Topology (Math 3281)

Solutions to Problem Set 1 24.10.14

1. (a) We have to check that d_A satisfies all three conditions of the definition of a metric. But this is the case because we have $d_A(a,b) = d(a,b)$ where we interpret $a, b \in A$ as elements of M. Since the conditions hold for d, they also have to hold for d_A. Or more formally for the first condition:

$$d_a(a,b) \quad = \quad d(a,b) \geq 0$$

with equality if and only if $a = b$.

(b) Let $a \in A$. We want to show that the inclusion $i : A \to X$ is continuous at a. So let $\varepsilon > 0$. Choose $\delta = \varepsilon$. Then if $d_A(a,b) < \delta$, we get $d(i(a), i(b)) = d(a,b) = d_A(a,b) < \delta = \varepsilon$, so i is continuous at a. As this holds for all $a \in A$, we get that inclusion is continuous.

2. First note that

$$d_1(x,y) \quad \geq \quad d_2(x,y).$$

To see this, write $a = |x_1 - y_1|$ and $b = |x_2 - y_2|$, so that $d_1(x,y) = a + b$ and $d_2(x,y) = \sqrt{a^2 + b^2}$. Since

$$(a+b)^2 \quad \geq \quad a^2 + b^2,$$

the previous inequality follows by taking the square-root.

Now $(a-b)^2 \geq 0$, so $a^2 + b^2 \geq 2ab$. Therefore

$$\begin{aligned} d_1(x,y)^2 &= a^2 + 2ab + b^2 \\ &\leq 2(a^2 + b^2) \\ &= 2d_2(x,y)^2, \end{aligned}$$

so

$$d_1(x,y) \quad \leq \quad \sqrt{2} d_2(x,y).$$

Now choose $R_1(x) = x$ and $R_2(x) = x/\sqrt{2}$. If $y \in B_1(x; R_1(r))$, then $d_1(x,y) < r$, so $d_2(x,y) < r$ and $y \in B_2(x;r)$. Also, if $y \in B_2(x; R_2(r))$, then $d_1(x,y) \leq \sqrt{2} d_2(x,y) < r$, so $y \in B_1(x,r)$.

Consider $\text{id}\colon (\mathbb{R}^2, d_1) \to (\mathbb{R}^2, d_2)$, so let $\varepsilon > 0$. Choose $\delta = \varepsilon$. Let $a \in \mathbb{R}^2$. Then if $d_1(a, x) < \delta$, we get $d_2(\text{id}(a), \text{id}(x)) \leq d_1(\text{id}(a), \text{id}(x)) = d_1(a, x) < \delta = \varepsilon$, which means the function is continuous at a.

Consider $\text{id}\colon (\mathbb{R}^2, d_2) \to (\mathbb{R}^2, d_1)$, so let $\varepsilon > 0$. Choose $\delta = \varepsilon/\sqrt{2}$. Let $a \in \mathbb{R}^2$. Then if $d_2(a, x) < \delta$, we get $d_1(\text{id}(a), \text{id}(x)) \leq \sqrt{2} d_2(\text{id}(a), \text{id}(x)) = d_2(a, x) < \sqrt{2}\delta = \varepsilon$, which means the function is continuous at a.

3. (a) Note that $d(x, y) = \sqrt{d_1(x, y)}$, so symmetry follows, and so does $d(x, y) \geq 0$ with equality if and only if $x = y$. For the triangle inequality, note that $\sqrt{a + b} \leq \sqrt{a} + \sqrt{b}$, so

$$\begin{aligned} d(x, y) &= \sqrt{d_1(x, y)} \\ &\leq \sqrt{d_1(x, z) + d_1(z, y)} \\ &\leq \sqrt{d_1(x, z)} + \sqrt{d_1(z, y)} \\ &= d(x, z) + d(z, y). \end{aligned}$$

(b) Note that

$$(\sqrt{a} + \sqrt{b})^2 = a + b + 2\sqrt{a}\sqrt{b}$$

which is bigger than $a + b$ unless a or b are 0. Therefore we get

$$d(x, y) < d(x, z) + d(z, y)$$

unless $z = x$ or $z = y$.