j} \wedge Z_p. \tag{2.8}$$

Since the merge map merging circles Z_1 and Z_2 to a new circle Z is induced by the projection $Z_1, Z_2 \mapsto Z$, $Z_i \mapsto Z_i$ for $i \neq 1, 2$, it is immediate that this commutes with the merge maps in the odd Khovanov cube; it also commutes with the split maps since splitting a circle Z into Z_1 and Z_2 corresponds to multiplying on the left by $Z_1 - Z_2$, which commutes with multiplying on the right by Z_p. Thus, $CKh_o(K)$ is a cochain complex over $\mathbb{Z}[X]/(X^2)$. There is also a left action of $\mathbb{Z}[X]/(X^2)$ defined by

$$X \cdot (Z_{i_1} \wedge \cdots \wedge Z_{i_j}) = (-1)^{\mathrm{split}(v)} Z_p \wedge Z_{i_1} \wedge \cdots \wedge Z_{i_j}. \tag{2.9}$$

where $Z_{i_1}, \ldots, Z_{i_j}$ are circles in the resolution K_v. Again, this evidently commutes with all merge maps; it is straightforward to check that it commutes with all split maps as well.

Here is the special case of Putyra's result [Put16, Theorem 6.7] that we use in this paper:

Theorem 2.10. *Given link diagrams K_1 and K_2, there is an isomorphism of chain complexes*

$$CKh_o(K_1 \amalg K_2) \cong CKh_o(K_1) \otimes_{\mathbb{Z}} CKh_o(K_2). \tag{2.11}$$

Given based link diagrams K_1 and K_2, there is an isomorphism of chain complexes

$$CKh_o(K_1 \# K_2) \cong CKh_o(K_1) \otimes_{\mathbb{Z}[X]/(X^2)} CKh_o(K_2). \tag{2.12}$$

Further, if we choose the basepoint for $CKh_o(K_1 \# K_2)$ to be on one of the two arcs where the connected sum occurred, then the chain isomorphism (2.12) respects the $\mathbb{Z}[X]/(X^2)$ bimodule structure. Finally, if we reduce the isomorphisms (2.11) and (2.12) modulo 2, we obtain the isomorphisms (2.4) and (2.3) with $R = \mathbb{F}_2$.

Proof. We start with the result about disjoint unions. Suppose K_i has n_i crossings. For convenience, fix an ordering of the crossings of each K_i, so we can identify the cube of resolutions of K_i with $\underline{2}^{n_i}$, where $\underline{2}^1 = (0 \to 1)$. Fix also an edge assignment ε_i for K_i.

Let $v = (v_1, v_2) \in \{0,1\}^{n_1+n_2}$, and let $CKh_o^{v_i}(K_i)$ denote the summand of $CKh_o(K_i)$ corresponding to the v_i-resolution. Given a basis element $w_i \in CKh_o^{v_i}(K_i)$, so w_i is a wedge product of circles in $(K_i)_{v_i}$, let $|w_i|$ denote the number of factors in the wedge product. Define $f\colon CKh_o^{v_1}(K_1) \otimes_{\mathbb{Z}} CKh_o^{v_2}(K_2) \to CKh_o^{v}(K_1 \sqcup K_2)$ by

$$f(w_1 \otimes w_2) = (-1)^{|w_1| \cdot \mathrm{split}(v_2)} w_1 \wedge w_2.$$

Clearly this induces an isomorphism on cochain groups

$$f\colon CKh_o^{i}(K_1) \otimes_{\mathbb{Z}} CKh_o^{j}(K_2) \to CKh_o^{i+j}(K_1 \sqcup K_2),$$

and thus we can define a differential on $CKh_o(K_1 \sqcup K_2)$ by $\partial^f = f \circ \partial^{\otimes} \circ f^{-1}$. We need to check that this differential is the differential of an odd Khovanov complex for $K_1 \sqcup K_2$; that is, we need to check that it differs from the usual merge and split maps by an edge assignment.

We first show that ∂^f restricted to an edge $v \to v'$ in the cube of resolutions differs from the merge or split map by a sign only depending on the edge, and then show that this sign function is indeed an edge assignment. To see the first part, we need to check the various cases with $v_1 \to v_1'$ a merge or split, and $v_2 \to v_2'$ a merge or split. The most interesting case is the case when $v_2 \to v_2'$ is a split, and this case will also make clear how the other easier cases work. We have

$$\begin{aligned}\partial^f_{v\to v'}(w_1 \wedge w_2) &= f\partial^{\otimes}_{v\to v'}((-1)^{|w_1|\cdot \mathrm{split}(v_2)} w_1 \otimes w_2)\\ &= f((-1)^{\mathrm{gr}_h(w_1)+\varepsilon_2(v_2\to v_2')+|w_1|\cdot\mathrm{split}(v_2)} w_1 \otimes (A-B)\wedge w_2)\\ &= (-1)^{\mathrm{gr}_h(w_1)+\varepsilon_2(v_2\to v_2')+|w_1|} w_1 \wedge (A-B)\wedge w_2\\ &= (-1)^{\mathrm{gr}_h(w_1)+\varepsilon_2(v_2\to v_2')}(A-B)\wedge w_1 \wedge w_2,\end{aligned}$$

where A and B are the two circles created by the split, and the orientation of the handle (crossing orientation) points from A to B. Since $\mathrm{gr}_h(w_1)$ only depends on v_1, the sign difference $(-1)^{\mathrm{gr}_h(w_1)+\varepsilon_2(v_2\to v_2')}$ between $\partial^f_{v\to v'}$ and the split map depends only on the edge $v \to v'$, and not on the particular generator involved, as desired. The other three cases are similar, but easier: for a $v_2 \to v_2'$ merge, the difference is $(-1)^{\mathrm{gr}_h(w_1)+\varepsilon_2(v_2\to v_2')}$, for a $v_1 \to v_1'$ split the difference is $(-1)^{\mathrm{split}(v_2)+\varepsilon_1(v_1\to v_1')}$, and for a $v_1 \to v_1'$ merge, the difference is $(-1)^{\varepsilon_1(v_1\to v_1')}$.

It remains to show that the sign function is indeed an edge assignment. For faces in $\underline{2}^{n_1+n_2}$ of type A or C, this follows from the fact that $\partial^f \circ \partial^f = 0$. For faces of type X and Y, observe that such faces have to be either in $\underline{2}^{n_1}$ or in $\underline{2}^{n_2}$. In either case, the product of the signs around the face is the same as in (K_i, ε_i), because there is either an extra contribution of $(-1)^{4\,\mathrm{gr}_h(w_1)}$ or of $(-1)^{2\,\mathrm{split}(v_2)}$.

Turning to the result for connected sums, note that there is a natural surjection on cochain groups

$$m\colon CKh_o(K_1 \sqcup K_2) \to CKh_o(K_1 \# K_2)$$

by merging the appropriate circles. The edge assignment on the disjoint union descends to an edge assignment on the connected sum; again, this follows from $\partial\circ\partial = 0$, since X- and Y-type faces have to be in $\underline{2}^{n_1}$ or $\underline{2}^{n_2}$ only. In particular, m is a cochain map with this choice.

Now,

$$g = m \circ f\colon CKh_o(K_1) \otimes_{\mathbb{Z}} CKh_o(K_2) \to CKh_o(K_1 \sqcup K_2) \to CKh_o(K_1 \# K_2).$$

is a cochain map. We claim that $g(w_1 X \otimes w_2) = g(w_1 \otimes X w_2)$. Indeed, if we let Z_p and $Z_{p'}$ be the components of $(K_1)_{v_1}$ and $(K_2)_{v_2}$ containing the basepoints, then

$$g(w_1 X \otimes w_2) = m(f(w_1 \wedge Z_p \otimes w_2) = m((-1)^{(|w_1|+1)\cdot\mathrm{split}(v_2)} w_1 \wedge Z_p \wedge w_2),$$

while

$$g(w_1 \otimes X w_2) = m(f((-1)^{\mathrm{split}(v_2)} w_1 \otimes Z_{p'} \wedge w_2) = m((-1)^{\mathrm{split}(v_2)+|w_1|\cdot\mathrm{split}(v_2)} w_1 \wedge Z_{p'} \wedge w_2).$$

After the merge, Z_p and $Z_{p'}$ represent the same element, so these expressions agree. Thus, g descends to a cochain map

$$\overline{g}\colon CKh_o(K_1) \otimes_{\mathbb{Z}[X]/(X^2)} CKh_o(K_2) \to CKh_o(K_1 \# K_2).$$

We also get that $\overline{g}$ preserves the left and right $\mathbb{Z}[X]/(X^2)$-module structures. For the right action, this is obvious. For the left action,

$$\overline{g}(X w_1 \otimes w_2) = \overline{g}((-1)^{\mathrm{split}(v_1)} Z_p \wedge w_1 \otimes w_2) = (-1)^{\mathrm{split}(v_1)+(|w_1|+1)\,\mathrm{split}(v_2)} m(Z_p \wedge w_1 \wedge w_2),$$

while

$$X\overline{g}(w_1 \otimes w_2) = Xm((-1)^{|w_1|\cdot\mathrm{split}(v_2)} w_1 \wedge w_2) = (-1)^{\mathrm{split}(v)+|w_1|\cdot\mathrm{split}(v_2)} m(Z_p \wedge w_1 \wedge w_2).$$

So, since $\mathrm{split}(v) = \mathrm{split}(v_1) + \mathrm{split}(v_2)$ the left action of X is also preserved. $\square$

The left and right module structures on CKh_o are related by an automorphism of the complex. Specifically, decompose $CKh_o(K)$ as $C'(K) \oplus C''(K)$ where $C'(K) = CKh_o(K) \cdot X$ is the image of multiplication by X (i.e., is spanned by wedge products containing Z_p as a factor) and $C''(K)$ is the complement to $C'(K)$ spanned by the empty wedge products and wedge products of the form $(Z_1 - Z_p) \wedge Z_2 \wedge \cdots \wedge Z_i$ (where the Z_j are arbitrary circles in the resolution not containing the basepoint). Define $h\colon C' \oplus C'' \to C' \oplus C''$

$$h(w', w'') = (w', (-1)^{\mathrm{split}(v)+|w''|}w''),$$

where v is the vertex of the cube that w'' lies over.

Lemma 2.13. *The map h is a chain isomorphism satisfying $h(X \cdot w) = h(w) \cdot X$.*

Proof. This is straightforward from the definitions. □

Corollary 2.14. *Given based link diagrams K_1 and K_2, if we view both $CKh_o(K_i)$ as right modules over the commutative ring $\mathbb{Z}[X]/(X^2)$ by using the action in Equation (2.8), then there is an isomorphism of chain complexes of $\mathbb{Z}[X]/(X^2)$-modules*

$$CKh_o(K_1 \# K_2) \cong CKh_o(K_1) \otimes_{\mathbb{Z}[X]/(X^2)} CKh_o(K_2)$$

where the action on the left-hand side is again the right action by $\mathbb{Z}[X]/(X^2)$ (and the basepoint is on one of the arcs where the connected sum occurred).

Proof. This is immediate from Theorem 2.10 and Lemma 2.13. □

So, for the rest of the paper, we will only think of $CKh_o(K)$ as a right module over $\mathbb{Z}[X]/(X^2)$, and ignore the left action.

The Künneth theorems for reduced Khovanov homology are slightly simpler:

Lemma 2.15. *There are chain isomorphisms*

$$\widetilde{CKh}(K_1 \# K_2; R) \cong \widetilde{CKh}(K_1; R) \otimes_R \widetilde{CKh}(K_2; R) \tag{2.16}$$

$$\widetilde{CKh}_h(K_1 \# K_2; R) \cong \widetilde{CKh}_h(K_1; R) \otimes_{R[h]} \widetilde{CKh}_h(K_2; R) \tag{2.17}$$

$$\widetilde{CKh}_o(K_1 \# K_2; R) \cong \widetilde{CKh}_o(K_1; R) \otimes_R \widetilde{CKh}_o(K_2; R). \tag{2.18}$$

Moreover, the second isomorphism mod h is the first one, and the first and third isomorphisms agree mod 2.

Proof. The result is immediate from Lemmas 2.2 and 2.6, Theorem 2.10, and the definitions of the reduced complexes. □

3. Even-odd local equivalence

In this section, we introduce a notion of local equivalence combining even and odd Khovanov homology. We work over the ground ring $R = \mathbb{Z}$, so in particular $\mathcal{R} = \mathbb{Z}[X, h]/(X^2 - Xh)$.

Definition 3.1. *A* local even-odd (LEO) triple *consists of a finitely generated, bigraded cochain complex C over $\mathbb{Z}[X]/(X^2)$, a finitely generated, bigraded cochain complex D over $\mathcal{R}$, and a bigraded chain homotopy equivalence*

$$f\colon C \otimes_{\mathbb{Z}} \mathbb{Z}/(2) \to D \otimes_{\mathcal{R}} \mathcal{R}/(2, h),$$

so that:

- *C is freely generated over $\mathbb{Z}[X]/(X^2)$,*
- *D is freely generated over $\mathcal{R}$,*
- *the map f is a homomorphism of cochain complexes over $\mathbb{F}_2[X]/(X^2)$, and*
- *the localization $h^{-1}D = D \otimes_{\mathcal{R}} h^{-1}\mathcal{R}$ is homotopy equivalent to a free graded module of rank 1 over $h^{-1}\mathcal{R}$, supported in homological grading 0 and odd quantum gradings.*

Given a knot diagram K, let $f\colon CKh_o(K) \otimes_{\mathbb{Z}} \mathbb{F}_2 \to CKh_h(K) \otimes_{\mathcal{R}} \mathbb{F}_2$ be the identification of the mod-2 reduction of the odd Khovanov complex with the mod-$(2,h)$ reduction of the Bar-Natan complex (both of which are $CKh(K;\mathbb{F}_2)$). It follows from the results cited in Section 2.1 that

$$LEO(K) = (CKh_o(K), CKh_h(K), f)$$

is a LEO triple, and this is our motivating example. A special case is the *trivial LEO triple*

$$LEO(U) = (\mathbb{Z}[X]/(X^2)\{1\}, \mathcal{R}\{1\}, \mathrm{id})$$

associated to the unknot U. (The set braces indicate quantum grading shifts.) Another LEO triple associated to K that uses only the even Khovanov homology is

$$LEE(K) = (CKh(K), CKh_h(K), f),$$

where f is the obvious identification of $CKh_h(K) \otimes_{\mathcal{R}} \mathbb{F}_2$ with $CKh(K;\mathbb{F}_2)$. While the data in $LEE(K)$ is completely determined by $LEO(K)$, we will use it later to put certain existing invariants into our framework.

Definition 3.2. *Given LEO triples (C, D, f) and (C', D', f'), a* local map *from (C, D, f) to (C', D', f') consists of bigrading-preserving chain maps $\alpha\colon C \to C'$ and $\beta\colon D \to D'$ (respecting the module structures) so that*

- *the induced map $\beta\colon h^{-1}D \to h^{-1}D'$ is a homotopy equivalence and*
- *the following diagram commutes up to homotopy:*

$$\begin{tikzcd} C \arrow[r] \arrow[d, "\alpha"] & C \otimes_{\mathbb{Z}} \mathbb{Z}/(2) \arrow[r, "f"] \arrow[d, "\alpha"] & D \otimes_{\mathcal{R}} \mathcal{R}/(2,h) \arrow[d, "\beta"] & D \arrow[l] \arrow[d, "\beta"] \\ C' \arrow[r] & C' \otimes_{\mathbb{Z}} \mathbb{Z}/(2) \arrow[r, "f'"] & D' \otimes_{\mathcal{R}} \mathcal{R}/(2,h) & D'. \arrow[l] \end{tikzcd} \tag{3.3}$$

(The first and third squares automatically commute on the nose, so the condition is equivalent to homotopy commutativity of the second square. Also, the homotopy $f'\circ\alpha \sim \beta\circ f$ is required to respect the action of $\mathbb{F}_2[X]/(X^2)$.) We say that (C, D, f) and (C', D', f') are locally equivalent *if there are local maps $(C, D, f) \to (C', D', f')$ and $(C', D', f') \to (C, D, f)$.*

A key motivation for Definition 3.2 is the following:

Proposition 3.4. *If K_0 and K_1 are (smoothly) concordant, then $LEO(K_1)$ and $LEO(K_2)$ are locally equivalent, as are $LEE(K_1)$ and $LEE(K_2)$. In particular, if K is slice then $LEO(K)$ and $LEE(K)$ are locally equivalent to $LEO(U)$.*

Proof. We prove the result for LEO; the statements for LEE follow since it is determined by LEO. Suppose that Σ is a concordance from K_0 to K_1. Fix a description of Σ as a movie.

Then there is a diagram:

$$
\begin{tikzcd}
CKh_o(K_0) \arrow[r, "/(2)"] \arrow[d, "\alpha"] & CKh(K_0;\mathbb{F}_2) \arrow[d] & CKh_h(K_0) \arrow[l, "/(2,h)"'] \arrow[r] \arrow[d, "\beta"] & h^{-1}CKh_h(K_0) \arrow[d, "\simeq"] \\
CKh_o(K_1) \arrow[r, "/(2)"] & CKh(K_1;\mathbb{F}_2) & CKh_h(K_1) \arrow[l, "/(2,h)"'] \arrow[r] & h^{-1}CKh_h(K_1).
\end{tikzcd} \tag{3.5}
$$

Here, all the vertical arrows are induced by Σ, and are bigrading-preserving; the second vertical arrow is the map on $CKh(\cdot;\mathbb{F}_2)$ induced by the movie. Commutativity of the middle and right squares follows from the principle at the end of Section 2.1, that the maps on the even Khovanov complex with R-coefficients (for any R) are induced by the maps over $\mathbb{Z}$, and the map on the Bar-Natan complex reduces to the usual map on the Khovanov complex. Commutativity of the left square is a special case of a result of Sarkar-Scaduto-Stoffregen about the odd Khovanov spectrum [SSS20, Lemma 5.11], and is implicit in Putyra's work [Put14]. In particular, all of the squares commute exactly, not just up to homotopy.

The fact that the vertical arrow at the right is a homotopy equivalence essentially follows from a result of Rasmussen's [Ras10, Corollary 4.2], though he was working over $\mathbb{Q}$, used a different Frobenius algebra, and only stated it for homology. The result for the Bar-Natan deformation, over arbitrary rings, is well known (e.g., [LS22, Proposition 3.4] where the result is stated only for homology, but proof gives the chain level statement).

To arrange that α commutes with the action of $\mathbb{Z}[X]$ and β commutes with the action of $\mathcal{R}$, we modify Σ and its movie representative slightly. Specifically, the action of X uses a basepoint on K_0 and K_1, though up to homotopy equivalence the resulting complexes of modules are independent of the choice of basepoint. Choose Σ and the movie representing it so that no Reidemeister move in the movie for Σ crosses the basepoint, and the component containing the basepoint never disappears (dies), and choose the basepoint on K_1 to be the image of the basepoint on K_0 under the movie. (To arrange the condition about Reidemeister moves, one can replace any move that crosses the basepoint by a sequence of moves going around the rest of the diagram and through infinity. Note that this may change the isotopy class of Σ in $[0,1]\times S^3$.) Then, the map associated to each step in the movie for Σ commutes with the X-action.

Thus, Σ induces a local map $LEO(K_0)\to LEO(K_1)$. Reading Σ backward gives a local map $LEO(K_1)\to LEO(K_0)$, so the $LEO(K_0)$ and $LEO(K_1)$ are equivalent. ☐

With this motivation in hand, we return to developing some general properties of LEO triples and local equivalence.

Lemma 3.6. *Local equivalence of LEO triples is an equivalence relation.*

Proof. This is straightforward. ☐

We can make the set of local equivalence classes of LEO triples into a group, as follows:

Definition 3.7. *The* tensor product *of LEO triples* (C,D,f) *and* (C',D',f') *is*

$$\big(C\otimes_{\mathbb{Z}[X]/(X^2)} C'\{-1\},\ D\otimes_{\mathcal{R}} D'\{-1\},\ f\otimes f'\big).$$

The dual *of* (C,D,f) *is* $(\mathrm{Hom}(C,\mathbb{Z}[X]/(X^2))\{2\},\ \mathrm{Hom}(D,\mathcal{R})\{2\},\ \overline{f})$ *where we grade the dual complexes so that* $\mathrm{Hom}(C,\mathbb{Z})_{i,j}=\mathrm{Hom}(C_{-i,-j},\mathbb{Z})$*, and similarly for* D*, and the* $\{2\}$ *denotes a quantum grading shift. The map* $\overline{f}$ *is the transpose (map of dual complexes)* *induced by the inverse homotopy equivalence to f. We will denote the dual by $(C,D,f)^*$ or (C^*,D^*,f^*). (The latter is a slight abuse of notation.)*

Theorem 3.8. *The set of local equivalence classes of LEO triples forms an abelian group $\mathcal{C}_{LEO}$, with addition the tensor product from Definition 3.7, identity $LEO(U)$, and inverse given by the dual from Definition 3.7. Moreover, the assignment $K \mapsto LEO(K)$ induces a homomorphism from the smooth concordance group $\mathcal{C}$ to $\mathcal{C}_{LEO}$.*

Proof. For the first statement, we must verify that:

(1) The tensor product of LEO triples is a LEO triple.
(2) Tensor product respects local equivalence.
(3) The dual of a LEO triple is a LEO triple.
(4) The dual respects local equivalence.
(5) The tensor product is associative.
(6) The triple $LEO(U)$ is a unit for tensor product.
(7) The tensor product is commutative up to local equivalence.
(8) For any LEO triple (C,D,f), $(C,D,f)\otimes(C,D,f)^*$ is locally equivalent to $LEO(U)$.

