Topology (Math 3281)

Solutions to Problem Set 2 07.11.14

1. Note that all τ_i contain $\emptyset$ and X. τ_3 is not closed under union and neither is τ_7. All others satisfy the conditions of a topology.

A homeomorphism between two spaces induces a bijection between the topologies. Now notice that τ_1 has four elements, τ_2 and τ_4 have five elements, and τ_5 and τ_6 have three elements. τ_2 and τ_4 are homeomorphic, with a homeomorphism given by $h(1) = 4$, $h(2) = 2$ and $h(4) = 1$. τ_5 and τ_6 are not homeomorphic, as a homeomorphism would have to satisfy $h(\{2, 4\}) = \{2\}$, but a bijection cannot satisfy this.

2. We can write $X_1 = f^{-1}((0, \infty) \times (0, \infty))$, where $f\colon \mathbb{R}^3 \to \mathbb{R}^2$ is given by

$$f(x, y, z) = (x - y, y - z).$$

As $(0, \infty) \times (0, \infty) \subset \mathbb{R}^2$ is open and f is continuous, we get X_1 is open. It is not closed, as $(0, 0, 0)$ is a limit point not contained in X_1.

For X_2 use $X_2 = g^{-1}(\{0\})$ with $g\colon \mathbb{R}^4 \to \mathbb{R}$ given by

$$g(x, y, z, w) = x^4 + y^3 + z^2 - w.$$

This shows that X_2 is closed. It is not open, as $(0, 0, 0, 0) \in X_1$, but for every $\varepsilon > 0$ we get that $(0, 0, 0, \varepsilon) \notin X_1$.

Now use $X_3 = h^{-1}([0, \infty))$ with $h\colon \mathbb{R}^2 \to \mathbb{R}$ given by

$$h(x_1, x_2) = x_1 - x_2^2$$

so that X_3 is closed. It is not open, as $(0, 0) \in X_3$, but $(0, \varepsilon) \notin X_3$ for every $\varepsilon > 0$.

The set X_4 is neither open nor closed. To see that it is not open, not that $(0, 0, 5) \in X_4$, but $(0, 0, 5 + \varepsilon) \notin X_4$ for every $\varepsilon > 0$. To see that it is not closed, note that $(0, 0, 0) \notin X_4$, but it is a limit point of X_4 because $(0, \varepsilon, 0) \in X_4$ for every $\varepsilon > 0$.

3. As $A \subset Y$ is closed in Y, there is an open set U in X with $U \cap Y = Y - A$. Now $U \cup (X - Y) = X - A$ (check that if $x \notin A$, but $x \in Y$, then $x \in U$; and if $x \in U$, then $x \notin A$). This is a union of open sets, as Y is closed in X. Therefore A is closed in X.

4. (a) We claim the set of limit points is $\{0\} \cup \{\frac{1}{n} \,|\, n \geq 1\}$.

First note that these points are limit points. In any neighborhood of 0 there will be infinitely many points of the form $\frac{1}{n} + \frac{1}{n}$. In any neighborhood of $\frac{1}{n}$ we can find infinitely many points $\frac{1}{n} + \frac{1}{m}$, $m \geq 1$, so $\frac{1}{n}$ is a limit point.

It remains to show that these are all the limit points. The basic idea is that if an interval around a point x contains only finitely many points of A, then x is not a limit point.

It is clear that points $x < 0$ and $x > 2$ cannot be limit points. Now let $x > 1$. Let $\varepsilon = x - 1 > 0$. Then there exist only finitely many points $a = \frac{1}{m} + \frac{1}{n}$ with $a \geq x - \varepsilon/2$ (note that either $n = 1$ or $m = 1$ is necessary for that). Hence x cannot be a limit point.

Now let $x \in (\frac{1}{k}, \frac{1}{k-1})$ with $k \geq 2$. If $n > k$, then only finitely many n, m with $\frac{1}{n} + \frac{1}{m} \in (\frac{1}{k}, \frac{1}{k-1})$ exist. So we look at points of the form $\frac{1}{k} + \frac{1}{m}$ with $m \geq 1$. But only finitely many of those can be within an ε-neighborhood of x by the same argument as above, if $\varepsilon > 0$ is such that $[x - \varepsilon, x + \varepsilon] \subset (\frac{1}{k}, \frac{1}{k-1})$. Thus x cannot be a limit point.

(b) We note that $|\frac{1}{n} \sin n| < \frac{1}{n}$, so points $x \neq 0$ have neighborhoods containing only finitely many elements of B. Thus the only possible limit point is 0. Given $\varepsilon > 0$, choose n such that $\frac{1}{n} < \varepsilon$. Then $\frac{1}{n} \sin n$ is within the ε-neighborhood of 0. Since $\sin n \neq 0$ for all $n \geq 1$ (by the irrationality of π), we get that 0 is a limit point.