Points (1), (2), (3), and (4) are straightforward from the definitions; note that for Point (4), a local map from (C,D,f) to (C',D',f') induces a map from $(C',D',f')^*$ to $(C,D,f)^*$. Points (5), (6), and (7) are immediate from the definitions. For Point (8), observe that

$$(C,D,f)\otimes(C,D,f)^* = (\mathrm{Hom}_{\mathbb{Z}[X]/(X^2)}(C,C)\{1\}, \mathrm{Hom}_{\mathcal{R}}(D,D)\{1\}, F)$$

where $F(\eta) = f\circ\eta\circ\overline{f}$ (and $\overline{f}$ is the homotopy inverse to f). Consider the map from $LEO(U)$ to this tensor product defined by

$$\alpha\colon \mathbb{Z}[X]/(X^2)\{1\} \to \mathrm{Hom}_{\mathbb{Z}[X]/(X^2)}(C,C)\{1\} \qquad \beta\colon \mathcal{R}\{1\} \to \mathrm{Hom}_{\mathcal{R}}(D,D)\{1\}$$

both of which send 1 to the identity map. We claim that Diagram (3.3) homotopy commutes. As noted above, it suffices to check that the second square, which has the form

$$\begin{tikzcd}
\mathbb{F}_2[X]/(X^2) \arrow[r, "\mathrm{id}"] \arrow[d, "\alpha"] & \mathbb{F}_2[X]/(X^2) \arrow[d, "\beta"] \\
\mathrm{Hom}_{\mathbb{Z}[X]/(X^2)}(C,C)\otimes_{\mathbb{Z}}\mathbb{Z}/(2) \arrow[r, "F"] & \mathrm{Hom}_{\mathcal{R}}(D,D)\otimes\mathcal{R}/(2,h),
\end{tikzcd}$$

homotopy commutes. (We are suppressing the quantum grading shifts, and will continue to do so for the rest of the proof.) Since D is free over $\mathcal{R}$,

$$\mathrm{Hom}_{\mathcal{R}}(D,D)\otimes\mathcal{R}/(2,h) = \mathrm{Hom}_{\mathcal{R}/(2,h)}(D\otimes\mathcal{R}/(2,h), D\otimes\mathcal{R}/(2,h))$$

If $k\colon D\otimes\mathcal{R}/(2,h) \to D\otimes\mathcal{R}/(2,h)$ is the homotopy from $f\circ\overline{f}$ to the identity, then the homotopy making this square commute sends $a\in\mathbb{F}_2[X]/(X^2)$ to $a\cdot k \in \mathrm{Hom}_{\mathcal{R}/(2,h)}(D\otimes\mathcal{R}/(2,h), D\otimes\mathcal{R}/(2,h))$ (i.e., sends 1 to k). Finally, after inverting h, the map

$$\beta\colon h^{-1}\mathcal{R} \to h^{-1}\mathrm{Hom}_{\mathcal{R}}(D,D) = \mathrm{Hom}_{h^{-1}\mathcal{R}}(h^{-1}D, h^{-1}D)$$

sends 1 to the identity map. So, up to homotopy, the map to

$$\mathrm{Hom}_{h^{-1}\mathcal{R}}(h^{-1}\mathcal{R}, h^{-1}\mathcal{R}) \cong h^{-1}\mathcal{R}$$

induced by the homotopy equivalence $h^{-1}D \simeq h^{-1}\mathcal{R}$ is an isomorphism. (Here, by $h^{-1}\mathcal{R}$ we really mean a free module of rank 1 over $h^{-1}\mathcal{R}$.) So, the map $\beta\colon h^{-1}\mathcal{R}\to h^{-1}\mathrm{Hom}_{\mathcal{R}}(D,D)$ is a homotopy equivalence.

Similarly, there is a local map from $(C, D, f) \otimes (C, D, f)^*$ to $LEO(U)$ given by letting $\alpha'\colon \mathrm{Hom}_{\mathbb{Z}[X]/(X^2)}(C, C) \to \mathbb{Z}[X]/(X^2)$ and $\beta'\colon \mathrm{Hom}_{\mathcal{R}}(D, D) \to \mathcal{R}$ be the following trace maps. On a basic tensor $c^\vee \otimes c' \in \mathrm{Hom}(C, \mathbb{Z}[X]/(X^2)) \otimes C$, we have $\alpha(c^\vee \otimes c') = c^\vee(c')$. (When checking this is a chain map, it is important to remember that the differential on the dual complex $\mathrm{Hom}(C, \mathbb{Z}[X]/(X^2))$ is given by $d(f) = (-1)^{\mathrm{gr}_h(f)+1} f \circ \partial$, and the differential on the tensor product of course inherits the Koszul sign.) The map β' is defined similarly. It is clear that α' and β' are chain maps. The two ways around the second square give the maps $c^\vee \otimes c' \mapsto c^\vee(c')$ and $c^\vee \otimes c' \mapsto c^\vee(\overline{f}(f(c')))$; if ℓ is the homotopy from $\overline{f} \circ f$ to the identity map then the homotopy making the second square commute is $c^\vee \otimes c' \mapsto c^\vee(\ell(c))$. Again, the facts that the evaluation map $\mathrm{Hom}_{h^{-1}\mathcal{R}}(h^{-1}\mathcal{R}, h^{-1}\mathcal{R}) \to h^{-1}\mathcal{R}$ is an isomorphism and that $h^{-1}D \simeq h^{-1}\mathcal{R}$ imply that the map $h^{-1}\mathrm{Hom}_{\mathcal{R}}(D, D) \to h^{-1}\mathcal{R}$ induced by β' is a homotopy equivalence.

For the second statement, in light of Proposition 3.4, all that remains is to show that if K_0 and K_1 are knots then $LEO(K_0 \# K_1)$ is locally equivalent to $LEO(K_0) \otimes LEO(K_1)$; but this is immediate from Lemma 2.6 and Theorem 2.10. □

Remark 3.9. The conditions we have placed on a LEO triple (C, D, f) are not strong enough to rule out some pathological behavior which does not occur for $LEO(K)$ (see Example 3.16). There is other structure in Khovanov homology, such as the automorphism I and chain map T of Section 6.1, which one might be able to abstract to exclude such behavior.

3.1. **Reduced versions.** Recall that the reduced Khovanov complexes of a based link are given by

$$\begin{aligned}
\widetilde{CKh}_h(K) &= CKh_h(K) \otimes_{\mathcal{R}} \mathbb{Z}[h]\{-1\}\\
\widetilde{CKh}_o(K) &= CKh_o(K) \otimes_{\mathbb{Z}[X]/(X^2)} \mathbb{Z}\{-1\}\\
\widetilde{CKh}(K) &= CKh(K) \otimes_{\mathbb{Z}[X]/(X^2)} \mathbb{Z}\{-1\}
\end{aligned}$$

where X acts by 0 on $\mathbb{Z}[h]$ and $\mathbb{Z}$. Here is an analogue of Definitions 3.1 and 3.2 motivated by the reduced Khovanov complexes:

Definition 3.10. *A* reduced LEO triple *consists of finitely generated, free, bigraded cochain complexes C over $\mathbb{Z}$ and D over $\mathbb{Z}[h]$, and a bigraded homotopy equivalence*

$$f\colon C \otimes_{\mathbb{Z}} \mathbb{Z}/(2) \to D \otimes_{\mathbb{Z}[h]} \mathbb{Z}[h]/(2, h),$$

so that $h^{-1}D$ is homotopy equivalent to a free graded module of rank 1 over $\mathbb{Z}[h, h^{-1}]$, supported in homological grading 0 and even quantum gradings.

A local map *of such triples from (C, D, f) to (C', D', f') consists of bigrading-preserving chain maps $\alpha\colon C \to C'$ and $\beta\colon D \to D'$ so that*

- *the induced map $\beta\colon h^{-1}D \to h^{-1}D'$ is a homotopy equivalence and*
- *the following diagram commutes up to homotopy:*

$$\begin{tikzcd}
C \otimes_{\mathbb{Z}} \mathbb{Z}/(2) \arrow[r, "f"] \arrow[d, "\alpha"] & D \otimes_{\mathbb{Z}[h]} \mathbb{Z}[h]/(2, h) \arrow[d, "\beta"] \\
C' \otimes_{\mathbb{Z}} \mathbb{Z}/(2) \arrow[r, "f'"] & D' \otimes_{\mathbb{Z}[h]} \mathbb{Z}[h]/(2, h).
\end{tikzcd} \tag{3.11}$$

We call (C, D, f) and (C', D', f') are locally equivalent *if there are local maps $(C, D, f) \to (C', D', f')$ and $(C', D', f') \to (C, D, f)$.*

This definition has many of the same formal properties as the unreduced case:

Proposition 3.12. *Reduced LEO triples satisfy the following:*

(1) *Local equivalence is an equivalence relation on reduced LEO triples.*
(2) *If we define the tensor product of reduced LEO triples (C, D, f) and (C', D', f') by $(C \otimes_{\mathbb{Z}} C', D \otimes_{\mathbb{Z}[h]} D', f \otimes f')$ and the inverse of (C, D, f) by*
$$(\mathrm{Hom}(C, \mathbb{Z}), \mathrm{Hom}(D, \mathbb{Z}[h]), \overline{f})$$
(where $\overline{f}$ is induced by the homotopy inverse to f), then the set of local equivalence classes of reduced LEO triples is an abelian group $\widetilde{\mathcal{C}}_{LEO}$.
(3) *There is a homomorphism $\pi\colon \mathcal{C}_{LEO} \to \widetilde{\mathcal{C}}_{LEO}$ given by*
$$(C, D, f) \mapsto \big(C \otimes_{\mathbb{Z}[X]/(X^2)} \mathbb{Z}\{-1\},\, D \otimes_{\mathcal{R}} \mathbb{Z}[h]\{-1\},\, \widetilde{f}\big),$$
where $\widetilde{f}$ is the map induced by f.
(4) *The map $\mathcal{C} \to \widetilde{\mathcal{C}}_{LEO}$ which sends K to $(\widetilde{CKh}_o(K), \widetilde{CKh}_h(K), \mathrm{id})$ is a well-defined group homomorphism.*

Proof. Point (1) is immediate from the definition. The proof of Point (2) is similar to the proof of the first part of Theorem 3.8, replacing $\mathbb{Z}[X]/(X^2)$ by $\mathbb{Z}$ and $\mathcal{R}$ by $\mathbb{Z}[h]$ throughout, and is left to the reader. Point (3) is immediate from the definitions. Point (4) follows from Proposition 3.4 and Point (3). □

We give one further specialization of LEO triples, which is perhaps the simplest formulation incorporating both the Bar-Natan deformation and a version of the odd Khovanov complex:

Definition 3.13. *A* two-reduced local even-odd triple *consists of finitely generated, free, bigraded cochain complexes C over $\mathbb{Z}$ and D over $\mathbb{F}_2[h]$, and a bigraded homotopy equivalence*
$$f\colon C \otimes_{\mathbb{Z}} \mathbb{Z}/(2) \to D \otimes_{\mathbb{F}_2[h]} \mathbb{F}_2[h]/(h)$$
so that $h^{-1}D$ is homotopy equivalent to a free graded module of rank 1 over $\mathbb{F}_2[h, h^{-1}]$, supported in homological grading 0 and even quantum gradings.

Local maps *and* local equivalence *of two-reduced local even-odd triples are defined as in Definition 3.10, replacing $\mathbb{Z}[h]$ by $\mathbb{F}_2[h]$ throughout.*

Proposition 3.14. *The local equivalence classes of two-reduced local even-odd triples form an abelian group $\widetilde{\mathcal{C}}^o_{LEO}$, and there is a homomorphism from the concordance group to $\widetilde{\mathcal{C}}^o_{LEO}$.*

The proof is the same that of Proposition 3.12, and is left to the reader.

3.2. **The s-invariant.** Here is a key property of LEO triples, which will allow us to define the s-invariant:

Lemma 3.15. *Let (C, D, f) be a LEO triple and let $\mathbb{F}$ be a field. Then the homology of $D \otimes_{\mathbb{Z}} \mathbb{F}$, viewed as a module over $\mathbb{F}[h]$, decomposes as $\mathbb{F}[h] \oplus \mathbb{F}[h] \oplus T$ where T is a torsion $\mathbb{F}[h]$-module. Similarly, if $(\widetilde{C}, \widetilde{D}, f)$ is a reduced LEO triple, then the homology of $\widetilde{D} \otimes_{\mathbb{Z}} \mathbb{F}$ decomposes as $\mathbb{F}[h] \oplus T$ where T is a torsion $\mathbb{F}[h]$-module.*

Proof. We prove the unreduced case; the reduced case is similar. By the classification of modules over a PID, the homology of $D \otimes_{\mathbb{Z}} \mathbb{F}$ is isomorphic to $\mathbb{F}[h]^m \oplus T$ for some m and some torsion module T. Since localization is exact, it follows that the homology of $D \otimes_{\mathbb{Z}[h]} \mathbb{F}[h, h^{-1}]$ is isomorphic to $\mathbb{F}[h, h^{-1}]^m$. On the other hand, by hypothesis, $D \otimes_{\mathbb{Z}[h]} \mathbb{Z}[h, h^{-1}] \simeq h^{-1}\mathcal{R} \cong \mathbb{Z}[h, h^{-1}]^2$. So, $D \otimes_{\mathbb{Z}[h]} \mathbb{F}[h, h^{-1}] \simeq \mathbb{F}[h, h^{-1}]^2$, and $m = 2$, as claimed. □

For the complexes coming from a knot, the generators of the two free $\mathbb{F}[h]$ summands of $Kh_h(K;\mathbb{F})$ lie in quantum gradings which are 2 apart, and the s-invariant is defined to be the average of these two gradings. (See, for instance, Schütz's paper [Sch, Section 2].) The following example shows that this is not true in general for LEO triples.

Example 3.16. Let k be a non-negative integer, and D the free $\mathcal{R}$-cochain complex concentrated in homological degrees -1 and 0 given by $D^{-1} = \mathcal{R}\{-1\}$, $D^0 = \mathcal{R}\{1\} \oplus \mathcal{R}\{-1+2k\}$ and $\partial\colon D^{-1} \to D^0$ given by $\partial(1) = (X - h, h^k)$. (Recall that the quantum grading shift $\{-1+2k\}$, for instance, means the generator $1 \in \mathcal{R}\{-1+2k\}$ satisfies $\mathrm{gr}_q(1) = -1+2k$.)

Since h^k is an isomorphism over $h^{-1}\mathcal{R}$, we can use Gaussian elimination [BN07, Lemma 3.2] to see that $h^{-1}D$ is homotopy equivalent to a free graded module of rank 1 supported in homological degree 0, namely $h^{-1}\mathcal{R}\{1\}$. If we restrict coefficients to $\mathbb{Z}[h]$, the complex D is given by

$$\begin{tikzcd}
 & \mathbb{Z}[h]\{1\} \\
\mathbb{Z}[h]\{-1\} \arrow[ur, "-h"] \arrow[r, "1"] \arrow[dr, "h^k"'] & \mathbb{Z}[h]\{-1\} \\
\mathbb{Z}[h]\{-3\} \arrow[dr, "h^k"] & \mathbb{Z}[h]\{-1+2k\} \\
 & \mathbb{Z}[h]\{-3+2k\}
\end{tikzcd}$$

Another Gaussian elimination shows this is homotopy equivalent to E given by $E^{-1} = \mathbb{Z}[h]\{-3\}$, $E^0 = \mathbb{Z}[h]\{1\} \oplus \mathbb{Z}[h]\{-1+2k\} \oplus \mathbb{Z}[h]\{-3+2k\}$ and $\partial_E^{-1}(1) = (0,0,h^k)$. Given a field $\mathbb{F}$ and the ring homomorphism $\mathbb{Z}[h] \to \mathbb{F}$ sending h to 1, we get $E \otimes_{\mathbb{Z}[h]} \mathbb{F}$ is homotopy equivalent to a complex with two generators, and their quantum gradings are 1 and $-1+2k$. In particular, the difference in quantum grading of the two generators can be arbitrarily large in general.

Definition 3.17. *Given a bigraded chain complex D, let $D^{(q)}$ be the subspace of D in quantum grading q. Since we are often working with graded rings, such as $\mathbb{Z}[X]/(X^2)$ or $\mathbb{Z}[h]$, $D^{(q)}$ is typically not a submodule of D. We will denote the homology of D in quantum grading q by*

$$H^{*,q}(D) = H(D^{(q)}).$$

Finally, for brevity, we will write $H^{i,j}(D;\mathbb{F})$ to denote $H^{i,j}(D \otimes_{\mathbb{Z}} \mathbb{F})$.

Definition 3.18. *Given a LEO triple (C, D, f) and a field $\mathbb{F}$, let*

$$s_{\mathbb{F}}^{+}(C,D,f) = \max\{q \mid H^{0,q}(D;\mathbb{F}) \to H^{0,q}(h^{-1}D;\mathbb{F}) \text{ non-zero}\} - 1$$
$$s_{\mathbb{F}}^{-}(C,D,f) = \max\{q \mid H^{0,q}(D;\mathbb{F}) \to H^{0,q}(h^{-1}D;\mathbb{F}) \text{ surjective}\} + 1.$$

Given a reduced LEO triple $(\widetilde{C}, \widetilde{D}, f)$, let

$$\begin{aligned} s_{\mathbb{F}}(\widetilde{C},\widetilde{D},f) &= \max\{q \mid H^{0,q}(\widetilde{D};\mathbb{F}) \to H^{0,q}(h^{-1}\widetilde{D};\mathbb{F}) \text{ non-zero}\} \\ &= \max\{q \mid H^{0,q}(\widetilde{D};\mathbb{F}) \to H^{0,q}(h^{-1}\widetilde{D};\mathbb{F}) \text{ surjective}\}. \end{aligned}$$

Given a LEO triple (C, D, f), let $s_{\mathbb{F}}(C,D,f)$ be $s_{\mathbb{F}}$ of the image of (C,D,f) in $\widetilde{\mathcal{C}}_{LEO}$.

Note that Definition 3.18 does not use C or f, only D.

Lemma 3.19. *The numbers $s^{\pm}_{\mathbb{F}}(C,D,f)$ and $s_{\mathbb{F}}(C,D,f)$ depend only on $\mathbb{F}$ and the local equivalence class of (C,D,f). Similarly, $s_{\mathbb{F}}(\widetilde{C},\widetilde{D},f)$ depends only on $\mathbb{F}$ and the local equivalence class of $(\widetilde{C},\widetilde{D},f)$.*

Proof. This is immediate from the definitions. □

If $(C,D,f) = LEO(K)$ then for each field $\mathbb{F}$ the elements $s^{\pm}_{\mathbb{F}}(C,D,f)$ and $s(C,D,f)$ are all equal (e.g., [LS14, Section 2.2], [Sch, Section 2]), and agree with the Rasmussen invariant over $\mathbb{F}$. In contrast:

Lemma 3.20. *The maps $s^+_{\mathbb{F}}$ and $s^-_{\mathbb{F}}$ are not group homomorphisms, but the map $s_{\mathbb{F}}$ is.*

Proof. To see that $s^{\pm}_{\mathbb{F}}$ are not group homomorphisms, consider the complex D from Example 3.16, and extend D arbitrarily to a LEO triple. Direct computation gives $s^+(D) = 2k-2$ and $s^-(D) = 2$. On the other hand, if we let $D^* = \mathrm{Hom}(D,\mathcal{R})\{2\}$ be the dual complex then direct computation again shows that $s^+(D^*) = -2$ and $s^-(D^*) = 2-2k$. By Part (8) of Theorem 3.8, $D\otimes D^*\{-1\}$ is locally equivalent to $LEO(U)$, so $s^{\pm}(D\otimes D^*\{-1\}) = s^{\pm}(U) = 0$. For any $k \neq 2$, this shows s^+ and s^- are not group homomorphisms.

To see that s is a group homomorphism, consider reduced LEO triples $(\widetilde{C}_i,\widetilde{D}_i,f_i)$, $i=1,2$. Abusing notation, let $\widetilde{D}_i$ denote the tensor product of $\widetilde{D}_i$ with $\mathbb{F}$. By the Künneth theorem, the free part of $\widetilde{D}_1 \otimes_{\mathbb{F}[h]} \widetilde{D}_2$ is the tensor product of the free parts of $\widetilde{D}_1$ and $\widetilde{D}_2$. The s-invariant is the quantum grading of a generator of this free part, so the result follows. □

Corollary 3.21. *The image of the smooth knot concordance group $\mathcal{C}$ in $\mathcal{C}_{LEO}$ from Theorem 3.8 is a proper subgroup of $\mathcal{C}_{LEO}$.*

Proposition 3.22. *The tuple $\vec{s} = (s_{\mathbb{Q}}, s_{\mathbb{F}_2}-s_{\mathbb{Q}}, s_{\mathbb{F}_3}-s_{\mathbb{Q}}, s_{\mathbb{F}_5}-s_{\mathbb{Q}},\dots)$ defines a surjective homomorphism $\widetilde{\mathcal{C}}_{LEO} \to (2\mathbb{Z})^\infty = \bigoplus_{n\in\mathbb{N}} 2\mathbb{Z}$.*

Proof. Let $(\widetilde{C},\widetilde{D},f)$ be a reduced LEO-triple. To see that $\vec{s}$ is well-defined, we need to show that for all but finitely many primes p we get $s_{\mathbb{F}_p}(\widetilde{C},\widetilde{D},f) = s_{\mathbb{Q}}(\widetilde{C},\widetilde{D},f)$.

Set $s = s_{\mathbb{Q}}(\widetilde{C},\widetilde{D},f)$ and denote by $i\colon \mathbb{Q}[h]\{s\} \to H^0(\widetilde{D})\otimes\mathbb{Q}$, $r\colon H^0(\widetilde{D})\otimes\mathbb{Q} \to \mathbb{Q}[h]\{s\}$ the grading-preserving maps with $r\circ i = \mathrm{id}_{\mathbb{Q}[h]}$, which exist by Lemma 3.15. Then

$$i(1) = \sum_{m=1}^{k} a_m \otimes q_m,$$

with $a_m \in H^0(\widetilde{D})$ and $q_m \in \mathbb{Q}$. There exists a localization $P^{-1}\mathbb{Z} \subset \mathbb{Q}$ of $\mathbb{Z}$ obtained by inverting finitely many primes P such that each $q_m \in P^{-1}\mathbb{Z}$, $m = 1,\dots,k$. Also, $\mathbb{Z}[h]$ is a Noetherian ring, so $H^0(\widetilde{D})$ is finitely generated. After possibly adding finitely many primes to P, we can assume that $r\big(H^0(\widetilde{D})\big) \subset P^{-1}\mathbb{Z}[h]\{s\}$. From the commutative diagram

$$\begin{tikzcd}
P^{-1}\mathbb{Z}[h]\{s\} \arrow[r, "i|"] \arrow[dr, equal] & H^0(\widetilde{D})\otimes P^{-1}\mathbb{Z} \arrow[r] \arrow[d, "r|"] & H^0(\widetilde{D})\otimes\mathbb{Q} \arrow[d, "r"] \\
 & P^{-1}\mathbb{Z}[h]\{s\} \arrow[r, hook] & \mathbb{Q}[h]\{s\}
\end{tikzcd}$$

it follows that $P^{-1}\mathbb{Z}[h]\{s\}$ is a direct summand in $H^0(\widetilde{D})\otimes P^{-1}\mathbb{Z}$. (Here, $i|$ and $r|$ denote the restrictions of i and r.) If p is a prime not in P, there is a surjective ring homomorphism

$P^{-1}\mathbb{Z} \to \mathbb{F}_p$, and therefore $\mathbb{F}_p[h]\{s\}$ is a direct summand in $H^0(\widetilde{D} \otimes \mathbb{F}_p)$. It follows that $s_{\mathbb{F}_p}(\widetilde{C}, \widetilde{D}, f) = s$ for all but finitely many primes.

To see that $\vec{s}$ is surjective, we will now show that 2 times each standard basis vector is in the image. The vector $(2, 0, 0, \dots)$ is the image of $(\widetilde{C}, \widetilde{D}, f)$ where $\widetilde{D} = \mathbb{Z}[h]\{2\}$ in homological grading 0 and $\widetilde{C}$ and f are any choice of extension to a reduced LEO triple (e.g., $\widetilde{C} = \mathbb{Z}\{2\}$ and f the obvious identification).

For the other basis vectors, let p be a prime. Consider the complex $\widetilde{D}$ with $\widetilde{D}^0 = \mathbb{Z}[h] \oplus \mathbb{Z}[h]\{2\}$ and $\widetilde{D}^1 = \mathbb{Z}[h]\{2\}$, and $\partial\colon \widetilde{D}^0 \to \widetilde{D}^1$ given by $\partial(x, y) = hx + py$. Extend $\widetilde{D}$ to a reduced LEO triple $(\widetilde{C}, \widetilde{D}, f)$ arbitrarily. Then

$$s_{\mathbb{F}}(\widetilde{C}, \widetilde{D}, f) = \begin{cases} 2 & \mathrm{char}(\mathbb{F}) = p \\ 0 & \text{otherwise,} \end{cases}$$

as desired. □

In particular, Proposition 3.22 implies that $\widetilde{\mathcal{C}}_{LEO}$ has a $\mathbb{Z}^\infty$ summand, and since $\mathcal{C}_{LEO} \twoheadrightarrow \widetilde{\mathcal{C}}_{LEO}$, the group $\mathcal{C}_{LEO}$ does as well. (We have not shown these complexes are induced by knots, so this does not re-prove the existence of a $\mathbb{Z}^\infty$ summand of $\mathcal{C}$, but it is conjectured to be so [Sch, Conjecture 1.3], [LZ22, Question 6.5].) The definition of $s_{\mathbb{F}_2}(C, D, f)$ extends to two-reduced LEO triples, giving a $\mathbb{Z}$-summand of $\widetilde{\mathcal{C}}^o_{LEO}$. Nontriviality of the invariants introduced in Section 4 implies that the map $\vec{s}\colon \widetilde{\mathcal{C}}_{LEO} \to (2\mathbb{Z})^\infty$ is not an isomorphism, nor is $s_{\mathbb{F}_2}\colon \widetilde{\mathcal{C}}^o_{LEO} \to \mathbb{Z}$ (cf. Remark 6.11).

Question 3.23. Is it possible to completely determine $\widetilde{\mathcal{C}}^o_{LEO}$? What about $\widetilde{\mathcal{C}}_{LEO}$? Also, what is the image of the smooth concordance group $\mathcal{C}$ in either $\widetilde{\mathcal{C}}^o_{LEO}$ or $\widetilde{\mathcal{C}}_{LEO}$?

3.3. **The graded integral s-invariant.** There is an integral version of the s-invariant, which is not a homomorphism but which gives a lower bound on $s_{\mathbb{F}}$ for any field $\mathbb{F}$. As we do not use these invariants until Section 6, the reader may wish to initially skip ahead to Section 4. Specifically, given a *LEO*-triple (C, D, f), let

$$\begin{aligned} s^+_{\mathbb{Z}}(C, D, f) &= \max\{q \mid \exists\, a \in H^{0,q}(D; \mathbb{Z}) \text{ with } i(a) \in H^{0,q}(h^{-1}D; \mathbb{Z}) \text{ primitive}\} - 1, \\ s^-_{\mathbb{Z}}(C, D, f) &= \max\{q \mid H^{0,q}(D; \mathbb{Z}) \to H^{0,q}(h^{-1}D; \mathbb{Z}) \text{ surjective}\} + 1. \end{aligned}$$

For a reduced *LEO*-triple $(\widetilde{C}, \widetilde{D}, f)$, we also set

$$s_{\mathbb{Z}}(\widetilde{C}, \widetilde{D}, f) = \max\{q \mid H^{0,q}(\widetilde{D}; \mathbb{Z}) \to H^{0,q}(h^{-1}\widetilde{D}; \mathbb{Z}) \text{ surjective}\}. \tag{3.24}$$

For a *LEO*-triple (C, D, f), we set $s_{\mathbb{Z}}(C, D, f)$ to be $s_{\mathbb{Z}}$ of its image in $\widetilde{\mathcal{C}}_{LEO}$. If K is a knot, we write $s_{\mathbb{Z}}(K)$ for $s_{\mathbb{Z}}(LEO(K))$, and similarly for the $\pm$ decorations. We will show in Lemma 6.21 that the three numbers $s^+_{\mathbb{Z}}(K)$, $s^-_{\mathbb{Z}}(K)$, and $s_{\mathbb{Z}}(K)$ agree.

To put these constructions in context, recall that Schütz [Sch] defined an s-invariant over $\mathbb{Z}$ as a finite sequence of numbers which encodes the E_∞-term of the reduced Bar-Natan–Lee–Turner spectral sequence of a knot K. The definition can be extended to reduced *LEO* triples (C, D, f) as follows. Consider the commutative diagram

$$\begin{tikzcd} H^{0,q+2}(D) \arrow[r] \arrow[d, "h"] & H^{0,q+2}(h^{-1}D) \arrow[r, "\cong"] \arrow[d, "h", "\cong"'] & \mathbb{Z} \arrow[d, "\mathrm{id}"] \\ H^{0,q}(D) \arrow[r] & H^{0,q}(h^{-1}D) \arrow[r, "\cong"] & \mathbb{Z} \end{tikzcd}$$

and define

$$H^0(h^{-1}D)_q = \mathrm{im}\left(H^{0,q}(D) \to H^{0,q}(h^{-1}D)\right)$$
$$H^0(h^{-1}D)^{(q)} = H^0\left(h^{-1}D\right)_q / h(H^0(h^{-1}D)_{q+2}\big).$$

Since $H^{0,q}(h^{-1}D) \cong \mathbb{Z}$, both $H^0(h^{-1}D)_q$ and $H^0(h^{-1}D)^{(q)}$ are cyclic groups. Furthermore, for large q, $H^0(h^{-1}D)_q = 0$, and for small q, $H^0(h^{-1}D)_q = H^{0,q}(h^{-1}D)$. In fact,

$$s_{\mathbb{Q}}(C, D, f) = \max\{q \in 2\mathbb{Z} \mid H^0(h^{-1}D)^{(q)} \neq 0\}$$
$$s_{\mathbb{Z}}(C, D, f) = \min\{q \in 2\mathbb{Z} \mid H^0(h^{-1}D)^{(q)} \neq 0\},$$

as can be seen straight from the definitions (Definition 3.18 and Formula (3.24)). We now define the *graded length of* (C, D, f) as

$$\mathrm{gl}(C, D, f) = \frac{s_{\mathbb{Q}}(C, D, f) - s_{\mathbb{Z}}(C, D, f)}{2}.$$

Lemma 3.25. *Let (C, D, f) be a reduced LEO triple, and q an even integer. The cyclic groups $H^0(h^{-1}D)_q$ and $H^0(h^{-1}D)^{(q)}$ only depend on the local equivalence class of (C, D, f).*

Proof. Let (α, β) be a local map from (C, D, f) to (C', D', f'). From the commutative diagram

$$\begin{tikzcd} H^{0,q}(D) \arrow[r, "\beta"] \arrow[d] & H^{0,q}(D') \arrow[d] \\ H^{0,q}(h^{-1}D) \arrow[r, "\beta", "\cong"'] & H^{0,q}(h^{-1}D') \end{tikzcd}$$

we get $\beta(H^0(h^{-1}D)_q) \subset H^0(h^{-1}D)_q$. Similarly, a local map (α', β') from (C', D', f') to (C, D, f) implies $\beta'(H^0(h^{-1}D')_q) \subset H^0(h^{-1}D)_q$. Since β and β' induce isomorphisms of $\mathbb{Z} = H^{0,q}(h^{-1}D)$ we get that $H^0(h^{-1}D)_q$ and $H^0(h^{-1}D')_q$ are isomorphic. As β commutes with h, we also get $H^0(h^{-1}D)^{(q)} \cong H^0(h^{-1}D')^{(q)}$. $\square$

Corollary 3.26. *Let (C, D, f) be a reduced LEO triple. Then $s_{\mathbb{Z}}(C, D, f)$ and $\mathrm{gl}(C, D, f)$ only depend on the local equivalence class of (C, D, f).*

The cyclic groups $H^0(h^{-1}D)^{(q)}$ can also be expressed in terms of the complex $D_{h=1}$. To see this, note that there is a filtration

$$\cdots \subset \widetilde{\mathcal{F}}_q \subset \widetilde{\mathcal{F}}_{q-2} \subset \cdots \subset D_{h=1}$$

given by

$$\widetilde{\mathcal{F}}_q = p(D^{(q)}) \subset D_{h=1},$$

where $p\colon D \to D_{h=1}$ is projection and $D^{(q)}$ is as in Definition 3.17. Since p restricted to $D^{(q)}$ is injective, we get $H^{0,q}(D) \cong H^0(\widetilde{\mathcal{F}}_q)$. Also, p extends to $p\colon h^{-1}D \to D_{h=1}$, and restricting this to $h^{-1}D^{(q)}$ induces an isomorphism $h^{-1}D^{(q)} \to D_{h=1}$ for all even q. In particular, for a knot K, the integral s-invariant $s^{\mathbb{Z}}(K)$ of Schütz [Sch] can be recovered using $D = \widetilde{CKh}_h(K)$.

Definition 3.27. *Let (C, D, f) be a reduced LEO triple. If $s_{\mathbb{Q}}(C, D, f) = s_{\mathbb{Z}}(C, D, f)$, define $s^{\mathbb{Z}}(C, D, f) = s_{\mathbb{Q}}(C, D, f)$. Otherwise, for $q = s_{\mathbb{Q}}(C, D, f) - 2, \dots, s_{\mathbb{Z}}(C, D, f)$, denote the cardinality of the finite cyclic group $H^0(h^{-1}D)^{(q)}$ by*

$$c(q) = \# H^0(h^{-1}D)^{(q)},$$

and define $s^{\mathbb{Z}}(C, D, f) \in \mathbb{Z}^{1+\mathrm{gl}(C,D,f)}$ as

$$s^{\mathbb{Z}}(C, D, f) = \Big(s_{\mathbb{Q}}(C, D, f), c\big(s_{\mathbb{Q}}(C, D, f) - 2\big), \dots, c\big(s_{\mathbb{Q}}(C, D, f) - 2\mathrm{gl}(C, D, f)\big)\Big).$$

These definitions can be carried over to unreduced *LEO* triples, although we do not necessarily get that the $H^0(h^{-1}D)^{(q)}$ are finite cyclic. Also, at least for knots, this does not give better information, as we will show in Remark 6.22.

4. Invariants of LEO triples

In this section, we introduce some algebraic invariants of local equivalence classes, in the spirit of the refinements of the s-invariant using the Khovanov stable homotopy type [LS14]. Unlike the invariants in the previous section, the ones defined here depend on both C and D, not just D. We give two classes of refinements, name the reduced variants of both, and then discuss some examples.

4.1. Bockstein-refined s-invariants.

For n a positive integer, we have the short exact sequence

$$0 \longrightarrow \mathbb{Z}/(2) \longrightarrow \mathbb{Z}/(2^{n+1}) \longrightarrow \mathbb{Z}/(2^n) \longrightarrow 0.$$

For a LEO triple (C, D, f), this gives rise to the Bockstein homomorphism

$$\beta_n \colon H^k(C; \mathbb{Z}/(2^n)) \to H^{k+1}(C; \mathbb{F}_2).$$

Now consider the configurations

$$\begin{tikzcd}
\langle \check{a} \rangle \arrow[r] \arrow[d, hook] & \langle \hat{a} \rangle \arrow[d, hook] & \langle a \rangle \arrow[l] \arrow[r] \arrow[d, hook] & \langle \overline{a} \rangle \arrow[d, hook] \\
H^{-1,q}(C; \mathbb{Z}/(2^n)) \arrow[r, "f\circ\beta_n"] & H^{0,q}(D_{h=0}; \mathbb{F}_2) & H^{0,q}(D; \mathbb{F}_2) \arrow[l, "p"'] \arrow[r, "i"] & H^{0,q}(h^{-1}D; \mathbb{F}_2) \\
\langle \check{a}, \check{b} \rangle \arrow[r] \arrow[u, hook] & \langle \hat{a}, \hat{b} \rangle \arrow[u, hook] & \langle a, b \rangle \arrow[l] \arrow[r] \arrow[u, hook] & \langle \overline{a}, \overline{b} \rangle \arrow[u, equal]
\end{tikzcd} \tag{4.1}$$

where $\langle \cdot \rangle$ refers to the $\mathbb{F}_2$-vector space generated by the elements.

Definition 4.2. *Given a LEO triple (C, D, f) and a positive integer n, we say an integer q is* β_n-half-full *if there exist $\check{a} \in H^{-1,q}(C; \mathbb{Z}/(2^n))$ and $a \in H^{0,q}(D; \mathbb{F}_2)$ such that $f \circ \beta_n(\check{a}) = p(a)$ and $i(a) \neq 0$.*

We say q is β_n-full *if there exist $\check{a}, \check{b} \in H^{-1,q}(C; \mathbb{Z}/(2^n))$ and $a, b \in H^{0,q}(D; \mathbb{F}_2)$ such that $f \circ \beta_n(\check{a}) = p(a)$, $f \circ \beta_n(\check{b}) = p(b)$, and $i(a), i(b)$ span $H^{0,q}(h^{-1}D; \mathbb{F}_2)$ as an $\mathbb{F}_2$-vector space.*

So, q being β_n-half-full means we have a commutative diagram as in the upper half of (4.1) and q being β_n-full means we have a commutative diagram as in the lower half of (4.1). We allow $\check{a}$ and $\check{b}$ to be 0 in this definition. In particular, if q is β_n-half-full, then $q - 2$ is half-full as well, using $ha \in H^{0,q-2}(D; \mathbb{F}_2)$ and $0 \in H^{-1,q}(C; \mathbb{Z}/(2^n))$.

Definition 4.3. *Given a LEO triple (C, D, f) and a positive integer n, let*

$$r^{\beta_n}(C, D, f) = \max\{q \in \mathbb{Z} \mid q \text{ is } \beta_n\text{-half-full}\} + 1,$$

and

$$s^{\beta_n}(C, D, f) = \max\{q \in \mathbb{Z} \mid q \text{ is } \beta_n\text{-full}\} + 3.$$

If $q > s^+(C, D, f) + 1$, then q cannot be β_n-half-full. Also, if $q < s^+(C, D, f)$ we can find $a \in H^{0,q+2}(D; \mathbb{F}_2)$ with $i(a) \in H^0(h^{-1}D; \mathbb{F}_2)$ non-zero. Then $ha \in H^{0,q}(D; \mathbb{F}_2)$ also satisfies $i(ha) \neq 0$ but $p(ha) = 0$ so q is β_n-half-full. In particular,

$$0 \leq r^{\beta_n}(C, D, f) - s^+_{\mathbb{F}_2}(C, D, f) \leq 2. \tag{4.4}$$

A similar argument shows that

$$0 \leq s^{\beta_n}(C, D, f) - s^-_{\mathbb{F}_2}(C, D, f) \leq 2. \tag{4.5}$$

For $k \leq n$, we have the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z}/(2) \arrow[r] \arrow[d, equal] & \mathbb{Z}/(2^{k+1}) \arrow[r] \arrow[d, "\cdot 2^{n-k}"] & \mathbb{Z}/(2^k) \arrow[r] \arrow[d, "\cdot 2^{n-k}"] & 0 \\
0 \arrow[r] & \mathbb{Z}/(2) \arrow[r] & \mathbb{Z}/(2^{n+1}) \arrow[r] & \mathbb{Z}/(2^n) \arrow[r] & 0
\end{tikzcd}$$

Given a *LEO*-triple, naturality implies that if q is β_k-(half-)full, then q is β_n-(half-)full so $r^{\beta_n} \geq r^{\beta_k}$ and $s^{\beta_n} \geq s^{\beta_k}$. Passing to the direct limit gives a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z}/(2) \arrow[r] \arrow[d, equal] & \mathbb{Z}/(2^{k+1}) \arrow[r] \arrow[d, "\cdot 2^{-k-1}"] & \mathbb{Z}/(2^k) \arrow[r] \arrow[d, "\cdot 2^{-k}"] & 0 \\
0 \arrow[r] & \mathbb{Z}/(2) \arrow[r] & \mathbb{Z}[\frac{1}{2}]/\mathbb{Z} \arrow[r] & \mathbb{Z}[\frac{1}{2}]/\mathbb{Z} \arrow[r] & 0
\end{tikzcd}$$

and a Bockstein homomorphism $\beta_\infty \colon H^k(C; \mathbb{Z}[\frac{1}{2}]/\mathbb{Z}) \to H^{k+1}(C; \mathbb{F}_2)$ giving rise to invariants $r^{\beta_\infty}(C, D, f)$ and $s^{\beta_\infty}(C, D, f)$ with

$$r^{\beta_\infty}(C, D, f) = \lim_{n\to\infty} r^{\beta_n}(C, D, f) \quad \text{and} \quad s^{\beta_\infty}(C, D, f) = \lim_{n\to\infty} s^{\beta_n}(C, D, f).$$

See Examples 4.12–4.14 for some examples of these invariants, as well as of the ones constructed later in this section.

In the case $n = 1$, we can use the isomorphism $f \colon H^{*,q}(C; \mathbb{F}_2) \to H^{*,q}(D_{h=0}; \mathbb{F}_2)$ to produce another operation $\beta \colon H^{-1,q}(C; \mathbb{F}_2) \to H^{0,q}(C; \mathbb{F}_2)$ given by

$$\beta = \beta_1 + f^{-1} \circ \beta_1 \circ f. \tag{4.6}$$

The notions of β-half-full and β-full carry over verbatim, and we can define $r^\beta(C, D, f)$ and $s^\beta(C, D, f)$ as before.

Remark 4.7. For a knot K, the map β from Equation (4.6) corresponds to the sum of the even and odd first Steenrod squares $\mathrm{Sq}^1 + \mathrm{Sq}^1_o \colon Kh^{-1,q}(K; \mathbb{F}_2) \to Kh^{0,q}(K; \mathbb{F}_2)$. (See also Lemma 6.8.) The sum of these Steenrod squares can also be realized via a connecting homomorphism involving unified Khovanov homology, as described by Putyra and Shumakovitch [PS16]. In Section 7.2, we use the associated refined s-invariant, s^β, to give an alternate proof that the five key knots from Manolescu-Piccirillo's paper [MP] are not slice.

4.2. **The comprehensive refinements.** Our second class of refinements use the change of coefficients map $j\colon H^{i,j}(C)\to H^{i,j}(C;\mathbb{F}_2)$ instead of the a Bockstein homomorphism, and are based on the diagrams

$$H^{0,q}(C)\xrightarrow{f\circ j}H^{0,q}(D_{h=0};\mathbb{F}_2)\xleftarrow{p}H^{0,q}(D;\mathbb{F}_2)\xrightarrow{i}H^{0,q}(h^{-1}D;\mathbb{F}_2)$$

and

$$H^{0,q}(C)\xrightarrow{f\circ j}H^{0,q}(D_{h=0};\mathbb{F}_2)\xleftarrow{p}H^{0,q}(D)\xrightarrow{i}H^{0,q}(h^{-1}D).$$

Definition 4.8. *Given a LEO triple (C,D,f), we say an integer q is* oddly half-full *if there exist $\check{a}\in H^{0,q}(C)$ and $a\in H^{0,q}(D;\mathbb{F}_2)$ such that $f\circ j(\check{a})=p(a)$ and $i(a)\neq 0$. We say q is* oddly full *if in addition there exist $\check{b}\in H^{0,q}(C)$ and $b\in H^{0,q}(D;\mathbb{F}_2)$ such that $f\circ j(\check{b})=p(b)$ and $i(a)$, $i(b)$ span $H^{0,q}(h^{-1}D;\mathbb{F}_2)$ as an $\mathbb{F}_2$-vector space.*

Similarly, q is completely half-full *if there exist $\check{a}\in H^{0,q}(C)$ and $a\in H^{0,q}(D)$ such that $f\circ j(\check{a})=p(a)$ and $i(a)$ is a nonzero primitive element of $H^{0,q}(h^{-1}D)$. Moreover, q is* completely full *if in addition there exist and $\check{b}\in H^{0,q}(C)$ $b\in H^{0,q}(D)$ such that $f\circ j(\check{b})=p(b)$ and $i(a)$, $i(b)$ generate $H^{0,q}(h^{-1}D)$ as a $\mathbb{Z}$-module.*

For a LEO-triple (C,D,f) define

$$\begin{aligned}
r_o(C,D,f)&=\max\{q\in\mathbb{Z}\mid q \textit{ is oddly half-full}\}-1,\\
s_o(C,D,f)&=\max\{q\in\mathbb{Z}\mid q \textit{ is oddly full}\}+1,\\
r_c(C,D,f)&=\max\{q\in\mathbb{Z}\mid q \textit{ is completely half-full}\}-1,\\
s_c(C,D,f)&=\max\{q\in\mathbb{Z}\mid q \textit{ is completely full}\}+1.
\end{aligned}$$

The shifts here are different from Definition 4.3, and are chosen so that they all vanish for the trivial element $LEO(U)$; see Example 4.12. The invariants r_o and s_o satisfy

$$0\le s^{+}_{\mathbb{F}_2}(C,D,f)-r_o(C,D,f)\le 2 \tag{4.9}$$

$$0\le s^{-}_{\mathbb{F}_2}(C,D,f)-s_o(C,D,f)\le 2. \tag{4.10}$$

The situation for the complete invariants is more complicated; see Theorem 6.3 in the case of knots.

4.3. **The reduced case.** All of the local equivalence invariants defined here have reduced analogues

$$\tilde{s}^{\alpha}(\widetilde{C},\widetilde{D},f)=\max\{q\in 2\mathbb{Z}\mid q\text{ is }\alpha\text{-reduced-full}\}+2$$

where α is any of the β_n or β, and α-reduced-full is defined as α-half-full, but with $\widetilde{C},\widetilde{D}$ in place of C,D. Also,

$$\tilde{s}_o(\widetilde{C},\widetilde{D},f)=\max\{q\in 2\mathbb{Z}\mid q\text{ is oddly reduced-full}\},$$

which satisfies $0\le s_{\mathbb{F}_2}(\widetilde{C},\widetilde{D},f)-\tilde{s}_o(\widetilde{C},\widetilde{D},f)\le 2$. Define $\tilde{s}_c$ similarly, where to be completely reduced-full we require that $i(a)$ be a generator for $H^{0,q}(h^{-1}\widetilde{D})$ as a $\mathbb{Z}$-module. Given an unreduced LEO triple (C,D,f), we will write $\tilde{s}^{\alpha}(C,D,f)$ or $\tilde{s}_{\gamma}(C,D,f)$ to denote the invariant of its image $\pi(C,D,f)$ in $\widetilde{\mathcal{C}}_{LEO}$.

We will sometimes refer to the invariants r^{α}, s^{α}, and $\tilde{s}^{\alpha}$ as the *Bockstein refinements* and r_{γ}, s_{γ}, and $\tilde{s}_{\gamma}$ as the *comprehensive refinements.* (The latter terminology is justified in Theorems 5.1, 6.3, and 6.5.)

4.4. **Invariance and examples.** Next, we establish the invariance under local equivalence of the various numbers defined in the first part of this section.

Lemma 4.11. *The integers r^{β_n}, s^{β_n}, r^{β}, s^{β}, r_o, s_o, r_c, and s_c are invariant under local equivalence of (C, D, f), i.e., descend to $\mathcal{C}_{LEO}$; their reduced versions $\tilde{s}^{\beta_n}$, $\tilde{s}^{\beta}$, $\tilde{s}_o$, and $\tilde{s}_c$ descend to $\widetilde{\mathcal{C}}_{LEO}$. Further, $\tilde{s}_o$ and $\tilde{s}^{\beta_n}$ descend to $\widetilde{\mathcal{C}}^o_{LEO}$.*

Proof. The proofs are all essentially the same, so we only spell out a representative case, r^{β_n}. Suppose $(h\colon C \to C', k\colon D \to D')$ is a local map from (C, D, f) to (C', D', f'). Then, we have a commutative diagram

$$\begin{tikzcd}
H^{-1,q}(C;\mathbb{Z}/(2^n)) \arrow[r, "f\circ\beta_n"] \arrow[d, "h_*"] & H^{0,q}(D_{h=0};\mathbb{F}_2) \arrow[d, "k_*"] & H^{0,q}(D;\mathbb{F}_2) \arrow[l, "p"'] \arrow[r, "i"] \arrow[d, "k_*"] & H^0(h^{-1}D;\mathbb{F}_2) \arrow[d, "k_*", "\cong"'] \\
H^{-1,q}(C';\mathbb{Z}/(2^n)) \arrow[r, "f\circ\beta_n"] & H^{0,q}(D'_{h=0};\mathbb{F}_2) & H^{0,q}(D';\mathbb{F}_2) \arrow[l, "p"'] \arrow[r, "i"] & H^0(h^{-1}D';\mathbb{F}_2),
\end{tikzcd}$$

where the arrow on the right is an isomorphism. So, if q is β_n-half-full for (C, D, f), witnessed by elements $\check{a}$, $\hat{a}$, a, and $\overline{a}$, then q is also β_n-half-full for (C', D', f'), witnessed by $h_*(\check{a})$, $k_*(\hat{a})$, $k_*(a)$, and $k_*(\overline{a})$. Hence, $r^{\beta_n}(C', D', f') \geq r^{\beta_n}(C, D, f)$. If (C, D, f) and (C', D', f') are locally equivalent, there is also a local map the other direction, giving the reverse inequality. □

We illustrate the definitions above with a few simple algebraic examples.

Example 4.12. Consider the trivial LEO triple $LEO(U) = (\mathbb{Z}[X]/(X^2)\{1\}, \mathcal{R}\{1\}, \mathrm{id})$. We write down the sequence in Diagram (4.1) for $q = -3$, -1, 1, and 3. For $q = -3$, it has the form

$$\begin{tikzcd}
H^{-1,-3}(C;\mathbb{Z}/(2^n)) \arrow[r, "\mathrm{id}\circ\beta_n"] \arrow[d, equal] & H^{0,-3}(D_{h=0};\mathbb{F}_2) \arrow[d, equal] & H^{0,-3}(D;\mathbb{F}_2) \arrow[l, "p"'] \arrow[r, "i"] \arrow[d, equal] & H^{0,-3}(h^{-1}D;\mathbb{F}_2) \arrow[d, equal] \\
0 \arrow[r] & 0 & \mathbb{F}_2\langle h^2, Xh\rangle \arrow[l] \arrow[r] & \mathbb{F}_2\langle h^2, Xh\rangle.
\end{tikzcd}$$

In particular, -3 is β_n-full, with $\check{a} = \check{b} = \hat{a} = \hat{b} = 0$, $a = \overline{a} = h^2$, and $b = \overline{b} = Xh$. The same applies to β (which also vanishes); indeed, β will be similar to β_n throughout this example. For r_c and s_c, we are interested instead in the diagram

$$\begin{tikzcd}
H^{0,-3}(C) \arrow[r, "f\circ j"] \arrow[d, equal] & H^{0,-3}(D_{h=0};\mathbb{F}_2) \arrow[d, equal] & H^{0,-3}(D) \arrow[l, "p"'] \arrow[r, "i"] \arrow[d, equal] & H^{0,-3}(h^{-1}D) \arrow[d, equal] \\
0 \arrow[r] & 0 & \mathbb{Z}\langle h^2, Xh\rangle \arrow[l] \arrow[r] & \mathbb{Z}\langle h^2, Xh\rangle.
\end{tikzcd}$$

So $q = -3$ is completely full, by taking $\check{a} = \check{b} = 0$ and $a = h^2$, $b = Xh$, and the same for oddly full.

For $q = -1$, the diagram for s^{β_n} is

$$0 \longrightarrow \mathbb{F}_2\langle X\rangle \xleftarrow{0 \leftarrow\!\shortmid h} \mathbb{F}_2\langle h, X\rangle \longrightarrow \mathbb{F}_2\langle h, X\rangle,$$

so $q = -1$ is not full for β_n (because we cannot find $\check{b}$, say) but is half-full via $\check{a} = \hat{a} = 0$ and $a = \overline{a} = h$. So, $s^{\beta_n} = -3 + 3 = 0$. For the complete case, the sequence is

$$\mathbb{Z}\langle X\rangle \longrightarrow \mathbb{F}_2\langle X\rangle \xleftarrow{0 \leftarrow\!\shortmid h} \mathbb{Z}\langle h, X\rangle \longrightarrow \mathbb{Z}\langle h, X\rangle,$$

and we can take $\check{a} = 0$, $a = h$, and $\check{b} = b = X$ to see that $q = -1$ is completely and oddly full.

For $q = 1$, the diagram for s^{β_n} is

$$0 \longrightarrow \mathbb{F}_2\langle 1\rangle \xleftarrow{\mathrm{id}} \mathbb{F}_2\langle 1\rangle \longrightarrow \mathbb{F}_2\langle 1, h^{-1}X\rangle,$$

so 1 is not β_n-half-full (because one cannot find $\check{a}$), so $r^{\beta_n} = -1 + 1 = 0$. For the complete case, the sequence is

$$\mathbb{Z}\langle 1\rangle \longrightarrow \mathbb{F}_2\langle 1\rangle \longleftarrow \mathbb{Z}\langle 1\rangle \longrightarrow \mathbb{Z}\langle 1, h^{-1}X\rangle$$

so 1 is completely half-full but not completely full. So $s_c = -1 + 1 = 0$, and the same for s_o.

For $q = 3$, the diagram for s^{β_n} is

$$0 \longrightarrow 0 \longleftarrow 0 \longrightarrow \mathbb{F}_2\langle h^{-1}, h^{-2}X\rangle,$$

so again 3 is not β_n-half-full (as must be the case), while for the complete case the sequence is similarly

$$0 \longrightarrow 0 \longleftarrow 0 \longrightarrow \mathbb{Z}\langle h^{-1}, h^{-2}X\rangle,$$

so 3 is not completely half-full, and $r_c = r_o = 1 - 1 = 0$.

Example 4.13. For this example, we look at a reduced LEO triple. Consider the two-step complex C over $\mathbb{Z}$ given by $C^{-1} = \mathbb{Z}$, $C^0 = \mathbb{Z}\{2\} \oplus \mathbb{Z}$, and $\partial(1) = (0, 2^n)$; and the two-step complex D over $\mathbb{Z}[h]$ given by $D^{-1} = \mathbb{Z}[h]$, $D^0 = \mathbb{Z}[h]\{2\} \oplus \mathbb{Z}[h]$, and $\partial(1) = (h, 0)$. There is an evident identification of $C \otimes \mathbb{F}_2$ and $D \otimes \mathbb{F}_2$, given in grading 0 by the 2×2 identity matrix. That is:

$$C: \begin{tikzcd} \mathbb{Z}\{2\} & \mathbb{Z} \\ & \mathbb{Z} \arrow[u, "2^n"'] \end{tikzcd} \qquad C/2 = D/(2,h): \begin{tikzcd} \mathbb{F}_2\{2\} & \mathbb{F}_2 \\ & \mathbb{F}_2 \end{tikzcd} \qquad D: \begin{tikzcd} \mathbb{Z}[h]\{2\} & \mathbb{Z}[h] \\ & \mathbb{Z}[h] \arrow[ul, "h"] \end{tikzcd}$$

Then $s_{\mathbb{F}_2}(C, D, f) = 0$. The Bockstein β_m vanishes for $m < n$, so $\tilde{s}^{\beta_m}(C, D, f) = 0$, but for $p \geq n$, 0 is β_p-reduced-full so $\tilde{s}^{\beta_p} = 2$. For this complex $\beta = \beta_1$ (since $D/(h)$ has no torsion) and $\tilde{s}^{\beta} = \tilde{s}^{\beta_1}$. Also, 0 is oddly and completely reduced-full, so $\tilde{s}_o = \tilde{s}_c = 0$.

If we dualize the complexes just described, we have $\tilde{s}^{\beta_p}(C^*, D^*, f^*) = s_{\mathbb{F}_2} = 0$ for all p. However, for the dual complex, 0 is no longer oddly or completely reduced-full, so $\tilde{s}_o = \tilde{s}_c = -2$.

Replacing each copy of $\mathbb{Z}$ in C by $\mathbb{Z}[X]/(X^2)$ and each copy of $\mathbb{Z}[h]$ in D by $\mathcal{R}$ gives an unreduced LEO triple (C, D, f) with examples with $s^{\beta_p}(C, D, f) = r^{\beta_p}(C, D, f) = \tilde{s}^{\beta_p}(C, D, f)$ and similarly for the other refined s-invariants.

Example 4.14. As another example, consider the reduced LEO triple given by

$$C: \begin{tikzcd} & \mathbb{Z}\{2\} & \\ \mathbb{Z} & \mathbb{Z} & \mathbb{Z}\{2\} \\ & \mathbb{Z} \arrow[ul, "2"] \arrow[u, "2"'] & \end{tikzcd} \qquad D: \begin{tikzcd} & \mathbb{Z}[h]\{2\} & \\ \mathbb{Z}[h] \arrow[ur, "h"] & \mathbb{Z}[h] & \mathbb{Z}[h]\{2\} \arrow[ul, "-2"'] \\ & \mathbb{Z}[h] \arrow[ul, "2"] \arrow[ur, "h"'] & \end{tikzcd}$$

with the obvious identification of $C/(2)$ and $D/(2,h)$. That is, $C^{-1} = \mathbb{Z}$, $C^0 = \mathbb{Z}\oplus\mathbb{Z}\oplus(\mathbb{Z}\{2\})$, and $C^1 = \mathbb{Z}\{2\}$, with the only nontrivial differential the one from C^{-1} to C^0 given by $\partial(1) = (2,2,0)$, and D is obtained from C by replacing each $\mathbb{Z}$ with $\mathbb{Z}[h]$, but has differential $D^{-1} \to D^0$ given by $\partial(1) = (2,0,h)$ and $D^0 \to D^1$ given by $\partial(a,b,c) = ha - 2c$.

The Bockstein maps on the cohomology of $D/(2,h)$ are given below, where the dashed arrows are β_1 and the dotted arrows are $f^{-1} \circ \beta_1 \circ f$:

$$\begin{tikzcd}
 & \mathbb{F}_2\{2\} & \\
\mathbb{F}_2 & \mathbb{F}_2 & \mathbb{F}_2\{2\} \arrow[ul, dashed] \\
 & \mathbb{F}_2 \arrow[ul, dashed, bend left] \arrow[ul, dotted, bend right] \arrow[u, dotted] &
\end{tikzcd} .$$

It follows that 0 is not reduced-full for the Bocksteins β_n induced by C, or for the Bocksteins induced by D, but is for the sum β from Equation (4.6). Thus, $\tilde{s}^{\beta_n}(C,D,f) = s_{\mathbb{F}_2}(C,D,f) = 0$ while $\tilde{s}^{\beta}(C,D,f) = 2$. It is also easy to check that $\tilde{s}_o(C,D,f) = \tilde{s}_c(C,D,f) = 0$.

If we dualize the complexes, to

$$C^* : \quad \begin{tikzcd}
 & \mathbb{Z} & \\
\mathbb{Z} \arrow[ur, "2"] & \mathbb{Z} \arrow[u, "2"'] & \mathbb{Z}\{-2\} \\
 & \mathbb{Z}\{2\} &
\end{tikzcd} \qquad D^* : \quad \begin{tikzcd}
 & \mathbb{Z}[h] & \\
\mathbb{Z}[h] \arrow[ur, "2"] & \mathbb{Z}[h] & \mathbb{Z}[h]\{-2\} \arrow[ul, "h"'] \\
 & \mathbb{Z}[h]\{-2\} \arrow[ul, "h"] \arrow[ur, "-2"'] &
\end{tikzcd},$$

we now have $\tilde{s}^{\beta_n}(C,D,f) = \tilde{s}^{\beta}(C,D,f) = \tilde{s}_o(C,D,f) = 0$ but $\tilde{s}_c(C,D,f) = -2$: any generator in bigrading $(0,0)$ mapping to a generator of $H^{0,0}(h^{-1}D^*)$ must have the form $(2a,b,0)$ with b odd, but the mod-$(2,h)$ reduction of such a class is not in the image of the homology of C^*.

5. Structural results on reduced LEO groups

In this section, we show that the invariant $\tilde{s}_c$, applied to a reduced LEO triple and its dual, detects the trivial local equivalence class, and $\tilde{s}_o$ does the same for two-reduced LEO triples. In the two-reduced case, a little more work shows that the local equivalence group $\widetilde{\mathcal{C}}^o_{LEO}$ is totally ordered.

Theorem 5.1. *A reduced LEO triple (C,D,f) is trivial in $\widetilde{\mathcal{C}}_{LEO}$ if and only if $\tilde{s}_c(C,D,f) = \tilde{s}_c(C^*,D^*,f^*) = 0$. Similarly, a two-reduced LEO triple (C,D,f) is trivial in $\widetilde{\mathcal{C}}^o_{LEO}$ if and only if $\tilde{s}_o(C,D,f) = \tilde{s}_o(C^*,D^*,f^*) = 0$.*

Consequently, (C,D,f) is locally equivalent to (C',D',f) if and only if

$$\tilde{s}_c\big((C,D,f)\otimes(C',D',f')^*\big) = \tilde{s}_c\big((C,D,f)^*\otimes(C',D',f')\big) = 0,$$

and similarly for the two-reduced case.

Proof. We prove the statement about $\widetilde{\mathcal{C}}_{LEO}$; the statement about $\widetilde{\mathcal{C}}^o_{LEO}$ is similar. The "only if" direction is Lemma 4.11. So, assume that $\tilde{s}_c(C,D,f) = \tilde{s}_c(C^*,D^*,f^*) = 0$. Suppose that $[\check{a}]$ and $[a]$ witness the fact that 0 is completely reduced-full, so $\check{a} \in C$ and $a \in D$ are cocycle representatives in quantum grading 0. Let $\widetilde{a}$ denote the image of $\check{a}$ in C. Define $\alpha\colon \mathbb{Z} \to C$ by $\alpha(1) = \widetilde{a}$ and $\beta\colon \mathbb{Z}[h] \to D$ by $\beta(1) = a$. Since the image of $[a]$ in $H^{0,0}(h^{-1}D)$ is a generator, β induces a quasi-isomorphism $\mathbb{Z}[h,h^{-1}] \to h^{-1}D$; since $h^{-1}D$ is a complex of free modules over $\mathbb{Z}[h,h^{-1}]$, this is the same as a homotopy equivalence. The images $[p(a)]$ of $[a]$ and $[f(\widetilde{a})]$ of $[\check{a}]$ in $H(D_{h=0};\mathbb{F}_2)$ agree, so there is an element $c \in D_{h=0} \otimes \mathbb{F}_2$ so that $\partial(c) = f(\widetilde{a}) - p(a)$. Define a homotopy $\gamma\colon \mathbb{F}_2 \to D_{h=0} \otimes \mathbb{F}_2$ by $\gamma(1) = c$. Then γ is a homotopy making Diagram (3.11) homotopy commute. In particular, (α,β) is a local map from $(\mathbb{Z},\mathbb{Z}[h],\mathrm{id})$ to (C,D,f).

To construct a local map the other direction, applying the discussion in the previous paragraph to (C^*,D^*,f^*) gives a local map $(\gamma,\delta)\colon (\mathbb{Z},\mathbb{Z}[h],\mathrm{id}) \to (C^*,D^*,f^*)$. The transpose of γ (respectively δ) is a map $\gamma^T\colon C \to \mathbb{Z}$ (respectively $\delta^T\colon D^* \to \mathbb{Z}[h]$), and Diagram (3.11) for (γ,δ) induces a corresponding diagram for (γ^T,δ^T) after replacing the homotopy equivalences with their inverses. Finally, the fact that the map $\delta\colon \mathbb{Z}[h,h^{-1}] \to h^{-1}D^*$ is a quasi-isomorphism implies that the transpose to δ is also a quasi-isomorphism. So, (γ^T,δ^T) is the desired local map $(C,D,f) \to (\mathbb{Z},\mathbb{Z}[h],\mathrm{id})$. ☐

Corollary 5.2. *Given a reduced LEO triple (C,D,f), $\tilde{s}_c(C^*,D^*,f^*) = -\tilde{s}_c(C,D,f)$ if and only if (C,D,f) is locally equivalent to $(\mathbb{Z}\{q\},\mathbb{Z}[h]\{q\},\mathrm{id})$, where $q = \tilde{s}_c(C,D,f)$.*

Proof. The "if" direction is trivial; for the other, suppose $\tilde{s}_c(C^*,D^*,f^*) = -\tilde{s}_c(C,D,f) = -q$. Tensoring (C,D,f) with $(\mathbb{Z}\{q\},\mathbb{Z}[h]\{q\},\mathrm{id})^*$ simply has the effect of shifting the gradings, so the result and its dual have $\tilde{s}_c = 0$. Applying Theorem 5.1 and then tensoring by $(\mathbb{Z}\{q\},\mathbb{Z}[h]\{q\},\mathrm{id})$ gives the result. ☐

Remark 5.3. For the unreduced case, it is not true that

$$r_c(C,D,f) = s_c(C,D,f) = r_c((C,D,f)^*) = s_c((C,D,f)^*) = 0$$

implies that (C,D,f) is locally equivalent to $LEO(U)$. Let D be the complex from Example 3.16 with $k=2$ and let $C = D_{h=0}$. Consider the triple $(C\{-2\},D\{-2\},\mathrm{id})$ and its dual. Since C is simply the modulo-h reduction of D, q is completely half-full if and only if there is an element of $H(D\{-2\})$ of grading q which is not h-torsion (and similarly for completely full). So, the computation in Example 3.16 shows that 1 is completely half-full, and -1 is completely full, so $r_c(C\{-2\},D\{-2\},\mathrm{id}) = s_c(C\{-2\},D\{-2\},\mathrm{id}) = 0$. On the other hand, the dual complex to D, with grading shifted up by 2 and viewed as a complex over $\mathbb{Z}[h]$, is given by

$$\begin{tikzcd}
5 & \mathbb{Z}[h]\langle a\rangle & & \\
3 & \mathbb{Z}[h]\langle Xa\rangle & \mathbb{Z}[h]\langle b\rangle \arrow[l] \arrow[ul, "h"'] & \\
1 & & \mathbb{Z}[h]\langle Xb\rangle & \mathbb{Z}[h]\langle c\rangle \arrow[uull, bend right, "h^2"'] \\
-1 & & & \mathbb{Z}[h]\langle Xc\rangle \arrow[uull, bend left, "h^2"]
\end{tikzcd}$$

where a, b, c are names for the generators of the copies of $\mathcal{R}$ and the numbers indicate the quantum gradings. Canceling Xa and b gives that

$$r_c\big((C\{-2\}, D\{-2\}, \mathrm{id})^*\big) = s_c\big((C\{-2\}, D\{-2\}, \mathrm{id})^*\big) = 0,$$

where the relevant generators are Xb and Xc.

There is no local map from $LEO(U)$ to $(C\{-2\}, D\{-2\}, \mathrm{id})$. A generator of $\mathcal{R}\{1\}$ in $LEO(U)$ would have to map to $(0, a) \in D^0 = \mathcal{R}\{-1\} \oplus \mathcal{R}\{1\}$ for some $a \in \mathbb{Z}$, but $Xh^2(0, a) = \partial(X)$ (where $X \in D^{-1} = \mathcal{R}\{-3\}$), so $(0, a)$ does not generate the homology of $h^{-1}D$.

A similar short computation shows that there is also no local map from $(C\{-2\}, D\{-2\}, \mathrm{id})$ to $LEO(U)$. This example turns out to show that the strategy used in Theorem 5.6 below to give a total order on $\widetilde{\mathcal{C}}^o_{LEO}$ does not work for $\mathcal{C}_{LEO}$.

Recall the variant $\widetilde{\mathcal{C}}^o_{LEO}$ of $\widetilde{\mathcal{C}}_{LEO}$ from Definition 3.13, which is defined like $\widetilde{\mathcal{C}}_{LEO}$ but where D is a complex over $\mathbb{F}_2[h]$ rather than $\mathbb{Z}[h]$. We show in Theorem 5.6 that $\widetilde{\mathcal{C}}^o_{LEO}$ is a totally ordered group. The main ingredient in the proof is the following:

Lemma 5.4. *Let (C, D, f) be a reduced LEO triple, with dual (C^*, D^*, f^*). If $\tilde{s}_o(C, D, f) = s_{\mathbb{F}_2}(C, D, f) - 2$ then $\tilde{s}_o(C^*, D^*, f^*) = s_{\mathbb{F}_2}(C^*, D^*, f^*)$.*

Proof. Since $\mathbb{Z}$ and $\mathbb{F}_2[h]$ are PIDs, C is homotopy equivalent to a direct sum of bigrading-shifted copies of $\mathbb{Z}$, $\mathbb{Z} \xrightarrow{2^n} \mathbb{Z}$, and $\mathbb{Z} \xrightarrow{p^n} \mathbb{Z}$ ($p > 2$ prime), while $D \otimes_{\mathbb{Z}} \mathbb{F}_2$ is homotopy equivalent to the direct sum of a copy of $\mathbb{F}_2[h]$ in bigrading $(0, q = s_{\mathbb{F}_2}(C, D, f))$ and some bigrading-shifted copies of $\mathbb{F}_2[h] \xrightarrow{h^n} \mathbb{F}_2[h]$. Since homotopy equivalences induce local equivalences, we may assume that C and D have this form. Moreover, since $\tilde{s}_o$ depends on C only through the image of the map $H(C) \to H(C; \mathbb{F}_2)$, we may assume that C has no summands of the form $\mathbb{Z} \xrightarrow{p^n} \mathbb{Z}$ for $p > 2$.

A copy of $\mathbb{Z} \xrightarrow{2^n} \mathbb{Z}$ where the first $\mathbb{Z}$ has bigrading (i, j) in C contributes copies of $\mathbb{F}_2$ to $H(C)$ in bigradings (i, j) and $(i + 1, j)$, with the second copy in the image of the map $H(C) \to H(C; \mathbb{F}_2)$. Similarly, a copy of $\mathbb{F}_2[h] \xrightarrow{h^n} \mathbb{F}_2[h]$ where the first $\mathbb{F}_2[h]$ is in bigrading (i, j) contributes copies of $\mathbb{F}_2$ to $H(D_{h=0})$ in bigradings (i, j) and $(i + 1, j + 2n)$, where the second copy is in the image of $H(D; \mathbb{F}_2) \to H(D_{h=0}; \mathbb{F}_2)$.

Suppose that $H^{0,q}(C \otimes \mathbb{F}_2) = \mathbb{F}_2^{c+d+e}$ where $\mathbb{F}_2^c \oplus 0^d \oplus 0^e$ comes from summands of $\mathbb{Z}$ in $C^{0,q}$, $0^c \oplus \mathbb{F}_2^d \oplus 0^e$ comes from summands of the form $\mathbb{Z} \xrightarrow{2^n} \mathbb{Z}$ with the second $\mathbb{Z}$ in bigrading $(0, q)$ (and, so, is in the image of the map $H(C) \to H(C; \mathbb{F}_2)$), and $0^c \oplus 0^d \oplus \mathbb{F}_2^e$ comes from summands of the form $\mathbb{Z} \xrightarrow{2^n} \mathbb{Z}$ with the first $\mathbb{Z}$ in bigrading $(0, q)$ (and, so, is not in the image of the map $H(C) \to H(C; \mathbb{F}_2)$). Suppose $H^{0,q}(D_{h=0}; \mathbb{F}_2) = \mathbb{F}_2^{1+a+b}$ where the first $\mathbb{F}_2$ comes from the $\mathbb{F}_2[h]$-summand, $0 \oplus \mathbb{F}_2^a \oplus 0^b$ comes from summands $\mathbb{F}_2[h] \xrightarrow{h^n} \mathbb{F}_2[h]$ where the second $\mathbb{F}_2[h]$ is generated in degree $(0, q)$ (hence, is in the image of $H(D; \mathbb{F}_2) \to H(D_{h=0}; \mathbb{F}_2)$), and $0 \oplus 0^a \oplus \mathbb{F}_2^b$ comes from summands $\mathbb{F}_2[h] \xrightarrow{h^n} \mathbb{F}_2[h]$ where the first $\mathbb{F}_2[h]$ is generated in degree $(0, q)$ (hence, is not in the image of $H(D; \mathbb{F}_2) \to H(D_{h=0}; \mathbb{F}_2)$). Here, $c + d + e = 1 + a + b$.

The hypothesis that $\tilde{s}_o(C, D, f) = s_{\mathbb{F}_2}(C, D, f) - 2$ means that q is not oddly reduced-full. So, $f(\{(x_1, \dots, x_c, y_1, \dots, y_d, 0, \dots, 0)\})$ does not intersect $\{(1, u_1, \dots, u_a, 0, \dots, 0)\}$. Let $E = \{(0, \dots, 0, 0, \dots, 0, z_1, \dots, z_e)\} \subset H^{0,q}(C; \mathbb{F}_2)$, and let $\pi_E \colon H^{0,q}(C; \mathbb{F}_2) \to E$ be the projection induced by our chosen basis. Let $A = \{(0, u_1, \dots, u_a, 0, \dots, 0)\} \subset H^{0,q}(D_{h=0}; \mathbb{F}_2)$. So, $\pi_E(f^{-1}(1, 0, \dots, 0, 0, \dots, 0)) \notin \pi_E(f^{-1}(A))$.

Let α_E be a linear functional on E such that $\alpha_E(f^{-1}(1,0,\dots,0,0,\dots,0)) = 1$ and $\alpha_E(\pi_E(f^{-1}(A))) = 0$, and let $\alpha = \alpha_E \circ \pi_E$; α is a class in $H^{0,-q}(C^*;\mathbb{F}_2)$. Since α vanishes on $\{(0,\dots,0,y_1,\dots,y_d,0,\dots,0)\}$, α is in the image of $H^{0,q}(C^*) \to H^{0,q}(C^*;\mathbb{F}_2)$. Further, $f^*\alpha$ vanishes on A, so $f^*\alpha$ is in the image of a class $\beta \in H^{0,q}(D^*;\mathbb{F}_2)$. Finally, since $(f^*\alpha)(1,0,\dots,0,0,\dots,0) = 1$, the image of β in $H^{0,q}(h^{-1}D^*;\mathbb{F}_2)$ is non-zero. Thus, $(0,-q)$ is oddly reduced-full, proving the result. □

Corollary 5.5. *If (C,D,f) is a reduced LEO triple, then at least one of $\tilde{s}_o(C,D,f)$ and $\tilde{s}_o(C^*,D^*,f^*)$ is non-negative.*

Proof. If $s_{\mathbb{F}_2}(C,D,f) \geq 2$, then $\tilde{s}_o(C,D,f) \geq 0$ as needed; if $s_{\mathbb{F}_2}(C,D,f) \leq -2$, then $s_{\mathbb{F}_2}(C^*,D^*,f^*) = -s_{\mathbb{F}_2}(C,D,f) \geq 2$, giving $\tilde{s}_o(C^*,D^*,f^*) \geq 0$. In the remaining case when $\tilde{s}_{\mathbb{F}_2}(C,D,f) = 0$, if $\tilde{s}_o(C,D,f) \neq 0$, then Lemma 5.4 gives that $\tilde{s}_o(C^*,D^*,f^*) = s_{\mathbb{F}_2}(C^*,D^*,f^*) = 0$, proving the claim. □

Define a relation on $\widetilde{\mathcal{C}}^o_{LEO}$ by declaring that $[(C,D,f)] \geq [(C',D',f')]$ if there is a local map from (C',D',f') to (C,D,f).

Theorem 5.6. *This definition specifies a translation-invariant total order on $\widetilde{\mathcal{C}}^o_{LEO}$. Further, the total order is characterized by $[(C,D,f)] \geq \pi(LEO(U))$ if and only if $\tilde{s}_o(C,D,f) \geq 0$. (Here, $\pi(LEO(U))$ denotes the identity in $\widetilde{\mathcal{C}}^o_{LEO}$.)*

Proof. This essentially follows from Corollary 5.5 and the fact that compositions and tensor products of local maps are local maps. In more detail, for the first statement, we must check that the order is transitive, anti-symmetric, translation-invariant, and that for any $[(C,D,f)]$ and $[(C',D',f)]$, either $[(C,D,f)] \geq [(C',D',f)]$ or $[(C',D',f)] \geq [(C,D,f)]$. Transitivity follows from the fact that a composition of local maps is a local map, and translation invariance follows from the fact that tensoring a local map $(C,D,f) \to (C',D',f')$ with the identity map of (C'',D'',f'') gives a local map of tensor products. Anti-symmetry is immediate: if there is a local map from (C,D,f) to (C',D',f') and from (C',D',f') to (C,D,f) then, by definition, (C,D,f) is locally equivalent to (C',D',f'). The claim that either $[(C,D,f)] \geq [(C',D',f')]$ or $[(C',D',f')] \geq [(C,D,f)]$ follows from the fact that for any (C,D,f) there is either a local map from $\pi(LEO(U))$ to (C,D,f) or vice-versa, which in turn follows from Corollary 5.5 and (the proof of) Theorem 5.1.

For the second statement, from the proof of Theorem 5.1, $\tilde{s}_o(C,D,f) \geq 0$ if and only if there is a local map from $\pi(LEO(U))$ to (C,D,f), so the property that $\tilde{s}_o(C,D,f) \geq 0$ characterizes the non-negative elements of $\widetilde{\mathcal{C}}^o_{LEO}$. By translation invariance, this in turn characterizes the total order. □

Corollary 5.7. *Every nontrivial element of $\widetilde{\mathcal{C}}^o_{LEO}$ has infinite order.*

Remark 5.8. As we will now show, the analogue of Corollary 5.5 does not hold for $\tilde{s}_c$, and hence the proof of Theorem 5.6 does not generalize to $\widetilde{\mathcal{C}}_{LEO}$. There is a knot K_1 with $s_{\mathbb{F}_2} = 2$ and $s_{\mathbb{F}_p} = 0$ for all other p, and a knot K_2 with $s_{\mathbb{F}_3} = -2$ and $s_{\mathbb{F}_p} = 0$ for all other p [LZ21, Sch]. Taking $K = K_1 \# K_2$, we claim that $\tilde{s}_c(K)$ and $\tilde{s}_c(\overline{K})$ are both negative. By Lemma 6.24, it suffices to show $s_{\mathbb{Z}}$ is negative for both. A result of Schütz [Sch, Corollary 4.9] gives that $s_{\mathbb{Z}} \leq s_{\mathbb{F}}$ for any field $\mathbb{F}$, so we have $s_{\mathbb{Z}}(K) \leq s_{\mathbb{F}_3}(K) = -2$ and $s_{\mathbb{Z}}(\overline{K}) \leq s_{\mathbb{F}_2}(\overline{K}) = -2$, as needed.

Remark 5.9. Theorem 5.1 reminds us of Hom's construction of her group $\mathcal{CFK}$ of knot Floer-like complexes modulo the complexes with $\varepsilon = 0$ [Hom15]. The role of her concordance invariant ε is played here by $\big(s_{\mathbb{F}_2}(C,D,f)-\tilde{s}_c(C,D,f), -s_{\mathbb{F}_2}(C,D,f)-\tilde{s}_c((C,D,f)^*)\big)$, which takes values in $\{(0,0),(2,0),(0,-2)\}$. For the analogue for s_o, Theorem 5.6 pushes this analogy further, showing that, like Hom's group $\mathcal{CFK}$, the version $\widetilde{\mathcal{C}}^o_{LEO}$ of the even-odd local equivalence group has a total order.

6. The case of knots

This section has two goals. The simpler is to connect the refined s-invariants with the concordance group and the slice genus; we do that near the outset. The other is to explore relations between these invariants that hold for knots but not general LEO triples. That is, for an arbitrary LEO triple, the invariants $s^+_{\mathbb{F}}$ and $s^-_{\mathbb{F}}$ are typically distinct, by Lemma 3.20, but for knots they agree, and agree with the reduced version $s_{\mathbb{F}}$. This fact, and the techniques underlying its proof, imply more relations between the refined s-invariants of LEO triples coming from knots. We state the key relations first, then develop the properties of the Khovanov complexes needed to prove them in Section 6.1, before giving the proofs themselves in Section 6.2. Finally, in Section 6.3, we use the structural results from Section 5 to study the relationships between the Bockstein and comprehensive refinements.

To start, we note:

Proposition 6.1. *Let $(C,D,f) = LEO(K)$ or $LEE(K)$. Then the numbers $r^\alpha(C,D,f)$, $s^\alpha(C,D,f)$, and $\tilde{s}^\alpha(C,D,f)$ for $\alpha \in \{\beta_n,\beta\}$ $(1 \leq n \leq \infty)$, and the numbers $r_\gamma(C,D,f)$, $s_\gamma(C,D,f)$, and $\tilde{s}_\gamma(C,D,f)$ for $\gamma \in \{o,c\}$, are all concordance invariants of K.*

Proof. This is immediate from Propositions 3.4 and 3.12 and Lemma 4.11. ☐

For the Bockstein refinements, the key relations are:

Theorem 6.2. *For any knot K and $\alpha \in \{\beta_n,\beta\}$, the invariants r^α, s^α, and $\tilde{s}^\alpha$ (applied to $LEO(K)$ or $LEE(K)$) lie in $\{s_{\mathbb{F}_2}(K), s_{\mathbb{F}_2}(K)+2\}$. Further,*

$$s_{\mathbb{F}_2}(K) \leq r^\alpha(LEO(K)) = \tilde{s}^\alpha(LEO(K)) = s^\alpha(LEO(K)) = s_{\mathbb{F}_2}(K)+2,$$
$$s^{\beta_1}(LEO(K)) \leq s^{\beta_2}(LEO(K)) \leq \cdots \leq s^{\beta_\infty}(LEO(K)).$$

(The analogue for $LEE(K)$ is given in Lemma 6.23.) The following theorem gives the analogous results for the comprehensive refinements:

Theorem 6.3. *For any knot K,*

$$s_{\mathbb{Z}}(K)-2 \leq r_c(K) = \tilde{s}_c(K) = s_c(K) \leq s_{\mathbb{Z}}(K)$$
$$s_{\mathbb{F}_2}(K)-2 \leq r_o(K) = \tilde{s}_o(K) = s_o(K) \leq s_{\mathbb{F}_2}(K)$$
$$\tilde{s}_c(K) \leq \tilde{s}_o(K).$$

(Here, we have shortened notation by writing K to mean $LEO(K)$.)

Both theorems are proved in Section 6.2, after we develop some more machinery in Section 6.1.

Remark 6.4. For an algebraic example illustrating that $s_{\mathbb{F}_2} - 2$ is not a lower bound for s_c and the other complete invariants, consider the complex D given by

$$\begin{tikzcd}
\mathcal{R}\{5\} & & \mathcal{R}\{3\} & & \mathcal{R}\{1\} \\
& \mathcal{R}\{3\} \arrow[ul, "h"] \arrow[ur, "3"'] & & \mathcal{R}\{1\} \arrow[ul, "h"] \arrow[ur, "3"'] &
\end{tikzcd}$$

with the top line in homological degree 0. One checks that $s_{\mathbb{F}_2}$ is 4, but $H^{0,q}(D;\mathbb{Z}) \to H^{0,q}(h^{-1}D;\mathbb{Z})$ is surjective only for $q \leq -1$. With $C = D \otimes_{\mathcal{R}} \mathbb{Z}[X]/(X^2)$ we get $s_c(C, D, \mathrm{id}) = r_c(C, D, \mathrm{id}) = 0$. While this example is algebraic, as noted in Remark 5.8, there exist knots K with $s_{\mathbb{F}_2}(K) > s_{\mathbb{F}_3}(K)$ ([LZ21, Sch]) and, by considering connected sums of such knots, we can make the difference between $s_{\mathbb{F}_2}$ and s_c arbitrarily large. Consequently, there are knots where s_c and s_o differ by arbitrarily large amounts.

The example of Remark 5.3 showed that r_c alone does not detect local equivalence in $\mathcal{C}_{LEO}$. One of the reasons Theorem 6.3 is powerful is because for knots, r_c does suffice to detect local equivalence:

Theorem 6.5. *If K is a knot so that $r_c(LEO(K)) = r_c(LEO(\overline{K})) = 0$ then $LEO(K)$ is locally equivalent to $LEO(U)$.*

Again, the proof is deferred to Section 6.2.

Corollary 6.6. *Let $LEO(\mathcal{C})$ be the image of the smooth concordance group in $\mathcal{C}_{LEO}$, and $\pi\colon \mathcal{C}_{LEO} \to \widetilde{\mathcal{C}}_{LEO}$ be the quotient map. Then the restriction of π to $LEO(\mathcal{C})$ is injective. In particular, $LEO(K)$ is trivial in $\mathcal{C}_{LEO}$ if and only if $\tilde{s}_c(LEO(K)) = \tilde{s}_c(LEO(\overline{K})) = 0$.*

Proof. By Theorem 5.1, a knot maps to zero in $\widetilde{\mathcal{C}}_{LEO}$ if and only if we have $\tilde{s}_c(LEO(K)) = \tilde{s}_c(LEO(\overline{K})) = 0$. By Theorem 6.5, a knot maps to zero in $\mathcal{C}_{LEO}$ if and only if $r_c(LEO(K)) = 0$. By Theorem 6.3, $\tilde{s}_c(LEO(K)) = r_c(LEO(K))$. ☐

Corollary 6.7. *Given a knot K, $\tilde{s}_c(LEO(\overline{K})) = -\tilde{s}_c(LEO(K))$ if and only if $LEO(K)$ is locally equivalent to $LEO(U)\{q\}$, where $q = \tilde{s}_c(LEO(K))$.*

Proof. The "if" direction is trivial. For the other direction, observe that, for any even $q \in \mathbb{Z}$, two LEO triples (C, D, f) and (C', D', f') are locally equivalent if and only if $(C, D, f)\{q\}$ is locally equivalent to $(C', D', f')\{q\}$. Also, by definition, we have $\tilde{s}_c((C, D, f)\{q\}) = \tilde{s}_c(C, D, f) + q$. Furthermore, if T is the trefoil knot with $s(T) = -2$ then, by direct computation, $LEO(K)$ is locally equivalent to $LEO(U)\{-2\}$. So, for any knot K and even integer q, $LEO(K)\{-q\} = LEO(K \# \frac{q}{2} T)$ is in the image of the smooth concordance group, so Corollary 6.6 applies to $LEO(K)\{-q\}$.

Now, if $\tilde{s}_c(LEO(K)) = -\tilde{s}_c(LEO(\overline{K})) = q$ then $\tilde{s}_c(LEO(K)\{-q\}) = \tilde{s}_c(LEO(\overline{K})\{q\}) = 0$. Hence, by Theorem 5.1, $\pi(LEO(K)\{-q\})$ is reduced locally equivalent to $\pi(LEO(U))$. Thus, by Corollary 6.6, $LEO(K)\{-q\}$ is locally equivalent to $LEO(U)$, and the result follows. ☐

We conclude this subsection with two simpler topological properties of these invariants. First, as mentioned at the beginning of Section 4, the invariants we have constructed are inspired by the refined s-invariants from the Khovanov stable homotopy type [LS14]. There, a refined s-invariant was associated to any cohomology operation, using the even Khovanov stable homotopy type. Analogous operations were constructed using the odd stable homotopy type by Sarkar-Scaduto-Stoffregen [SSS20]. The relationship of these earlier invariants to the ones of Section 4 is:

Lemma 6.8. *Let r^{Sq^1} and s^{Sq^1} be the refined s-invariants associated to the operation* Sq^1 *on even Khovanov homology* [LS14]*, and $r^{\mathrm{Sq}^1_o}$ and $s^{\mathrm{Sq}^1_o}$ be the refined s-invariants associated to the operation* Sq^1 *on odd Khovanov homology* [SSS20]*. Then*

$$r^{\mathrm{Sq}^1}(K) = r^{\beta_1}(LEE(K)) \qquad s^{\mathrm{Sq}^1}(K) = s^{\beta_1}(LEE(K))$$
$$r^{\mathrm{Sq}^1_o}(K) = r^{\beta_1}(LEO(K)) \qquad s^{\mathrm{Sq}^1_o}(K) = s^{\beta_1}(LEO(K))$$

Proof. Recall that the first Steenrod square Sq^1 is the Bockstein homomorphism β_1. So, the only difference between the two constructions is that here we have viewed the Bar-Natan complex as a graded complex over $\mathbb{F}_2[h]$, while the earlier papers considered the filtered Bar-Natan complex corresponding to $D_{h=1}$. If we write $\mathcal{F}_q D_{h=1} \subset D_{h=1}$ for the image of $D^{(q)}$ under the quotient map $D \to D_{h=1}$, we get that the quotient map restricted to the graded subcomplex $D^{(q)}$ is injective, so that $H^{0,q}(D;\mathbb{F}_2) \cong H^0(\mathcal{F}_q D_{h=1};\mathbb{F}_2)$. Also, $\mathcal{F}_q D_{h=1}/\mathcal{F}_{q+2}D_{h=1} \cong D^{(q)}_{h=0}$, and we can define β_n-half-full and β_n-full with $H^0(\mathcal{F}_q D_{h=1};\mathbb{F}_2)$ in place of $H^{0,q}(D;\mathbb{F}_2)$ and $H^0(D_{h=1};\mathbb{F}_2)$ in place of $H^0(h^{-1}D;\mathbb{F}_2)$. Because of the commutative diagram

$$\begin{tikzcd}
H^{0,q}(D_{h=0};\mathbb{F}_2) \arrow[d, "\cong"] & H^{0,q}(D;\mathbb{F}_2) \arrow[l] \arrow[r] \arrow[d, "\cong"] & H^{0,q}(h^{-1}D;\mathbb{F}_2) \arrow[d] \\
H^0(\mathcal{F}_q D_{h=1}/\mathcal{F}_{q+2}D_{h=1};\mathbb{F}_2) & H^0(\mathcal{F}_q D_{h=1};\mathbb{F}_2) \arrow[l] \arrow[r] & H^0(D_{h=1};\mathbb{F}_2)
\end{tikzcd}$$

we get that q being β_n-(half-)full is implied by the same in this filtered sense. To see that these notions are equivalent, note that $h^{-1}D$ is also graded, so the image of $H^{0,q}(D;\mathbb{F}_2)$ is contained in $H^{0,q}(h^{-1}D;\mathbb{F}_2) \cong \mathbb{F}_2 \oplus \mathbb{F}_2$, and this image is mapped isomorphically to $H^0(D_{h=1};\mathbb{F}_2)$. □

Remark 6.9. It is natural to ask whether the other refined invariants from the Khovanov stable homotopy type, like s^{Sq^2}, are invariants of local equivalence. This seems unlikely to us, but we do not know a counterexample.

Second, while they are not concordance homomorphisms, the refined s-invariants do give slice genus bounds:

Lemma 6.10. *Let Σ be a smooth, connected, orientable cobordism from K_0 to K_1, and let s' be any of the invariants in Proposition 6.1. Then,*

$$|s'(K_0) - s'(K_1)| \leq -\chi(\Sigma).$$

(For $s'(K) = s^{\beta_n}(LEE(K))$, say, this is a special case of a known result [LS14, Theorem 1].)

Proof. The proof of Proposition 3.4 shows that if there is a smooth, connected, orientable cobordism Σ from K_0 to K_1 then there is a local map from $LEO(K_0)$ to $LEO(K_1)$, except shifting the quantum grading by $\chi(\Sigma)$. Given a configuration showing that q is full or half-full in whatever sense for K_0, the image of that configuration under the local map shows that $q + \chi(\Sigma)$ is full or half-full in the same sense for K_1, so $s'(K_1) \geq s'(K_0) + \chi(\Sigma)$. Reversing the cobordism, the same argument gives $s'(K_0) \geq s'(K_1) + \chi(\Sigma)$, proving the result. □

Remark 6.11. It follows from Theorems 6.2 and 6.3, and the computations in Section 7 proving the refinements all differ from $s_{\mathbb{F}_2}$, that none of the refined s-invariants discussed in this section are homomorphisms from $\mathcal{C}_{LEO}$ (or $\widetilde{\mathcal{C}}_{LEO}$), or even from the subgroup generated by knots. Their nontriviality (Section 7) does imply that the homomorphism $\vec{s}$ from Proposition 3.22 has nontrivial kernel. It would be interesting to construct new homomorphisms from $\mathcal{C}_{LEO}$ to $\mathbb{Z}$ giving new concordance homomorphisms. One strategy (inspired by Dai-Hom-Stoffregen-Truong's work [DHST21]) might be to find an explicit answer to Question 3.23, or to prove a structure theorem for some other quotient of $\mathcal{C}_{LEO}$.

6.1. **Conjugation on the Khovanov complexes.** The special features of the Khovanov complexes of a knot which allows us to prove Theorem 6.3 are a conjugation action and particular generators for $h^{-1}Kh_h(K)$.

Define $\overline{\cdot}\colon \mathcal{R} \to \mathcal{R}$ as the $\mathbb{Z}[h]$-linear map with $\overline{1} = 1$ and $\overline{X} = h - X$. It is straightforward to check that this is a ring homomorphism with $\overline{\overline{x}} = x$ for all $x \in \mathcal{R}$. We call this automorphism the *conjugation of* $\mathcal{R}$.

Conjugation gives rise to a bigrading-preserving involution $I\colon CKh_h(K) \to CKh_h(K)$ as follows. If v is a vertex in the cube of resolutions for K, and $x = x_1 \otimes x_2 \otimes \cdots \otimes x_k \in CKh_h(K)$ is an element over v, define

$$I(x) = (-1)^{\mathrm{split}(v)}\overline{x}_1 \otimes \overline{x}_2 \otimes \cdots \otimes \overline{x}_k.$$

This commutes with the differential on $CKh_h(K)$ [Sch, §2] and satisfies $I(r \cdot x) = \overline{r}I(x)$ for all $r \in \mathcal{R}$. Also, if we write $CKh_{h=1}(K)$ for $CKh_h(K)_{h=1}$, the map I descends to $I\colon CKh_{h=1}(K) \to CKh_{h=1}(K)$, and the crucial property of this I is that there exists $\varepsilon_q \in \{\pm 1\}$ with

$$x + \varepsilon_q I(x) \in \mathcal{F}_{q+2} \tag{6.12}$$

for all $x \in \mathcal{F}_q \subset CKh_{h=1}(K)$ [Sch, Lemma 2.1]. Here, as in the reduced case, $\mathcal{F}_q = p(CKh_h^q(K))$ with $p\colon CKh_h(K) \to CKh_{h=1}(K)$. This property can be lifted to $CKh_h(K)$:

Lemma 6.13. *There is a chain map $T\colon CKh_h^q(K) \to CKh_h^{q+2}(K)$ over $\mathbb{Z}$ so that*

$$\mathrm{id} + \varepsilon_q I = h \cdot T. \tag{6.14}$$

Furthermore, at the level of homology, there is a commutative diagram

$$\begin{tikzcd} Kh_h^{i,q}(K) \arrow[r, "T"] \arrow[d] & Kh_h^{i,q+2}(K) \arrow[d] \\ Kh^{i,q}(K;\mathbb{F}_2) \arrow[r, "\nu"] & Kh^{i,q+2}(K;\mathbb{F}_2) \end{tikzcd} \tag{6.15}$$

where ν is Shumakovitch's acyclic differential on Khovanov homology over $\mathbb{F}_2$ [Shu14, §3.2].

Proof. For $x \in CKh_h^q(K)$ we get $p(x) + \varepsilon_q I(p(x)) = p(y)$ for some $y \in CKh_h^{q+2}(K)$ by Formula (6.12). Since $p|_{CKh_h^{(q+2)}}$ is injective, this uniquely determines y. But we also have $x + \varepsilon_q I(x) \in CKh_h^q(K)$, since I preserves gradings. Also, $hy \in CKh_h^q(K)$ and therefore

$$p(x + \varepsilon_q I(x)) = p(y) = p(hy).$$

Hence $x + \varepsilon_q I(x) - hy \in CKh^q(K)$ is in $\ker p$. But p restricted to $CKh_h^q(K)$ is injective, so $x + \varepsilon(q)I(x) - hy = 0$, and we can define $T(x) = y \in CKh_h^{q+2}$ and obtain Formula (6.14). Since ε_q is fixed and I is a chain map, we get that $h \cdot T$ is a chain map. Therefore $T \circ \partial - \partial \circ T$

lands in $\ker(h\colon CKh_h(K) \to CKh_h(K))$. Hence, since multiplication by h is injective on the free complex $CKh_h(K)$, we see T is a chain map.

For the commutative diagram (6.15), note that after passing to $\mathbb{F}_2$ coefficients, we can ignore ε_q. In particular, we then have a chain map $T\colon CKh_h(K;\mathbb{F}_2) \to CKh_h(K;\mathbb{F}_2)$ which respects the action of $\mathbb{F}_2[h]$. To see that setting $h = 0$ turns T into ν requires us to look more carefully at the definition. A typical basis element of $CKh_h(K;\mathbb{F}_2)$ over $\mathbb{F}_2[h]$ is of the form $b = \eta_1 \otimes \cdots \otimes \eta_k$ with $\eta_j \in \{1, X\}$ for all $j = 1, \ldots k$. Then $I(b)$ replaces each $\eta_j = X$ with $h + X = h + \eta_j$. In particular,

$$I(b) = b + hT_1(b) + h^2T_2(b) + \cdots + h^kT_k(b),$$

where $T_j(b)$ is the sum of all $\bar{\eta}_1 \otimes \cdots \otimes \bar{\eta}_k$, where exactly j of the η_m equal to X have been replaced by 1, and all $k - i$ other $\bar{\eta}_m = \eta_m$. In particular, T_1 agrees with Shumakovitch's ν [Shu14, Definition 3.1.A] and

$$T = \nu + hT_2 + \cdots + h^{k-1}T_k.$$

Setting $h = 0$ gives the desired result. □

Let $\varphi\colon h^{-1}\mathcal{R}\{1\} \to h^{-1}CKh_h(K)$ be a grading-preserving $\mathcal{R}$-linear homotopy equivalence. Then $\varphi(1)$ is a cycle which represents a homology class in $h^{-1}Kh_h^{0,1}(K)$ that generates $h^{-1}Kh_h(K)$ as a free $h^{-1}\mathcal{R}$-module. We can also pull back the involution $I\colon h^{-1}Kh_h(K) \to h^{-1}Kh_h(K)$ to $I_{h^{-1}\mathcal{R}}\colon h^{-1}\mathcal{R}\{1\} \to h^{-1}\mathcal{R}\{1\}$. This is a $\mathbb{Z}[h]$-linear grading-preserving involution on $h^{-1}\mathcal{R}\{1\}$ which satisfies $I_{h^{-1}\mathcal{R}}(rx) = \bar{r}I_{h^{-1}\mathcal{R}}(x)$ for all $r \in h^{-1}\mathcal{R}$ and $x \in h^{-1}\mathcal{R}\{1\}$. This implies that $I_{h^{-1}\mathcal{R}}$ is either plus or minus the conjugation.

Define

$$\begin{aligned}\mathcal{O} &= (h - X)[\varphi(1)] \in h^{-1}Kh_h^{0,-1}(K)\\ \overline{\mathcal{O}} &= I(\mathcal{O}) = \pm X[\varphi(1)] \in h^{-1}Kh_h^{0,-1}(K).\end{aligned}$$

Then $\mathcal{O}, \overline{\mathcal{O}}$ generate $h^{-1}Kh_h^{0,-1}(K) \cong \mathbb{Z}^2$ as an abelian group. Observe that $(h - X)\mathcal{O} = h\mathcal{O}$ and $X\mathcal{O} = 0$. Furthermore, each combination $m\mathcal{O} + n\overline{\mathcal{O}}$ with $\gcd(n, m) = 1$ is a primitive element in $h^{-1}Kh_h^{0,-1}(K)$, but only for $m, n \in \{\pm 1\}$ does this element generate $h^{-1}Kh_h(K)$ as an $h^{-1}\mathcal{R}$-module. This is because $\pm h$ and $\pm(h - 2X)$ are the only units of $h^{-1}\mathcal{R}$ in quantum degree -2 (the inverse of the latter is $\pm(h - 2X)h^{-2}$).

Lemma 6.16. *If $i\colon Kh_h^{0,q-2}(K) \to h^{-1}Kh_h^{0,q-2}(K)$ is surjective, then there exists $a \in Kh_h^{0,q}(K)$ such that $i(a)$ generates $h^{-1}Kh_h(K)$ as an $h^{-1}\mathcal{R}$-module. Furthermore, if $b_1, b_2 \in Kh_h^{0,q-2}(K)$ are such that $i(b_1), i(b_2)$ generate $h^{-1}Kh_h^{0,q-2}(K)$, then $a = T(b)$ for some linear combination b of b_1 and b_2.*

Proof. There exists j such that $h^j\mathcal{O}, h^j\overline{\mathcal{O}}$ generate $h^{-1}Kh_h^{0,q-2}(K)$. By assumption, there exists a cycle $b \in CKh_h^{0,q-2}(K)$ such that $i(b)$ represents $h^j\mathcal{O}$. Then, $i(I(b))$ represents $h^j\overline{\mathcal{O}}$. By Formula (6.14), $a = T(b)$ is a cycle in $CKh_h^{0,q}(K)$ such that $i(a)$ represents $h^{j-1}(\mathcal{O} \pm \overline{\mathcal{O}})$, which generates $h^{-1}Kh_h(K)$ as an $h^{-1}\mathcal{R}$-module. The second statement follows by construction of a. □

Lemma 6.17. *If there is an element $a \in Kh_h^{0,q}(K)$ with $i(a) \in h^{-1}Kh_h^{0,q}(K)$ primitive, then $i\colon Kh_h^{0,q-2}(K) \to h^{-1}Kh_h^{0,q-2}(K)$ is surjective. Furthermore, if $i(a)$ does not generate $h^{-1}Kh_h(K)$ as an $h^{-1}\mathcal{R}$-module, then $i\colon Kh_h^{0,q+2}(K) \to h^{-1}Kh_h^{0,q+2}(K)$ is non-zero.*

Proof. We can write $i(a) = mh^{j-1}\mathcal{O} + nh^{j-1}\overline{\mathcal{O}}$ with $m, n \in \mathbb{Z}$ such that $\gcd(m, n) = 1$.

Denote the image of $i\colon Kh_h^{0,q}(K) \to h^{-1}Kh_h^{0,q}(K) \cong \mathbb{Z}\oplus\mathbb{Z}$ by H. Then $mh^j\mathcal{O}+nh^j\overline{\mathcal{O}} \in H$. We also get $i((h-X)a) = mh^j\mathcal{O} \in H$ and therefore also $nh^j\overline{\mathcal{O}} \in H$. Applying I to $(h-X)a$ and Xa also shows that $I(mh^j\mathcal{O}) = mh^j\overline{\mathcal{O}} \in H$ as well as $I(nh^j\overline{\mathcal{O}}) = nh^j\mathcal{O} \in H$. This implies

$$mh^j\mathcal{O}, nh^j\mathcal{O}, nh^j\overline{\mathcal{O}}, mh^j\overline{\mathcal{O}} \in H. \tag{6.18}$$

In particular, if $(m, n) = (1, 0)$ or $(m, n) = (0, 1)$ we get $H = \mathbb{Z} \oplus \mathbb{Z}$. Otherwise, we have $|m| \geq |n|$ or $|m| < |n|$. In both cases, Formula (6.18), $\gcd(m, n) = 1$ and the Euclidean Algorithm imply $\mathcal{O}, \overline{\mathcal{O}} \in H$. This shows surjectivity of i in quantum grading $q - 2$.

By Lemma 6.16 there exists an $a' \in Kh_h^{0,q}(K)$ such that $i(a')$ generates $h^{-1}Kh_h(K)$ as an $h^{-1}\mathcal{R}$-module. In particular, we can assume that $i(a') = h^{j-1}(\mathcal{O} \pm \overline{\mathcal{O}})$. If we assume that $i(a)$ does not generate $h^{-1}Kh_h(K)$ as an $h^{-1}\mathcal{R}$-module, then $|m| \neq |n|$, and for some linear combination of a and a' we get that $(n\pm m)h^{j-1}\mathcal{O} \neq 0$ is in the image of i. Let $b \in CKh_h^q(K)$ be a cycle such that $i(b)$ represents this multiple of $h^{j-1}\mathcal{O}$. Using Formula (6.14), we get that $T(b) \in CKh_h^{q+2}(K)$ is a cycle with $i(T(b))$ representing $(n\pm m)h^{j-2}(\mathcal{O}\pm\overline{\mathcal{O}})$, a non-zero element in $h^{-1}Kh_h^{0,q+2}(K)$. ☐

The reduced complex is defined as

$$\widetilde{CKh}_h(K) = CKh_h(K) \otimes_{\mathcal{R}} \mathbb{Z}[h]\{-1\}$$

where X acts on $\mathbb{Z}[h]$ as 0. This gives rise to a short exact sequence of chain complexes

$$0 \longrightarrow \widetilde{CKh}_h(K)\{-1\} \xrightarrow{S} CKh_h(K) \xrightarrow{\pi} \widetilde{CKh}_h(K)\{1\} \longrightarrow 0 \tag{6.19}$$

with $S(x) = XI(x)$ (where we view x as a labeling of the circles that labels the marked circle 1) and $\pi(x) = x \otimes 1$. After inverting h this gives rise to another short exact sequence

$$0 \longrightarrow h^{-1}\widetilde{Kh}_h^{0,q+1}(K) \xrightarrow{S} h^{-1}Kh_h^{0,q}(K) \xrightarrow{\pi} h^{-1}\widetilde{Kh}_h^{0,q-1}(K) \longrightarrow 0 \tag{6.20}$$

with $\pi(h^k\overline{\mathcal{O}}) = 0$ and $\pi(h^k\mathcal{O})$ generating $h^{-1}\widetilde{Kh}_h^{0,q-1}(K)$. Here, k is an integer ensuring the right quantum degree.

6.2. **Applications of conjugation to the refined s-invariants.** Recall from Section 3.3 that for a knot K we have

$$\begin{aligned}
s^+_{\mathbb{Z}}(K) &= \max\{q \mid \exists\, a \in Kh_h^{0,q}(K) \text{ with } i(a) \in h^{-1}Kh_h^{0,q}(K) \text{ primitive}\} - 1,\\
s^-_{\mathbb{Z}}(K) &= \max\{q \mid Kh_h^{0,q}(K) \to h^{-1}Kh_h^{0,q}(K) \text{ surjective}\} + 1,\\
s_{\mathbb{Z}}(K) &= \max\{q \mid \widetilde{Kh}_h^{0,q}(K) \to h^{-1}\widetilde{Kh}_h^{0,q}(K) \text{ surjective}\}.
\end{aligned}$$

Lemma 6.21. *Let K be a knot. Then*

$$s^+_{\mathbb{Z}}(K) = s^-_{\mathbb{Z}}(K) = s_{\mathbb{Z}}(K).$$

Proof. Let $q = s^-_{\mathbb{Z}}(K) - 1$, so that $i\colon Kh_h^{0,q}(K) \to h^{-1}Kh_h^{0,q}(K) \cong \mathbb{Z} \oplus \mathbb{Z}$ is surjective. By Lemma 6.16, $h^j\mathcal{O} \pm h^j\overline{\mathcal{O}}$, one of the generators of $h^{-1}Kh_h(K)$ and hence primitive, is $i(a)$ for some a in quantum degree $q + 2$. Then $\pi(h^j\mathcal{O} \pm h^j\overline{\mathcal{O}}) = h^j\pi(\mathcal{O}) \in h^{-1}\widetilde{Kh}_h^{0,q+1}(K)$ is $i(\pi(a))$, so is also in the image, and generates this abelian group. It follows that

$$s^-_{\mathbb{Z}}(K) \leq s^+_{\mathbb{Z}}(K) \text{ and } s^-_{\mathbb{Z}}(K) \leq s_{\mathbb{Z}}(K).$$

Now let $q = s_{\mathbb{Z}}(K)$, so that $i\colon \widetilde{Kh}_h^{0,q}(K) \to h^{-1}\widetilde{Kh}_h^{0,q}(K)$ is surjective. Then there exists a with $i(a) = h^{j-1}\pi(\mathcal{O}) \in h^{-1}\widetilde{Kh}_h^{0,q}(K)$. Since $S(\pi(\mathcal{O})) = \overline{\mathcal{O}}$, we get $b = \pm S(a) \in Kh_h^{0,q-1}(K)$ with $i(b) = h^j\overline{\mathcal{O}}$. Also, $i(I(b)) = \pm h^j\mathcal{O}$ and hence $i\colon Kh_h^{0,q-1}(K) \to h^{-1}Kh_h^{0,q-1}(K)$ is surjective. Therefore

$$s_{\mathbb{Z}}(K) \leq s_{\mathbb{Z}}^-(K) \text{ and hence } s_{\mathbb{Z}}(K) = s_{\mathbb{Z}}^-(K).$$

It remains to show that for $q = s_{\mathbb{Z}}^+(K)-1$ we get $i\colon Kh_h^{0,q}(K) \to h^{-1}Kh_h^{0,q}(K)$ is surjective. This follows from Lemma 6.17 and therefore

$$s_{\mathbb{Z}}^+(K) \leq s_{\mathbb{Z}}^-(K). \qquad \square$$

Remark 6.22. The graded groups $H^0(h^{-1}\widetilde{CKh}_h(K))^{(q)}$ contain p-torsion if and only if the analogous unreduced graded groups contain p-torsion [Sch, Lemma 4.15]. Combined with Lemma 6.21, this shows that an unreduced graded s-invariant contains the same information as the $s^{\mathbb{Z}}(K)$ defined by Schütz [Sch].

The first step towards proving Theorem 6.2 is:

Lemma 6.23. *For any knot K and $\alpha \in \{\beta_n, \beta\}$, the invariants r^α, s^α, and $\tilde{s}^\alpha$ (applied to $LEO(K)$ or $LEE(K)$) lie in $\{s_{\mathbb{F}_2}(K), s_{\mathbb{F}_2}(K)+2\}$. Further,*

$$s_{\mathbb{F}_2}(K) \leq r^\alpha(LEO(K)) \leq \tilde{s}^\alpha(LEO(K)) \leq s^\alpha(LEO(K)) \leq s_{\mathbb{F}_2}(K)+2,$$
$$s^{\beta_1}(LEO(K)) \leq s^{\beta_2}(LEO(K)) \leq \cdots \leq s^{\beta_\infty}(LEO(K)),$$

and similarly for $LEE(K)$.

Proof. Most of this is a rephrasing of observations from Section 4, keeping in mind that for knots, $s_{\mathbb{F}_2}^{\pm}(LEO(K)) = s_{\mathbb{F}_2}^{\pm}(LEE(K)) = s_{\mathbb{F}_2}(K)$. To see the inequalities between r^α, $\tilde{s}^\alpha$, and s^α, assume that $r^\alpha(LEO(K)) = s_{\mathbb{F}_2}(K) + 2 =: s+2$. Then there exists $a \in Kh_h^{0,s+1}(K;\mathbb{F}_2)$ and $\check{a} \in Kh_o^{-1,s+1}(K;\mathbb{A})$ with $0 \neq i(a) \in h^{-1}Kh^{0,s+1}(K;\mathbb{F}_2)$ and $\alpha(\check{a}) = p(a) \in Kh^{0,s+1}(K;\mathbb{F}_2)$. Here, $\mathbb{A}$ is the coefficient group appropriate for α. Notice that i cannot be surjective in quantum degree $s+1$, and therefore $i(I(a)) = i(a)$. This implies $0 \neq \pi(i(a)) \in h^{-1}\widetilde{Kh}_h^{0,s}(K;\mathbb{F}_2)$ and the naturality of the Bockstein homomorphism shows that $\pi(a)$ and $\pi(\check{a})$ witness that s is α-reduced full.

To see that $\tilde{s}^\alpha(LEO(K)) \leq s^\alpha(LEO(K))$, assume that s is α-reduced-full, witnessed by $a \in \widetilde{Kh}_h^{0,s}(K;\mathbb{F}_2)$ and $\check{a} \in \widetilde{Kh}_o^{-1,s}(K;\mathbb{A})$. Then $S(a) \in Kh_h^{0,s-1}(K;\mathbb{F}_2)$ and $S(\check{a}) \in Kh_o^{-1,s-1}(K;\mathbb{A})$ witness that $s-1$ is α-half-full. Using $b = I(S(a)) + S(a) \in Kh_h^{0,s-1}(K;\mathbb{F}_2)$ and $\check{b} = 0 \in Kh_o^{-1,s-1}(K;\mathbb{A})$ we see that $s-1$ is in fact α-full.

The proof for $LEE(K)$ is identical. $\square$

The analogous first step towards Theorem 6.3 is slightly harder:

Lemma 6.24. *For any knot K,*

$$\begin{tikzcd}
s_{\mathbb{Z}}(K)-2 \arrow[r, "\leq"] \arrow[dr, "\leq"] & r_c(K) \arrow[r, "\leq"] \arrow[dr, "\leq"] & \tilde{s}_c(K) \arrow[r, "\leq"] \arrow[dr, "\leq"] & s_c(K) \arrow[r, "\leq"] \arrow[dr, "\leq"] & s_{\mathbb{Z}}(K) \arrow[dr, "\leq"] & \\
& s_{\mathbb{F}_2}(K)-2 \arrow[r, "\leq"] & r_o(K) \arrow[r, "\leq"] & \tilde{s}_o(K) \arrow[r, "\leq"] & s_o(K) \arrow[r, "\leq"] & s_{\mathbb{F}_2}(K).
\end{tikzcd}$$

Proof. The fact that $r_c(K) \leq r_o(K)$, and similarly for $\tilde{s}$ and s, is immediate from the definitions. The claims $s_{\mathbb{F}_2}(K) - 2 \leq r_o(K)$ and $s_o(K) \leq s_{\mathbb{F}_2}(K)$ follow from Formulas (4.9) and (4.10) and the fact that for knots, $s^+_{\mathbb{F}_2} = s^-_{\mathbb{F}_2} = s_{\mathbb{F}_2}$. The inequality between $s_{\mathbb{Z}}$ and $s_{\mathbb{F}_2}$ follows from a result of Schütz [Sch, Corollary 4.9]. (In particular, this part also applies to $s_{\mathbb{F}}$ for any field $\mathbb{F}$: $s_{\mathbb{Z}}(K) \leq s_{\mathbb{F}}(K)$ for any field $\mathbb{F}$.) The fact that $s_c(K) \leq s_{\mathbb{Z}}(K)$ is immediate from the definitions and the fact that $s_{\mathbb{Z}}(K) = s^-_{\mathbb{Z}}(K)$ by Lemma 6.21.

To see that $s_{\mathbb{Z}}(K) - 2 \leq r_c(K)$, let q be the maximal grading so that there is an $a \in Kh_h^{0,q}(K;\mathbb{Z})$ with $i(a) \in h^{-1}Kh_h^{0,q}(K;\mathbb{Z})$ primitive. Since $s^+_{\mathbb{Z}}(K) = s_{\mathbb{Z}}(K)$ by Lemma 6.21, $q = s_{\mathbb{Z}}(K) + 1$. Then ha also has $i(ha) \in h^{-1}Kh_h^{0,q-2}(K;\mathbb{Z})$ primitive, but now $p(ha) = 0 \in Kh^{0,q-2}(K;\mathbb{F}_2)$, so in particular $p(a) = j(0)$, and hence $q-2$ is completely half-full. A similar argument shows that $s_{\mathbb{Z}}(K) - 2 \leq \tilde{s}_c(K)$ and the same for $s_c(K)$.

Finally, we show that $r_c(K) \leq \tilde{s}_c(K) \leq s_c(K)$. The proof is similar to the proof of Lemma 6.21.

To see that $r_c(K) \leq \tilde{s}_c(K)$, recall that we showed in Lemma 6.24 that $s_{\mathbb{Z}}(K) - 2 \leq r_c(K), \tilde{s}_c(K), s_c(K) \leq s_{\mathbb{Z}}(K)$; hence, if $r_c(K) = s_{\mathbb{Z}}(K) - 2$ then there is nothing to show. So, let $q = s^+_{\mathbb{Z}}(K) + 1$ and assume that q is completely half-full, witnessed by elements $a \in Kh_h^{0,q}(K)$ and $\check{a} \in Kh_o^{0,q}(K)$. We can write $i(a) \in h^{-1}Kh_h^{0,q}(K)$ as $mh^j\mathcal{O} + nh^j\overline{\mathcal{O}}$, for some j, where $\gcd(m,n) = 1$. The image $\pi(i(a)) = i(\pi(a))$ in $h^{-1}\widetilde{Kh}_h^{0,1}(K)$ is $mh^j\pi(\mathcal{O})$, which is not a generator unless $m = \pm 1$. So, consider the element $I(a)$. The image of $I(a)$ in $Kh_o^{0,q}(K)$ agrees with the image of $\pm a$, and hence of $\pm\check{a}$, and $i(I(a)) = nh^j\mathcal{O} + mh^j\overline{\mathcal{O}}$. Since $\gcd(m,n) = 1$, there are integers α, β so that $\pi(i(\alpha a + \beta I(a))) = h^j\pi(\mathcal{O})$. Then $\pi(\alpha a + \beta I(a)) \in \widetilde{Kh}_h^{0,1}(K)$ and $\pi((\alpha \pm \beta)\check{a}) \in \widetilde{Kh}_o^{0,q}(K)$ witness the fact that $q-1$ is completely reduced-full, so $\tilde{s}_c(K) = s_{\mathbb{Z}}(K)$.

Similarly, to see that $\tilde{s}_c(K) \leq s_c(K)$, if suffices to consider the case that $q = s_{\mathbb{Z}}(K)$ is completely reduced-full. Then we can find $a \in \widetilde{Kh}_h^{0,q}(K)$ and $\check{a} \in \widetilde{Kh}_o^{0,q}(K)$ which map to the same element of $\widetilde{Kh}^{0,q}(K;\mathbb{F}_2)$ and so that $i(a) = h^{j-1}\pi(\mathcal{O}) \in h^{-1}\widetilde{Kh}_h^{0,q}(K)$. Since $S(\pi(\mathcal{O})) = \overline{\mathcal{O}}$, there is an element $b = \pm S(a) \in Kh_h^{0,q-1}(K)$ with $i(b) = h^j\overline{\mathcal{O}}$. Also, $i(I(b)) = \pm h^j\mathcal{O}$. So, $i(b)$ and $i(I(b))$ span $h^{-1}Kh_h^{0,q-1}(K)$. Further, the images of b and $S(\check{a}) \in Kh_o^{0,q-1}(K)$ in $Kh^{0,q-1}(K;\mathbb{F}_2)$ agree, as do the images of $I(b)$ and $S(\check{a})$, so $q-1$ is completely full, as desired.

The analogues of these last inequalities for the o variants are similar but easier to prove, following an argument of Schütz [Sch, Proposition 5.5]. □

We can now prove Theorem 6.5, that for knots r_c applied to the knot and its mirror detects local equivalence:

Proof of Theorem 6.5. Write $LEO(K) = (C, D, f)$. Since $r_c(LEO(K)) = 0$, there are elements $a \in H^{0,1}(D)$ and $\check{a} \in H^{0,1}(C)$ with $f \circ j(\check{a}) = p(a) \in H^{0,1}(D_{h=0};\mathbb{F}_2)$ and $i(a) \in H^{0,1}(h^{-1}D)$ primitive. We claim that $i(a)$ generates $H(h^{-1}D)$ as a module over $h^{-1}\mathcal{R}$. Suppose not. By Lemma 6.17, the map $i\colon Kh_h^{0,3}(K) \to h^{-1}Kh_h^{0,3}(K) \cong \mathbb{Z}^2$ is non-zero. Hence, by the universal coefficient theorem, the map $i\colon Kh_h^{0,3}(K;\mathbb{Q}) \to h^{-1}Kh_h^{0,3}(K;\mathbb{Q}) \cong \mathbb{Q}^2$ is non-zero, so $s_{\mathbb{Q}}(K) \geq 2$. Hence, $s_{\mathbb{Q}}(\overline{K}) \leq -2$, but by construction $s_{\mathbb{Z}}(\overline{K}) \leq s_{\mathbb{Q}}(\overline{K})$, so, by Lemma 6.24, $s_c(\overline{K}) \leq -2$, a contradiction.

Now, we proceed as in the proof of Theorem 5.1. Fix cocycle representatives for a and $\check{a}$ and define $\alpha\colon \mathbb{Z}[X]/(X^2)\{1\} \to C$ by $\alpha(1) = \check{a}$ and $\beta\colon \mathcal{R}\{1\} \to D$ by $\beta(1) = a$. We just verified that β induces a quasi-isomorphism $h^{-1}\mathcal{R} \to h^{-1}D$; since both sides are free, β is also a homotopy equivalence. The homotopy between $f \circ \alpha$ and $\beta \circ \mathrm{id}$ is induced by the fact that $[p(a)] = [f \circ j(\check{a})] \in H(D_{h=0}; \mathbb{F}_2)$. So, (α, β) induces a local map $LEO(U) \to LEO(K)$.

Applying the same argument with $\overline{K}$ in place of K gives a local map $LEO(U) \to LEO(\overline{K}) = LEO(K)^*$. Dualizing gives a local map $LEO(K) \to LEO(U)^* = LEO(U)$, concluding the proof that $LEO(K)$ is locally equivalent to $LEO(U)$. ☐

The second component of the proofs of Theorems 6.2 and 6.3 is showing that, in fact, the three versions each comprehensive invariant agree, as do the three versions of the β_n-refined invariant for $LEO(K)$:

Lemma 6.25. *Let K be a knot and $1 \leq n \leq \infty$. Then for $\gamma \in \{c, o\}$ we have*

$$r_\gamma(LEO(K)) = s_\gamma(LEO(K)) \quad \text{and} \quad r^{\beta_n}(LEO(K)) = s^{\beta_n}(LEO(K)).$$

Proof. We focus on the case $r_c(LEO(K)) = s_c(LEO(K))$; the other cases are similar. By Lemma 6.24, it suffices to show that if $s_c(LEO(K)) = s_{\mathbb{Z}}(K)$, then $s_c(LEO(K)) \leq r_c(LEO(K))$. So assume that $q = s_{\mathbb{Z}}(K) - 1$ is completely full and let $a, b \in Kh_h^{0,q}(K)$ and $\check{a}, \check{b} \in Kh_o^{0,q}(K)$ witness this fact.

By Lemma 6.16, there exist $m, n \in \mathbb{Z}$ with $i(T(ma + nb))$ primitive in $h^{-1}Kh_h^{0,q}(K)$. By Diagram (6.15), we have $p(T(ma+nb)) = \nu(mj(\check{a})+nj(\check{b}))$. It remains to lift Shumakovitch's ν to odd Khovanov homology.

Khovanov homology with coefficients in $\mathbb{F}_2$ splits [Shu14, Corollary 3.2.C] and so does Bar-Natan homology over $\mathbb{F}_2$ [Wig16]. We want to analyze the matrix for the map

$$\tilde{\nu} \colon \widetilde{Kh}^{0,q-1}(K; \mathbb{F}_2) \oplus \widetilde{Kh}^{0,q+1}(K; \mathbb{F}_2) \to \widetilde{Kh}^{0,q+1}(K; \mathbb{F}_2) \oplus \widetilde{Kh}^{0,q+3}(K; \mathbb{F}_2)$$

induced by ν and Shumakovitch's splitting.

The sequence (6.19) can be split over $\mathbb{F}_2$ by using $r \colon \widetilde{CKh_h}(K; \mathbb{F}_2)\{1\} \to CKh_h(K; \mathbb{F}_2)$, $r(u \otimes 1) = u + XT(u)$. After setting $h = 0$, T turns into ν and because $X\nu + \nu X = \mathrm{id}$ [Shu14, §3.2], the splitting $\tilde{r} \colon \widetilde{CKh}(K; \mathbb{F}_2)\{1\} \to CKh(K; \mathbb{F}_2)$ is given by $\tilde{r}(u \otimes 1) = \nu(Xu)$. In particular, for $u \otimes 1 \in \widetilde{Kh}^{0,q-1}(K; \mathbb{F}_2)$ we get $\tilde{\nu}(u \otimes 1) = \nu(\nu(Xu)) = 0$, as $\nu^2 = 0$ [Shu14, §3.2].

For $u \otimes 1 \in \widetilde{Kh}^{0,q+1}(K; \mathbb{F}_2)$ observe that inclusion into $Kh^{0,q}(K; \mathbb{F}_2)$ is given by sending $u \otimes 1$ to $XI(u)$. As $h = 0$ in our situation, $XI(u) = Xu$ by Lemma 6.13. Then $\nu(Xu) = u + X\nu(u)$. Projecting $Kh^{0,q+2}(K; \mathbb{F}_2)$ to the summand $\widetilde{Kh}^{0,q+3}(K; \mathbb{F}_2)$ is done by sending v to $\nu(v) \otimes 1$, so $\nu(Xu)$ is sent to 0 in this summand. Finally, projection from $Kh^{0,q+2}(K; \mathbb{F}_2)$ to the summand $\widetilde{Kh}^{0,q+1}(K; \mathbb{F}_2)$ is just the standard change of coefficients $w \mapsto w \otimes 1$. Therefore $\nu(Xu) = u + X\nu(u)$ is sent to $u \otimes 1$.

It follows that $\tilde{\nu}$ is the identity on $\widetilde{Kh}^{0,q+1}(K; \mathbb{F}_2)$ and zero between all other summands. The splitting of odd Khovanov homology with integral coefficients [ORS13] descends to the splitting over $\mathbb{F}_2$ described above, so we can lift ν to odd Khovanov homology by using the same matrix as for $\tilde{\nu}$ (i.e., $(a, b) \mapsto (b, 0)$). This finishes the proof. ☐

Lemmas 6.23 and 6.25 imply that the reduced invariants $\tilde{s}^{\beta_n}(LEO(K))$ and $\tilde{s}_o(LEO(K))$ also agree with $r^{\beta_n}(LEO(K))$ and $r_o(LEO(K))$.

Remark 6.26. We do not know examples of knots K where $s^{\beta_n}(LEE(K)) \neq r^{\beta_n}(LEE(K))$ (respectively $s^{\beta}(LEO(K)) \neq r^{\beta}(LEO(K))$), and suspect they may also be equal in general.

Proof of Theorem 6.2. This is immediate from Lemmas 6.23 and 6.25. □

Proof of Theorem 6.3. This is immediate from Lemmas 6.24 and 6.25. □

6.3. **More relations between the refined s-invariants.** There are two main uses for concordance invariants: proving that a knot K is not slice (concordant to the unknot), and proving that knots K_1 and K_2 are not concordant. Given concordance invariants α and β, we say that α is a *stronger slice obstruction* than β if whenever β detects non-sliceness of K, so does α. We say α and β are *independent* if there are knots $K_1, \dots, K_4$ so that $\alpha(K_1) = \alpha(K_2)$, $\beta(K_1) \neq \beta(K_2)$, $\alpha(K_3) \neq \alpha(K_4)$, and $\beta(K_3) = \beta(K_4)$. It is possible for α to be a stronger slice obstruction than β but for α and β to be independent; depending on the problem, one property may be more relevant than the other. This subsection is focused on showing that some of the concordance invariants described above are stronger slice obstructions than others. Independence of many of these invariants follows from the computations in Section 7 (see also Examples 4.13 and 4.14).

We already know that some of the invariants are stronger slice obstructions than others. For example, Theorem 6.2 implies that for knots with $s_{\mathbb{F}_2}(K) = 0$, if $n > m$ then s^{β_n} is a stronger slice obstruction than s^{β_m}. On the other hand, the computations in Section 7 show that neither s^{β_∞} nor s^{β} is a stronger slice obstruction than the other. Similarly, for knots with $s_{\mathbb{F}_2}(K) = 0$, Lemma 6.24 implies that r_c is a stronger slice obstruction than the others in that lemma. (See also Theorem 6.5.)

We turn now to the relationship between the two families of invariants (s^α and s_γ).

Knowing $H^0(C)$ as an abelian group leads to an understanding of the image of the Bockstein homomorphism β_n, namely, the image is generated by the $\mathbb{Z}/(2^k)$-summands in $H^0(C)$ with $k \leq n$ via the map $H^0(C) \otimes_{\mathbb{Z}} \mathbb{F}_2 \to H^0(C; \mathbb{F}_2)$. So, $q = s_{\mathbb{F}_2}(K)$ is β_n-reduced-full if there is a non-h-torsion element of $H^{0,q}(D; \mathbb{F}_2)$ whose image in $H^{0,q}(C; \mathbb{F}_2)$ is in the image of the $\mathbb{Z}/(2^k)$-summands with $k \leq n$. In particular, as noted above, if $s_{\mathbb{F}_2}(K)$ is β_n-reduced-full (so $\tilde{s}^{\beta_n}(LEO(K)) = s_{\mathbb{F}_2}(K) + 2$), it is also β_p-reduced-full for $p \geq n$. This also tells us that q is oddly reduced-full, so $\tilde{s}_o(K) = s_{\mathbb{F}_2}(K)$. So, we have proved:

Lemma 6.27. *If $\tilde{s}^{\beta_n}(LEO(K)) = s_{\mathbb{F}_2}(K) + 2$ then $\tilde{s}_o(LEO(K)) = s_{\mathbb{F}_2}(K)$, and similarly for $LEE(K)$.*

On the other hand, if we dualize the complex, the Bockstein homomorphism reverses direction, so the corresponding $\mathbb{Z}/(2^k)$-summand no longer lifts to an integral class. So, it seems reasonable to expect that, often, if $\tilde{s}^{\beta_n}(LEO(K)) = s_{\mathbb{F}_2}(K) + 2$ then $\tilde{s}_o(\overline{K}) = s_{\mathbb{F}_2}(\overline{K}) - 2$ (where $\overline{K}$ denotes the mirror). A precise results along these lines is the following:

Proposition 6.28. *The concordance invariant $\tilde{\boldsymbol{s}}_c(K) = \big(\tilde{s}_c(LEO(K)), \tilde{s}_c(LEO(\overline{K}))\big)$ is a stronger slice obstruction than $s_{\mathbb{F}_2}(K)$ and all of s^{β_n}, s^{β}, s_o, $\tilde{s}^{\beta_n}, \tilde{s}^{\beta}, \tilde{s}_o, r^{\beta_n}$, and r^{β} applied to either $LEO(K)$ or $LEE(K)$. Also, the concordance invariant $\big(\tilde{s}_o(LEO(K)), \tilde{s}_o(LEO(\overline{K}))\big)$ is a stronger slice obstruction than $s_{\mathbb{F}_2}(K)$ and $\tilde{s}^{\beta_n}(LEO(K))$.*

Proof. For the first statement, by Theorems 6.3 and 6.5, if $\tilde{\boldsymbol{s}}_c(K) = (0, 0)$ then $LEO(K)$ is locally equivalent to $LEO(U)$. It follows that $LEE(K)$ is also locally equivalent to $LEO(U)$, since $LEE(K)$ is determined by $LEO(K)$. So, all of the local equivalence invariants for $LEO(K)$ and $LEE(K)$ must vanish. The proof of the second statement is similar, using two-reduced local equivalence and Theorem 5.1 in place of Theorem 6.5. □

Corollary 6.29. *If $\tilde{s}^{\beta_n}(LEO(K)) = s_{\mathbb{F}_2}(K) + 2$ then $\tilde{s}^{\beta_n}(LEO(\overline{K})) = -s_{\mathbb{F}_2}(K)$.*

crossings	$\boldsymbol{s}^{\mathrm{Sq}^1_e}(K)$	$\boldsymbol{s}^{\mathrm{Sq}^1_o}(K)$	$\boldsymbol{s}^{\beta}(K)$	$\boldsymbol{s}^{\beta_{15}}(K)$	$\tilde{\boldsymbol{s}}_c(K)$
9	0	1	1	1	1
10	0	2	2	2	2
11	0	10	10	10	10
12	0	49	49	50	50
13	0	286	285	297	297
14	2	1,718	1,717	1,797	1,797
15	41	11,244	11,239	11,808	11,819
16	162	73,814	73,787	77,873	77,929

Table 1. This table shows the number of prime knots with a specific number of crossings where each invariant is non-constant and hence carries more information than $s_{\mathbb{F}_2}$. The total number where $\tilde{\boldsymbol{s}}_c(K)$ is non-constant is 5.4% of the 1.7 million prime knots with at most 16 crossings.

Proof. By Lemma 6.27, if $\tilde{s}^{\beta_n}(LEO(K)) = s_{\mathbb{F}_2}(K) + 2$ and $\tilde{s}^{\beta_n}(LEO(\overline{K})) = -s_{\mathbb{F}_2}(K) + 2$ then $\tilde{s}_o(LEO(K)) = s_{\mathbb{F}_2}(K)$ and $\tilde{s}_o(LEO(\overline{K})) = s_{\mathbb{F}_2}(\overline{K}) = -s_{\mathbb{F}_2}(K)$, but then Corollary 5.2 implies that $\tilde{s}^{\beta_n}(LEO(K)) = s_{\mathbb{F}_2}(K)$, a contradiction. □

Finally, for alternating knots, the techniques in this paper do not give new information:

Proposition 6.30. *Let K be an alternating knot. Then, $LEO(K)$ is locally equivalent to $LEO(U)\{\sigma(K)\}$, where $\sigma(K)$ denote the signature of K. In particular, if we let $(C, D, f) = LEO(K)$ or $LEE(K)$, then*

$$s^{\alpha}(C, D, f) = \tilde{s}^{\alpha}(C, D, f) = r^{\alpha}(C, D, f) = \sigma(K)$$

for all $\alpha \in \{\beta_n, \beta\}$ with $1 \leq n \leq \infty$, and

$$s_{\gamma}(C, D, f) = \tilde{s}_{\gamma}(C, D, f) = r_{\gamma}(C, D, f) = \sigma(K)$$

for $\gamma \in \{o, c\}$.

Proof. For the first statement, let $(C, D, f) = LEO(K)$ and consider $r_c(C, D, f)$. By an earlier result [Sch, Lemma 4.11], $s_{\mathbb{Z}}(K) = s_{\mathbb{F}_2}(K) =: s$, and hence by Lemma 6.21, there exists $a \in H^{0,s+1}(D)$ with $i(a) \in H^{0,s+1}(h^{-1}D)$ primitive. Since odd Khovanov homology is torsion-free for alternating knots, we get that $f \circ j \colon H^{0,s+1}(C) \to H^{0,s+1}(D_{h=0}; \mathbb{F}_2)$ is surjective, and therefore $s+1$ is completely half-full, that is, $r_c(LEO(K)) = s$. Applying the same argument to the mirror of K gives $r_c(LEO(K)^*) = -s$. So, by Corollary 6.7, $LEO(K)$ is locally equivalent to $LEO(U)\{\sigma(K)\}$. Hence, their local equivalence invariants agree. □

7. Computations

Many of the knot invariants in Section 6 have been implemented in the latest version of `KnotJob` [Sch23], including the strongest among them (cf. Corollary 6.6). Specifically, `KnotJob` can compute[1]

$$\boldsymbol{s}^{\beta_{15}}(K) = \big(r^{\beta_{15}}(LEO(K)),\ -r^{\beta_{15}}(LEO(\overline{K}))\big),$$

[1] `KnotJob` uses the notations 'BLS-odd' for $\boldsymbol{s}^{\beta_{15}}$, '$\mathrm{Sq}^1$-sum' for $\boldsymbol{s}^{\beta}$, and 'complete LS-Inv' for $\tilde{\boldsymbol{s}}_c$.

$$\begin{aligned}\boldsymbol{s}^{\beta}(K) &= \big(r^{\beta}(LEO(K)),\ s^{\beta}(LEO(K)),\ -r^{\beta}(LEO(\overline{K})),\ -s^{\beta}(LEO(\overline{K}))\big),\\ \tilde{\boldsymbol{s}}_c(K) &= \big(\tilde{s}_c(LEO(K)),\ -\tilde{s}_c(LEO(\overline{K}))\big).\end{aligned}$$

The invariants

$$\begin{aligned}\boldsymbol{s}^{\mathrm{Sq}^1_o}(K) &= \big(r^{\beta_1}(LEO(K)),\ s^{\beta_1}(LEO(K)),\ -r^{\beta_1}(LEO(\overline{K})),\ -s^{\beta_1}(LEO(\overline{K}))\big),\\ \boldsymbol{s}^{\mathrm{Sq}^1_e}(K) &= \big(r^{\beta_1}(LEE(K)),\ s^{\beta_1}(LEE(K)),\ -r^{\beta_1}(LEE(\overline{K})),\ -s^{\beta_1}(LEE(\overline{K}))\big)\end{aligned}$$

had already been implemented in an earlier version of `KnotJob`. (By Lemma 6.25, we now know there is some redundancy in $\boldsymbol{s}^{\mathrm{Sq}^1_o}(K)$.)

For alternating knots, these tuples are constant with all entries equal to $s_{\mathbb{F}_2}(K)$ by Proposition 6.30. Also, for at least $\boldsymbol{s}^{\beta_n}(K)$, if the tuple is constant, then the common value is $s_{\mathbb{F}_2}(K)$; this follows from Theorem 6.2 and the fact that $s_{\mathbb{F}_2}(\overline{K}) = -s_{\mathbb{F}_2}(K)$. Table 1 shows the number of knots with a non-constant invariant with a specific number of crossings ≤ 16. For knots in this range, the invariant for $\beta = \mathrm{Sq}^1_o + \mathrm{Sq}^1_e$ is very similar to the invariant for Sq^1_o. This is not surprising, as Sq^1_e has little impact for these knots. There are only four knots with at most 15 crossings such that $\boldsymbol{s}^{\beta}$ is non-constant while $\boldsymbol{s}^{\mathrm{Sq}^1_o}$ is constant, and for all of these $\boldsymbol{s}^{\mathrm{Sq}^1_e}$ is also non-constant. Among 16-crossing knots, there are three with non-constant $\boldsymbol{s}^{\beta}$ for which both $\boldsymbol{s}^{\mathrm{Sq}^1_o}$ and $\boldsymbol{s}^{\mathrm{Sq}^1_e}$ are constant. For these, we also have $\boldsymbol{s}^{\beta_{15}}$ constant. For all these knots, we found that if any of the invariants $\boldsymbol{s}^{\mathrm{Sq}^1_e}(K)$, $\boldsymbol{s}^{\mathrm{Sq}^1_o}(K)$, $\boldsymbol{s}^{\beta}(K)$, and $\boldsymbol{s}^{\beta_{15}}(K)$ are non-constant, then so is $\tilde{\boldsymbol{s}}_c$ (cf. Proposition 6.28).

We have computed $\boldsymbol{s}^{\beta_{15}}$ instead of $\boldsymbol{s}^{\beta_\infty}$ because its implementation is faster, and the two only differ if the odd Khovanov homology has torsion of order $> 2^{15}$. In particular, for all the knots in Table 1, $\boldsymbol{s}^{\beta_{15}} = \boldsymbol{s}^{\beta_\infty}$. Computation times for $\boldsymbol{s}^{\beta_{15}}$ and $\boldsymbol{s}^{\beta}$ are faster than for $\tilde{\boldsymbol{s}}_c$, because the computations involve modular, rather than integer, arithmetic.

7.1. **Obstructing sliceness.** There are 352 million prime knots with at most 19 crossings [Bur20]. Dunfield and Gong have an ongoing project to identify which of these knots are smoothly slice [DGon]. As of April 2023, there were only 17,991 knots (0.005%) whose slice status remained unknown. In particular, each of these knots has signature 0, an Alexander polynomial that satisfies the Fox-Milnor criterion, all of Herald-Kirk-Livingston's twisted Alexander polynomials [HKL10] that were computed are consistent with the knot being slice, the Heegaard Floer invariants τ, ν, and ε are all 0, as are the even s-invariant over the fields $\mathbb{F}_2$, $\mathbb{F}_3$, and $\mathbb{Q}$, the even and odd Sq^1-refined s-invariants, and Schütz's $s^{\mathbb{Z}}$ invariant [Sch]. Of these 17,991 inscrutable knots, 826 have non-zero $\boldsymbol{s}^{\beta_{15}}$ invariant, 64 have non-zero $\boldsymbol{s}^{\beta}$ invariant, and 890 have nonvanishing $\tilde{\boldsymbol{s}}_c$ invariant. The knots where $\boldsymbol{s}^{\beta_{15}}$ and $\boldsymbol{s}^{\beta}$ are non-zero are disjoint, and $\tilde{\boldsymbol{s}}_c$ provided no new information, but was non-zero if and only if one of $\boldsymbol{s}^{\beta_{15}}$ and $\boldsymbol{s}^{\beta}$ was. Thus these invariants collectively obstruct sliceness of 890 knots, reducing the number of mystery knots by 5%. Of these 890 knots, at least 832 are topologically slice as they have Alexander polynomial equal to 1 [FQ90, Theorem 11.7B]. By Corollary 6.6, no further slice obstructions for these inscrutable knots can be obtained from $LEO(K)$.

Another way to put these 890 knots in context is that Dunfield-Gong found 1.6 million slice knots and showed 350.5 million are not even topologically slice [DGon]. The smooth slice invariants from Khovanov homology mentioned, namely $(s_{\mathbb{F}_2}, s_{\mathbb{F}_3}, s_{\mathbb{Q}}, s^{\mathbb{Z}}, \boldsymbol{s}^{\mathrm{Sq}^1}, \boldsymbol{s}^{\mathrm{Sq}^1_o})$, provide additional slice obstructions for only about 12,100 knots. Thus, the $\boldsymbol{s}^{\beta_{15}}$ and $\boldsymbol{s}^{\beta}$ invariants increase that total by 8.1% to about 13,000.

Also, Owens and Swenton studied *alternating* knots with at most 21 crossings, determining sliceness for all but 3,276 (0.0003%) of the 1.2 billion such knots [OS23]. By Proposition 6.30, the invariants $\boldsymbol{s}^{\beta_{15}}$, $\boldsymbol{s}^{\beta}$, and $\tilde{\boldsymbol{s}}_c$ give the same information as $s_{\mathbb{F}_2}$ for alternating knots, and so cannot help resolve these remaining cases.

7.2. **Manolescu-Piccirillo knots.** Manolescu and Piccirillo [MP] gave five topologically slice knots such that if any of them were smoothly slice, then one would obtain an exotic 4-sphere. Nakamura [Nak22] then showed that none of these knots are smoothly slice by using a 0-surgery homeomorphism to relate slice properties of two knots stably after a connected sum with some 4-manifold. We found the $\boldsymbol{s}^{\beta}$ invariant can also be used to check directly that these five knots are not slice; in contrast, the $\boldsymbol{s}^{\beta_{15}}$, $\boldsymbol{s}^{\mathrm{Sq}^1}$, and $\boldsymbol{s}^{\mathrm{Sq}^1_o}$ invariants of these knots all vanish.

Manolescu-Piccirillo's strategy was based on finding pairs of knots K and K' with the same 0-surgery where $s_{\mathbb{F}_2}(K) = 0$ and $s_{\mathbb{F}_2}(K') \neq 0$. From Dunfield-Gong's data [DGon], we found knots K and K' with the same 0-surgery where $\boldsymbol{s}^{\beta}(K) = 0$ and $\boldsymbol{s}^{\beta}(K') \neq 0$, but where all the invariants $(s_{\mathbb{F}_2}, s_{\mathbb{F}_3}, s_{\mathbb{Q}}, s^{\mathbb{Z}}, \boldsymbol{s}^{\mathrm{Sq}^1}, \boldsymbol{s}^{\mathrm{Sq}^1_o})$ vanish for both knots; here K had 17 crossings and K' had 40 crossings. We also found analogous such pairs for $\boldsymbol{s}^{\beta_{15}}$, with the caveat that we could not verify that $s^{\mathbb{Z}}(K') = 0$ as K' had 50 crossings.

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